HIGHER SCHOOL CERTIFICATE EXAMINATION TRIAL PAPER

2012

EXTENSION 1 - MATHEMATICS

General Instructions

- Reading Time – 5 minutes
- Working Time – 2 hours
- Write using black or blue pen
  Black is preferred
- Board approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in Questions 11 – 14

Total marks – 70

Section I: Multiple Choice

Questions 1 – 10 10 marks

- Attempt all questions
- Answer on the Answer Sheet provided
- Allow about 15 minutes for this section

Section II: Extended Response

Questions 11 – 14 60 marks

- Attempt all questions
- Allow about 1 hours 45 minutes for this section

Section I

Questions 1 – 10 (1mark for each question)

Read each question and choose an answer A, B, C or D.
Record your answer on the Answer Sheet provided.
Allow about 15 minutes for this section

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1. Given that $\sin\alpha = \frac{1}{3}$, where α is not an acute angle.

   What is the value of $\sin 2\alpha$?

   A) $\frac{2}{3}$ B) $\frac{4\sqrt{2}}{9}$ C) $-\frac{2\sqrt{2}}{9}$ D) $-\frac{4\sqrt{2}}{9}$

2. The point R divides the interval joining P (– 2, – 4) and Q (4, 6) externally in the ratio 2:3.

   Which of these are coordinates of R?

   A) (– 14, – 24) B) (– 14, 0) C) $(\frac{8}{5}, 2)$ D) $(\frac{2}{5}, 0)$

3. $\lim\limits_{x\to 0} \frac{\sin x \cos x}{2x}$

   A) 2 B) 1 C) $\frac{1}{2}$ D) $\frac{1}{4}$

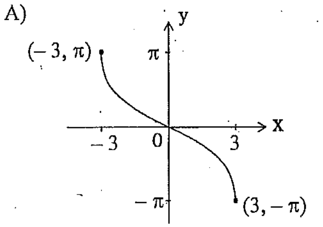
4. $\int \frac{dx}{4+9x^2} =$

   A) $\frac{1}{6}\tan^{-1}\frac{2x}{3}+c$ B) $\frac{1}{6}\tan^{-1}\frac{3x}{2}+c$

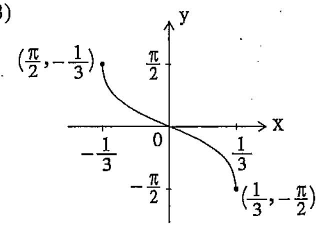
   C) $\frac{2}{27}\tan^{-1}\frac{3x}{2}+c$ D) $\frac{1}{9}\tan^{-1}\frac{3x}{2}+c$

5. Which of the following is the graph of $y = -2\sin^{-1} 3x$?

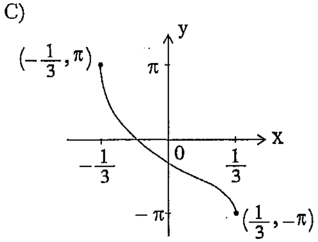
A)



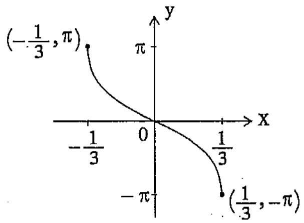
B)



C)



D)



6. $\int \left(\frac{1}{(1-x)^2} - \frac{2}{1-x} \right) dx$

A) $2\ln(1-x) + \frac{1}{x-1} + c$ B) $-2\ln(1-x) + \frac{1}{x-1} + c$

C) $2\ln(1-x) - \frac{1}{x-1} + c$ D) $-2\ln(1-x) - \frac{1}{x-1} + c$

7. Differentiate $y = \tan^{-1}\frac{1}{x}$

A) $-\frac{1}{1+x^2}$ B) $\frac{1}{1+x^2}$ C) $-\frac{x^2}{1+x^2}$ D) $\frac{x^2}{1+x^2}$

8. The velocity of a particle moving along the x axis is $V = \sqrt{x(x+2)}$.
Find its acceleration a ms^{-2}, when $x = 2$.

