SYDNEY GRAMMAR SCHOOL

2009 Half-Yearly Examination

FORM V
MATHEMATICS EXTENSION 1

Wednesday 13th May 2009

General Instructions
- Writing time — 2 hours
- Write using black or blue pen.
- Board-approved calculators may be used.
- All necessary working should be shown in every question.
- Start each question in a new book.

Structure of the paper
- Total marks — 84
- All seven questions may be attempted.
- All seven questions are of equal value.

Collection
- Write your name, class and master clearly on each leaflet.
- Hand in the seven questions in a single well-ordered pile.
- Hand in a leaflet for each question, even if it has not been attempted.
- If you use a second leaflet for a question, place it inside the first.
- The question papers will be collected separately.

Checklist
- Writing leaflets: 7 per boy.
- Candidature — 150 boys

Examiner
MLS

QUESTION ONE (12 marks) Use a separate writing booklet. **Marks**

(a) Factorise $9x^2 - 16$. [1]

(b) Write down the value of $\log_2 8$. [1]

(c) Write down the exact value of $\cos 225°$. [2]

(d) Solve $5 - 3x < 7$. [2]

(e) Solve $|x - 1| = 4$. [2]

(f) The line $6x - ky = 4$ passes through the point $(3, 2)$. Find the value of k. [2]

(g) Find the equation of a line that has an angle of inclination of $60°$ and an x intercept of 3. [2]

QUESTION TWO (12 marks) Use a separate writing booklet. **Marks**

(a) The first term of an AP is 6. The fifth term is 22.

(i) Show that the common difference is 4. [1]

(ii) Find the tenth term. [1]

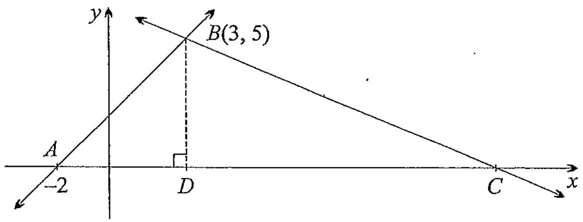
(iii) Find the sum of the first twenty terms. [1]

(iv) If the last term is 202, how many terms are there in this AP? [1]

Question 2 Continues on the Next Page

Exam continues next page ...

(b)



The diagram shows the points $A(-2,0)$, $B(3,5)$ and the point C which lies on the x-axis. The point D also lies on the x-axis such that BD is perpendicular to AC.

(i) Show that the gradient of AB is 1. [1]

(ii) Find the equation of the line AB. [1]

(iii) What is the size of $\angle BAC$? [1]

(iv) The length of BC is 13 units. Show that the length of DC is 12 units. [1]

(v) Calculate the area of $\triangle ABC$. [2]

(vi) Calculate the size of $\angle ABC$, to the nearest degree. [2]

QUESTION THREE (12 marks) Use a separate writing booklet. **Marks**

(a) Differentiate the following functions:

(i) x^3+4 [1]

(ii) $2\sqrt{x}$ [1]

(iii) $\dfrac{2}{x^4}$ [1]

(iv) $\dfrac{6x-5}{x}$ [1]

(b) Use the product rule to differentiate $(3x-1)(6x^2+5)$. [2]

(c) Differentiate $\dfrac{3x^2}{7x-1}$. [2]

(d) The equation of a parabola is $y=x^2-4x+1$.

(i) Find the gradient of the tangent to this parabola at the point $P(5,6)$. [1]

(ii) Find the equation of the tangent to the parabola at the point P. [1]

(iii) This tangent cuts the x-axis at A and the y-axis at B. Find the coordinates of M, the midpoint of AB. [2]

Exam continues overleaf ...