A) $\frac{3\sqrt{2}}{4}$ B) $\sqrt{2}$ C) 3   D) 6

9. A particle is moving in a simple harmonic motion equation $x = 3 + 2\sin^2 t$,
where x is the displacement.
The centre and amplitude of this particle are:

A) centre = 3, amplitude = 2   B) centre = 4, amplitude = 1

C) centre = 2, amplitude = 3   D) centre = 3, amplitude = 5

10. The value of $\binom{50}{1} + 2\binom{50}{2} + 3\binom{50}{3} + \ldots + 50\binom{50}{50} =$

A) 2^{49} B) 2^{50} C) $2^{50} - 1$ D) $2^{50} \times 5^2$

Section II

Question 11 – 14 (15 marks each)

Allow about 1 hour 45 minutes for this section

Question 11 **MARKS**

a) Solve $\frac{x}{2-x} > 1$ 2

b) The function $f(x) = \sqrt{x+1} - 2\sin x$ has a zero near $x = \frac{3}{4}$. 3

Use one application of Newton's method to obtain another approximation to this zero.

Give your answer correct to two significant figures.

c) Consider the polynomial $P(x) = 2x^3 - 3x^2 + kx - 8$, where k is a real number.

Given that the roots of the polynomial are $\alpha, \frac{1}{\alpha}$ and β,

i) find the roots of this polynomial. 2

ii) find the value of k. 2

d) Vanessa wants to invite 30 of her friends to her birthday party.
She wants to give the invitation cards to each of them individually.

In how many ways can she give out these cards

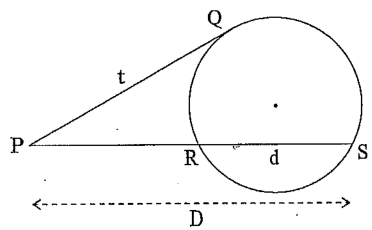
i) if there are no restrictions ? 1

ii) if she wants to give first her best four friends their cards in any order? 1

Question 11 (continued) **MARKS**

e) In the diagram the chord SR with length d is produced to meet the tangent at Q to the circle at P.

PQ has length t and PS has length D as shown.



Show $D^2 - dD - t^2 = 0$. 2

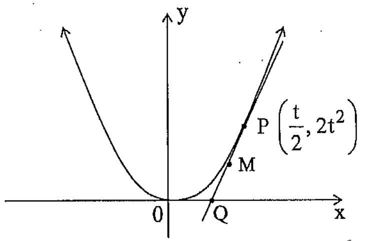
f) $f(x) = x(x+4)$ for $x \le -2$

i) Explain why $f(x)$ has an inverse function. 1

ii) Find the inverse function of $f(x)$. 1

Question 12 **MARKS**

a) The tangent at $P\left(\frac{t}{2}, 2t^2\right)$ to the parabola $y = 8x^2$ intersects the line x axis at Q. The point M is the midpoint of PQ.



i) Show that the coordinates of $M\left(\frac{3t}{8}, t^2\right)$ 3

ii) Find the locus of M as t varies. 1

b) The displacement x of a particle moving in a straight line at a time t seconds is given by $x = 3\cos\frac{t}{2} + \sqrt{3}\sin\frac{t}{2} - 2$

i) Prove that the particle is moving in a simple harmonic motion. 2

ii) Find the amplitude. 1

iii) Find the first time the particle reached its maximum velocity while moving in a positive direction. 2

c) Consider the function $f(x) = (e^{2x} - 1)(e^x - 1)$. hard for 3 unit

i) Find the coordinate of the turning point and determine its nature. 3

ii) Describe the behaviour of $f(x)$ as $x \to -\infty$ 1

iii) Sketch the curve $y = f(x)$ showing all the above information. You are not required to find any point of inflexion. 2

Question 13 **MARKS**

a) Find the exact values of x and y which are the solutions of the simultaneous equations 3

$$2\tan^{-1}x - \cos^{-1}y = \frac{\pi}{2}$$

$$\tan^{-1}x + 3\cos^{-1}y = \frac{5\pi}{6}$$

b) Use mathematical induction to prove that $(n+3)! > 4^{n+1}$, for all integers $n \geq 1$. tricky! 3

c) James placed a hot metal rod he was welding in a room with the surrounding air temperature of 18°C and allowed it to cool.