QUESTION FOUR (12 marks) Use a separate writing booklet. **Marks**

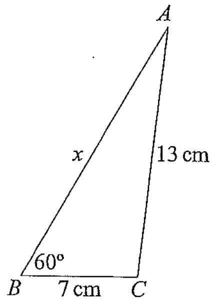
(a) (i) Sketch the parabola $y=6+x-2x^2$, showing the x and y intercepts. [2]

(ii) Solve $6+x-2x^2>0$. [1]

(b) Find the values of k for which the roots of $3x^2+2kx+4k=0$ are real and distinct. [3]

(c) Show algebraically that the line $y=2-x$ does not cut the curve $(x-2)^2+(y-2)^2=1$. [2]

(d)



The diagram shows a triangle with sides 7 cm, 13 cm and x cm, and an angle of 60° as marked.

(i) Show that $x^2-7x=120$. [1]

(ii) Find the value of x. [2]

(iii) Find the exact value of $\sin\angle ACB$. [1]

QUESTION FIVE (12 marks) Use a separate writing booklet. **Marks**

(a) Determine whether $f(x)=\dfrac{2x}{x^2+2}$ is odd, even or neither. [1]

(b) The line $y=mx+b$ is the tangent to the curve $y=x^3-3x+1$ at the point $(-2,-1)$. Find m and b. [2]

(c) (i) Write down the gradient of the line $ax+by+c=0$. [1]

(ii) Without finding the point of intersection, find the equation of the line that passes through the intersection of the lines $3x-y+4=0$ and $x+2y+3=0$ and is parallel to $2x-3y-7=0$. [3]

(d) (i) If $x=\sec\theta-\tan\theta$, show that $x+\dfrac{1}{x}=2\sec\theta$. [2]

(ii) Hence, or otherwise, solve the equation $x+\dfrac{1}{x}=\dfrac{4}{\sqrt{3}}$. [3]

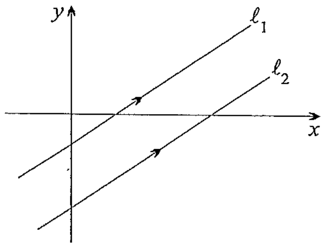
Exam continues next page ...

QUESTION SIX (12 marks) Use a separate writing booklet. **Marks**

(a) Solve $\sin 3\theta = \frac{1}{2}$, for $0° \le \theta \le 180°$. **3**

(b) Solve $2\sin^2\theta + \cos\theta = 2$, for $0° \le \theta \le 360°$. **3**

(c)



In the diagram:

ℓ_1 is the line $(r\sin\theta)x - (r\cos\theta)y = c_1$, and
ℓ_2 is the line $(r\sin\theta)x - (r\cos\theta)y = c_2$,

where $c_2 > c_1 > 0, r > 0$ and θ is acute.

(i) Copy this diagram onto your answer sheet.

(α) Find the x-intercept of ℓ_1 and ℓ_2. **1**

(β) Mark on the diagram an angle that is θ. **1**

(ii) Show that the distance between ℓ_1 and ℓ_2 is $\dfrac{c_2 - c_1}{r}$. **2**

(iii) Hence, or otherwise, find the distance between the lines **2**

$$x + \sqrt{3}\,y = \frac{3}{2} \text{ and } x + \sqrt{3}\,y = 1.$$

The Examination Continues on the Next Page

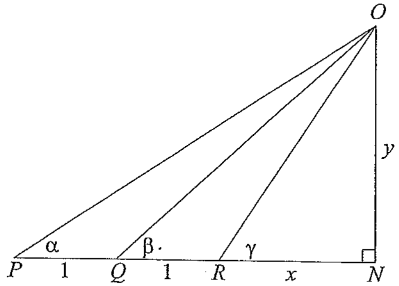
Exam continues overleaf ...

QUESTION SEVEN (12 marks) Use a separate writing booklet. **Marks**

(a) Show that $1^3 + 3^3 + 5^3 + ... + 25^3 = \sum_{j=1}^{25} j^3 - 8\sum_{k=1}^{12} k^3$. **2**

(b) Find the derivative of $y = \left(1 + (x^2 - 1)^3\right)^{\frac{1}{3}}$. **2**

(c)



In the figure above, $PQ = QR = 1$ and $ON \perp PN$. The acute angles α, β and γ are marked in the diagram. Let $RN = x$ and $ON = y$.

(i) Prove that $\cot\beta = \frac{1}{2}(\cot\alpha + \cot\gamma)$. **2**

(ii) It is given that $\gamma = 2\alpha$ and that $\cot\alpha + \cot 2\alpha = \dfrac{2}{\sqrt{3}}$.

(α) Show that $\triangle POR$ is isosceles. **1**

(β) Show that $\beta = 60°$. **1**

(γ) Find an exact value for x. **3**

(δ) Find an exact value for $\tan\alpha$. Simplify your answer. **1**

END OF EXAMINATION

Average: 68

Solutions and Marking Scheme. H.Y. 2009.
Form V 3 U

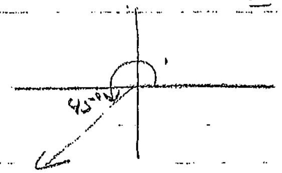
Q1.

a) $9x^2-16 = (3x-4)(3x+4)$ ✓

b) $\log_2 8 = \log_2 2^3 = 3$ ✓

c) $\cos 225^\circ = -\cos 45^\circ$ ✓ (for negative)

$= -\frac{1}{\sqrt{2}}$ ✓ (for $\frac{1}{\sqrt{2}}$).



(d) $5-3x<7$ ✓

$-3x<2$

$x>-\frac{2}{3}$ ✓ (one ✓ if $x<-\frac{2}{3}$)

(e) $|x-1| = 4$

$x-1=4$ or $x-1=-4$

$x=5$ ✓ or $x=-3$ ✓

(f) $(3,2)$ satisfies $6x-ky=4$.

$18-2k=4$ ✓

$-2k=-14$

$k=7$ ✓

(g) gradient $= \tan 60^\circ$

$= \sqrt{3}$ ✓

goes through $(3,0)$

so $y-0 = \sqrt{3}(x-3)$

$y = \sqrt{3}x - 3\sqrt{3}$ ✓

or $\sqrt{3}x - y - 3\sqrt{3} = 0$

Q2

(a) (i) $a=6$,

$T_5 = 22 = 6+4d$

$4d = 16$

$d = 4$ ✓

(ii) $T_{10} = 6+9\times 4$

$= 6+36$

$= 42$ ✓

(iii) $S_{20} = \frac{20}{2}(12+19\times 4)$

$= 10(88)$

$= 880$ ✓

(iv) $T_n = 202 = 6+(n-1)4$

$202 = 6+4n-4$

$202 = 2+4n$

$101 = 1+2n$

$n = 50$ ✓

(b)

(i) $m = \frac{5-0}{3--2}$ ✓ for substitution into $\frac{y_1-y_2}{x_1-x_2}$

$= \frac{5}{5}$

$= 1$

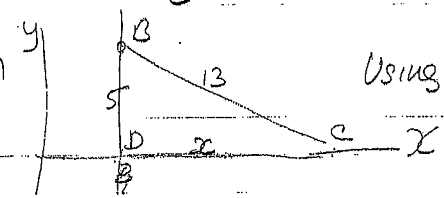
(ii) $y-0 = 1(x+2)$

$y = x+2$ ✓

(iii) $\tan\theta = 1$

$\theta = 45^\circ$ ✓

(iv)



Using Pythagoras $5^2+x^2=13^2$ ✓

$x^2 = 13^2-5^2$

$x = 12$

v) AC is $2+3+12=17$ units

Area $= \frac{1}{2} \times 17 \times 5$

$= 42\frac{1}{2}\ u^2$

vi) $\cos \angle CBD = \frac{5}{13}$

$\angle CBD = 67^\circ$ or use sine rule

$\angle ABC = 45^\circ + 67^\circ$

$= 112^\circ$

Q3.