After 9 minutes, the temperature of the rod was 82°C, and after a further 3 minutes it was 50°C.

If the temperature t of the rod decreases according the equation $\frac{dT}{dt} = k(T - A)$ where k is a constant and A is the surrounding air temperature.

i) Verify that $T = A + Be^{kt}$ is a solution of the above equation. 1

ii) Find the initial temperature of the rod. 3

d) This year Rebecca is to play 6 tennis games each month. She earns $1000 bonus if she wins more than half the games she plays in any given month.

The probability that Rebecca wins any of the games is $\frac{4}{5}$.

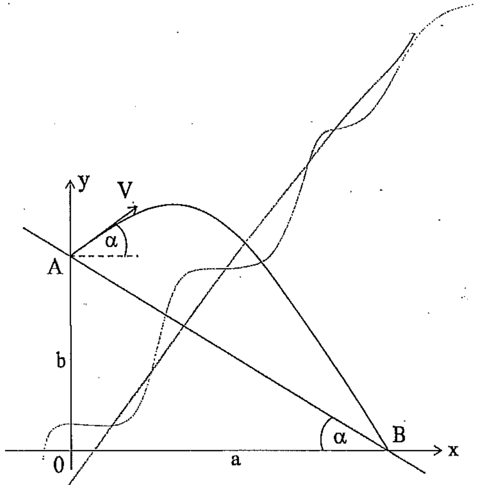
i) Show that the probability that Rebecca earns the $1000 bonus for any month is 0.90112 correct to 5 decimal places. 2

ii) Find the probability that Rebecca earns the bonus every month for the entire year. Give your answer correct to 3 decimal places. 1

iii) Find the probability that Rebecca earns at least $2000 in bonuses during this year. 2

Question 14 MARKS

a)



From a point A on a road a golf ball is projected with velocity V at an angle α to the horizontal.
It lands on the road at the point B.

Given that the road AB is inclined at an angle α to the horizontal and using the axes as shown, such that OA = b and OB = a:

i) Show that the cartesian equation of the path in terms of α is 2

$$y = x\tan\alpha - \frac{gx^2}{2V^2}(1+\tan^2\alpha)+b$$

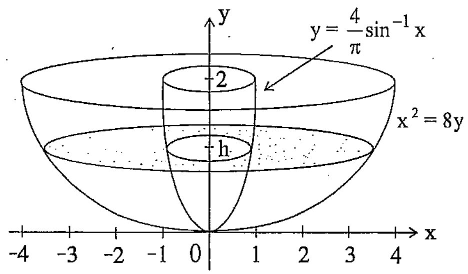
ii) Show that the distance AB travelled by the ball along the road is 3

$$AB = 2V\sqrt{\frac{b}{g}}$$

Question 14 (continued) MARKS

b) The area bounded by the curve $y = \frac{4}{\pi}\sin^{-1}x$, the curve $x^2 = 8y$ and the line $y = 2$ is rotated about the y axis to make a tank.