a) (i) $y = x^3 + 4$

$\frac{dy}{dx} = 3x^2$

ii) $y = 2x^{\frac{1}{2}}$

$\frac{dy}{dx} = x^{-\frac{1}{2}}$

$= \frac{1}{\sqrt{x}}$

iii) $y = 2x^{-4}$

$\frac{dy}{dx} = -8x^{-5}$

$= -\frac{8}{x^5}$

iv) $y = \frac{6x^2 - 5x}{x^2}$

$= 6 - 5x^{-1}$

$\frac{dy}{dx} = 5x^{-2}$

$= \frac{5}{x^2}$

(b) $y = (3x-1)(6x^2+5)$

$\frac{dy}{dx} = (3x-1)(12x) + 3(6x^2+5)$

$= 36x^2 - 12x + 18x^2 + 15$

$= 54x^2 - 12x + 15$

no need to simplify

(c) $y = \frac{3x^2}{7x-1}$

$\frac{dy}{dx} = \frac{(7x-1)6x - 7(3x^2)}{(7x-1)^2}$

$= \frac{42x^2 - 6x - 21x^2}{(7x-1)^2}$

no need to simplify

$= \frac{21x^2 - 6x}{(7x-1)^2} = \frac{3x(7x-2)}{(7x-1)^2}$

(d)

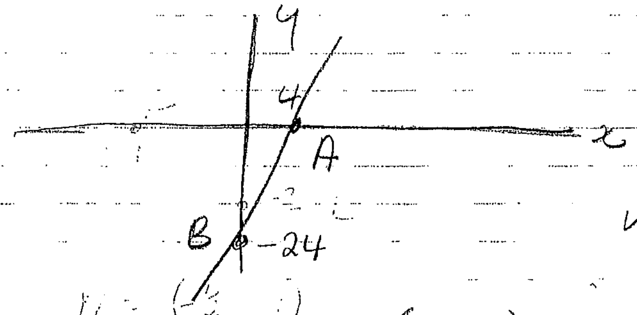
i) $\frac{dy}{dx} = 2x - 4$

at $x = 5$ $\quad m = 10 - 4$
$= 6$

ii) $y - 6 = 6(x - 5)$
$y - 6 = 6x - 30$
$y = 6x - 24$

or $6x - y - 24 = 0$

iii)



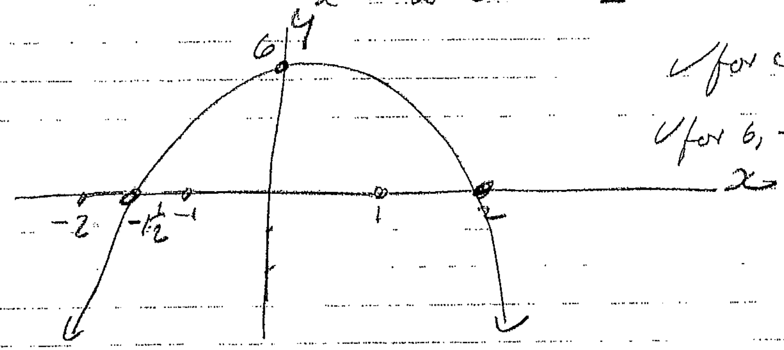
A is $(4, 0)$ $\quad$ B is $(0, -24)$

✓ for coord of A & B either on diag or written down

$M = (2, -12)$

Q4.

a) (i) $6 + x - 2x^2 = 0$ on x axis.
$(3 + 2x)(2 - x) = 0$
$x = 2$ or $-3/2$



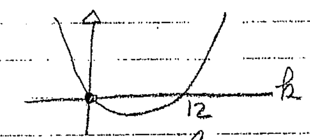
✓ for concave down

✓ for 6, $-1\frac{1}{2}$, 2.

ii) $-1\frac{1}{2} < x < 2$

(b) $3x^2 + 2kx + 4k = 0$
$\Delta = 4k^2 - 4 \times 3 \times 4k$
$= 4k^2 - 48k > 0$ for real & distinct

solve $4k^2 - 48k > 0$
$k(k - 12) > 0$



$k > 12$ or $k < 0$

(c) Solve $(x-2)^2 + (2-x-2)^2 = 1$
$(x-2)^2 + x^2 = 1$
$x^2 - 4x + 4 + x^2 = 1$.
$2x^2 - 4x + 3 = 0$
$\Delta = 16 - 4 \times 2 \times 3$
$= -8$ so no solution

or this is a circle centre (2,2), radius 1. ✓

Perp dist of line $x+y-2=0$ to (2,2) is d,

$$d = \left|\frac{2+2-2}{\sqrt{2}}\right|$$ ✓

$$= \frac{2}{\sqrt{2}} > 1$$

so line does not cut circle

(d)

i) Using cosine rule

$13^2 = x^2+7^2-2\times x\times 7\times\cos 60^\circ$

$169 = x^2+49 - 14x\times\frac{1}{2}$ } ✓

$120 = x^2-7x$

ii) $x^2-7x-120=0$

$\therefore x = \frac{7\pm\sqrt{49+480}}{2}$

$= \frac{7\pm\sqrt{529}}{2}$ $(23^2=529)$

$= \frac{7\pm 23}{2}$ ✓

$= 15$ or -8.

factorise:

$(x-15)(x+8)=0$

But x is a length so $x=15$ ✓ chosen $x>0$.

iii) $\frac{\sin C}{15} = \frac{\sin 60}{13}$

$\sin C = \frac{\sin 60}{13}\times 15$

$= \frac{\sqrt{3}}{2}\times\frac{1}{13}\times 15 = \frac{15\sqrt{3}}{26}$ ✓

Q5.

a) $f(x) = \frac{2x}{x^2+2}$

$f(-x) = \frac{2(-x)}{(-x)^2+2}$

$= \frac{-2x}{x^2+2}$

$= -f(x)$ ✓

so odd

b) $\frac{dy}{dx} = 3x^2-3$

at $x=-2$, m of curve $= 12-3$
$= 9$.

so gradient of line is 9 and $m=9$ ✓

$(-3,-1)$ lies on $y = 9x+b$

$-1 = -18+b$

so $b = 17$ ✓

(c) (i) $m = -\frac{a}{b}$ ✓

new line has the form

$3x-y+4+k(x+2y+3)=0$ ✓ $k\in\mathbb{R}$.

$(3+k)x+(2k-1)y+4+3k=0$

$m = \frac{-(3+k)}{2k-1}$ ✓

gradient of $3y=2x-7$ is $\frac{2}{3}$.

$$\frac{3+k}{1-2k} = \frac{2}{3}$$ (or equivalent)

$$9+3k = 2-4k$$
$$7k = -7$$
$$k = -1.$$

new line is $2x - 3y + 1 = 0$

(d) i) $x = \sec\theta - \tan\theta$

$$\frac{1}{x} = \frac{1}{\sec\theta - \tan\theta}$$

$$= \frac{1}{\sec\theta - \tan\theta} \times \frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}$$

$$= \frac{\sec\theta + \tan\theta}{\sec^2\theta - \tan^2\theta}$$

$$= \sec\theta + \tan\theta$$

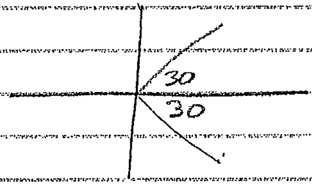
so $x + \frac{1}{x} = \sec\theta - \tan\theta + \sec\theta + \tan\theta$

$$= 2\sec\theta.$$

ii) $x + \frac{1}{x} = \frac{4}{\sqrt{3}} = 2\sec\theta$

$$\sec\theta = \frac{2}{\sqrt{3}}$$

$$\cos\theta = \frac{\sqrt{3}}{2}$$



$\theta = 30°$ or $330°$

so $\theta = 30°$, $x = \sec 30° - \tan 30°$

$$= \frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}}$$

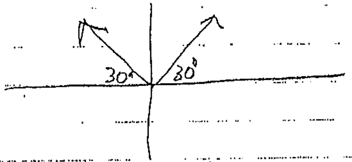
or $\theta = 330°$, $x = \sec 330° - \tan 330°$

$$= \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}.$$

$$= \frac{3}{\sqrt{3}} = \sqrt{3}$$

Q6.