The tank is being filled with water at constant rate of $\frac{9\pi}{2}\text{m}^3\text{s}^{-1}$.



i) Show that the volume of the water in cubic metres when the depth is h m can be expressed by 3

$$V = 4\pi h^2 - \frac{\pi}{2}h - \sin\frac{\pi h}{2}$$

ii) Calculate the rate at which the water is rising when h = 1m. 2

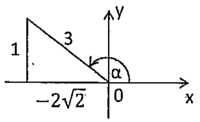
iii) Calculate the rate at which the area of the surface of the water is increasing when h = 1m 2

c) Show that 3

$$\frac{1}{3}\,{}^nC_0 - \frac{1}{4}\,{}^nC_1 + \frac{1}{5}\,{}^nC_2 + \ldots - \frac{(-1)^{n+3}}{n+3}\,{}^nC_n = \frac{2(n!)}{(n+3)!}$$

2012 Extension 1 Mathematics Solutions

1. D

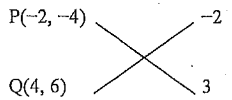


$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$= 2 \times \frac{1}{3} \times -\frac{2\sqrt{2}}{3}$

$= -\frac{4\sqrt{2}}{9}$

2. A



$x = \frac{-2\times 4 + 3 \times -2}{-2+3} = -14$

$y = \frac{-2\times 6 + 3\times -4}{-2+3} = -24$

Hence, R (−14, −24)

3. C

$\lim_{x\to 0} \frac{\sin x \cos x}{2x} = \lim_{x\to 0} \frac{2\sin x . \cos x}{2x} \times \frac{1}{2}$

$= \lim_{x\to 0} \frac{\sin 2x}{2x} \times \frac{1}{2}$

$= 1 \times \frac{1}{2}$

$= \frac{1}{2}$

4. B

$\int \frac{dx}{4+9x^2} = \frac{1}{9}\int \frac{dx}{\frac{4}{9}+x^2}$

$= \frac{1}{9} \times \frac{3}{2} \int \frac{\frac{2}{3}dx}{\left(\frac{2}{3}\right)^2 + x^2}$

$= \frac{1}{6} \tan^{-1}\frac{3x}{2} + c$

5. D

$y = -2\sin^{-1}3x$

Domain: $-1 \le 3x \le -1$ that is $-\frac{1}{3} \le x \le \frac{1}{3}$

Range: $-\frac{\pi}{2} \le \sin^{-1}3x \le \frac{\pi}{2}$ that is $-\pi \le y \le \pi$

Also, the curve passes the origin.

Hence, the only graph that matches is D

6. C

$\int \left(\frac{1}{(1-x)^2} - \frac{2}{1-x}\right)dx = \int \left((1-x)^{-2} + \frac{2\times -1}{1-x}\right)dx$

$= (1-x)^{-1} + 2\ln(1-x) + c$

$= \frac{1}{1-x} + 2\ln(1-x) + c$

$= \frac{-1}{x-1} + 2\ln(1-x) + c$

7. A

Let $u = \frac{1}{x}$ $\qquad y = \tan^{-1}u$

$\frac{du}{dx} = -1x^{-2}$ $\qquad \frac{dy}{du} = \frac{1}{1+u^2}$

$\frac{du}{dx} = -\frac{1}{x^2}$ $\qquad \frac{dy}{du} = \frac{1}{1+\left(\frac{1}{x}\right)^2}$

Hence, $\frac{dy}{dx} = -\frac{1}{x^2} \times \frac{1}{1+\left(\frac{1}{x}\right)^2} = -\frac{1}{x^2+1}$

8. C

$v = \sqrt{x^2+2x}$

$v^2 = x^2 + 2x$

$\frac{1}{2}v^2 = \frac{1}{2}x^2 + x$

$a = \frac{d\left(\frac{1}{2}v^2\right)}{dx} = x + 1$

Hence, when x = 2 then a = 3

9. B

$\cos 2t = 1 - 2\sin^2 t$ then $2\sin^2 t = 1 - \cos 2t$

So $x = 3 + 2\sin^2 t$

$= 3 + 1 - \cos 2t$

$= 4 - \cos 2t.$

Hence, the centre of oscillation is 4 and the amplitude is 1.