a) $\sin 3\theta = \frac{1}{2}$, $\quad 0 \le \theta \le 180$, $\quad 0 \le 3\theta \le 540^\circ$



related angle is 30°

$3\theta = 30^\circ, 150^\circ, 390^\circ, 510^\circ$

$\theta = 10^\circ, 50^\circ, 130^\circ, 170^\circ$

(for answers like $10^\circ, 50^\circ, 130^\circ$ get $2/3$)

b) $2\sin^2\theta + \cos\theta = 2$

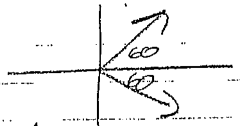
$2(1-\cos^2\theta) + \cos\theta - 2 = 0$

$-2\cos^2\theta + \cos\theta = 0$

$\cos\theta(2\cos\theta - 1) = 0$

$\cos\theta = 0$ or $\cos\theta = \frac{1}{2}$

$\theta = 90^\circ, 270^\circ$ or



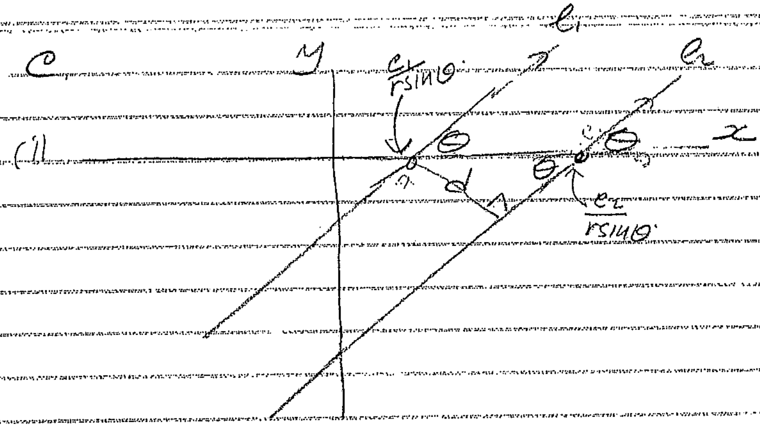
related angle is 60°

$\theta = 60^\circ$ or 300°.

$\theta = 90^\circ, 270^\circ, 60^\circ$ or 300°.

c

(i)



α. l_1 is $(r\sin\theta)x - (r\cos\theta)y = c_1$

$y=0$, $x = \frac{c_1}{r\sin\theta}$ x intercept of l_1

By symmetry, x intercept of l_2 is $\frac{c_2}{r\sin\theta}$

β. $\frac{r\sin\theta}{r\cos\theta} = \tan\theta =$ gradient of l_1 & l_2

(ii) Find d in the diagram

$\sin\theta = \dfrac{d}{\frac{c_2}{r\sin\theta} - \frac{c_1}{r\sin\theta}}$

$\sin\theta = \dfrac{r d \sin\theta}{(c_2 - c_1)}$

so $d = \dfrac{c_2 - c_1}{r}$

(iii) Find r: $x + \sqrt{3}y = \frac{3}{2}$ and $x + \sqrt{3}y = 1$

$r\sin\theta = 1$ and $r\cos\theta = \sqrt{3}$

$r^2\sin^2\theta = 1$ and $r^2\cos^2\theta = 3$

so $r^2 = 1+3$

and $r = 2$

so $d = \dfrac{\frac{3}{2} - 1}{2} = \frac{1}{4}$

Alternative (iii)

Take a pt on l_1, $x+\sqrt{3}y-\frac{3}{2}=0$

e.g. $(3/2, 0)$

& find its distance to $x+\sqrt{3}y-1=0$

$$d = \left|\frac{3/2-0-1}{\sqrt{1+\sqrt{3}^2}}\right|$$

$$= \left|\frac{1/2}{2}\right|$$

$$= \frac{1}{4}$$

Note since θ is acute, (iii) was meant to be
$x-\sqrt{3}y=\frac{3}{2}$ and $x-\sqrt{3}y=1$
but, it made no difference to the answer in (iii)

Q2.

(a) $1^3+2^3+3^3+4^3+\cdots 25^3 = (1^3+3^3+5^3+\cdots 25^3)$
$+(2^3+4^3+6^3+\cdots 24^3)$

ie $\sum_{r=1}^{25} r^3 = \sum_{r=1}^{13}(2r-1)^3 + \sum_{r=1}^{12}(2r)^3$

so $\sum_{r=1}^{13}(2r-1)^3 = \sum_{r=1}^{25} r^3 - \sum_{r=1}^{12}(2r)^3$, $2^3=8$.

ie $1^3+3^3+5^3+\cdots 25^3 = \sum_{r=1}^{25} r^3 - 8\sum_{r=1}^{12} r^3$

(b) $y = \left[1+(x^2-1)^3\right]^{\frac{1}{3}}$

let $y = u^{\frac{1}{3}}$ $\quad u = 1+v^3$ $\quad v = x^2-1$

$\frac{dy}{du} = \frac{1}{3}u^{-2/3}$ $\quad \frac{du}{dv} = 3v^2$ $\quad \frac{dv}{dx} = 2x$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx}$$

$$= \frac{1}{3}u^{-2/3} \times 3v^2 \times 2x$$

$$= \left(1+(x^2-1)^3\right)^{-2/3}(x^2-1)^2\, 2x$$

$$= \frac{2x(x^2-1)^2}{\left(1+(x^2-1)^3\right)^{2/3}}$$

C.

(i) $\cot\alpha = \frac{x+2}{y}$

$\cot\beta = \frac{x+1}{y}$

$\cot\gamma = \frac{x}{y}$

so $\frac{1}{2}(\cot\alpha + \cot\gamma) = \frac{1}{2}\left(\frac{x+2}{y} + \frac{x}{y}\right)$

$= \frac{x+1}{y}$

$= \cot\beta$

(ii) (α) $\gamma = 2\alpha$

so $\angle POR = \alpha$, exterior angle of $\triangle POR$ equals sum of int. opp angles.

Hence $\triangle POR$ is isosceles since it has two equal angles.

(β) $\cot\beta = \frac{1}{2}(\cot\alpha + \cot 2\alpha)$

$= \frac{1}{2} \times \frac{2}{\sqrt{3}}$

so $\cot\beta = \frac{1}{\sqrt{3}}$ and $0° < \beta < 180°$

so $\beta = 60°$

(γ) $\frac{x+1}{y} = \frac{1}{\sqrt{3}}$

$y = \sqrt{3}(x+1)$

And, $OR = PR = 2$ since $\triangle POR$ is isosceles

so $x^2 + y^2 = 4$

so $x^2 + 3(x+1)^2 = 4$

$4x^2 + 6x - 1 = 0$

$x = \frac{-6 \pm \sqrt{36+16}}{8}$

$= \frac{-6 \pm 2\sqrt{13}}{8}$

$= \frac{-3 \pm \sqrt{13}}{4}$

But $x > 0$

So $x = \frac{1}{4}(\sqrt{13} - 3)$

(δ) $\tan\alpha = \frac{y}{2+x}$

$= \frac{\sqrt{3}(x+1)}{2+x}$

$= \frac{\sqrt{3}\left(\frac{\sqrt{13}-3}{4} + 1\right)}{2 + \frac{\sqrt{13}-3}{4}}$

$= \frac{\sqrt{3}(\sqrt{13} - 3 + 4)}{8 + \sqrt{13} - 3}$

$= \frac{\sqrt{3}(\sqrt{13} + 1)}{\sqrt{13} + 5}$

$= \frac{\sqrt{3}(\sqrt{13}+1)(5 - \sqrt{13})}{25 - 13}$

$= \frac{\sqrt{3}}{3}(\sqrt{13} - 2)$