10. D

Consider the following binomial

$(1+x)^{50} = \binom{50}{0} + \binom{50}{1}x + \binom{50}{2}x^2 + \ldots + \binom{50}{50}x^{50}$

Differentiate both sides with respect to x, we get:

$50(1+x)^{49} = \binom{50}{1} + 2\binom{50}{2}x + \ldots + 50\binom{50}{50}x^{49}$

Let x = 1, we get:

$50\times 2^{49} = \binom{50}{1} + 2\binom{50}{2} + \ldots\ldots + 50\binom{50}{50}$

But $50\times 2^{49} = 25\times 2\times 2^{49} = 5^2 \times 2^{50}$

Hence, $\binom{50}{1} + 2\binom{50}{2} + \ldots + 50\binom{50}{50} = 2^{50}\times 5^2$

QUESTION 11

a) $\frac{x}{2-x} > 1$

$(2-x)^2 \times \frac{x}{2-x} > 1\times(2-x)^2$

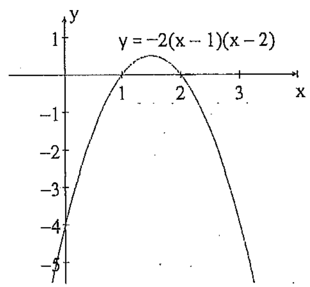
$(2-x)x > 4 - 4x + x^2$

$-x^2 + 4x - 4 + 2x - x^2 > 0$

$-2x^2 + 6x - 4 > 0$

$-2(x^2 - 3x + 2) > 0$

$-2(x-1)(x-2) > 0$



On the graph it can be seen that y > 0 when $1 < x < 2$,

Hence, $\frac{x}{2-x} > 1$ when $1 < x < 2$.

b) $f(x) = (x+1)^{\frac{1}{2}} - 2\sin x$

$f'(x) = \frac{1}{2}(x+1)^{-\frac{1}{2}} - 2\cos x$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$x_2 = 0.75 - \frac{f(0.75)}{f'(0.75)}$

$x_2 = 0.75 - \frac{(1.75)^{\frac{1}{2}} - 2\sin(0.75)}{\frac{1}{2}(1.75)^{-\frac{1}{2}} - 2\cos(0.75)}$

$x_2 = 0.71277743\ldots..$

$x_2 = 0.71$ (2 significant figures)

c) i) $P(x) = 2x^3 - 3x^2 + kx - 8$

$\alpha \times \frac{1}{\alpha} \times \beta = 4$ then $\beta = 4$

Now, $\alpha + \frac{1}{\alpha} + 4 = \frac{3}{2}$

$\alpha + \frac{1}{\alpha} = -\frac{5}{2}$

$2\alpha^2 + 2 = -5\alpha$

$2\alpha^2 + 5\alpha + 2 = 0$

$(2\alpha+1)(\alpha+2) = 0$

So $\alpha = -\frac{1}{2}$ or -2

Hence, the roots of P(x) = 0 are 4, −2 and $-\frac{1}{2}$

ii) As −2 is a root then P (−2) = 0 this means

$2\times(-2)^3 - 3\times(-2)^2 + (-2)\times k - 8 = 0$

$-16 - 12 - 2k - 8 = 0$

$-2k = 36$

$k = -18$

d) i) If 30 cards are to be handed out with no restrictions then the number of different ways is 30!

ii) If 4 cards are to be handed out to her 4 best friends first, this can be done in 4! ways. The remaining 16 cards could be handed out in 16! ways. Hence, the total number of ways is 4! × 16!

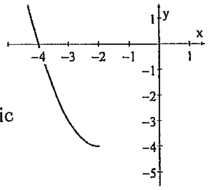
e) $PQ^2 = PS \times PR$ (the square of the length of the tangent from an external point is equal to the product of the intercepts)

So $t^2 = D\times(D-d)$

$t^2 = D^2 - dD$

Hence, $D^2 - dD - t^2 = 0$

f) i) The portion of the curve $f(x) = x(x+4)$ in the domain $x \le -2$ is shown. As this curve is monotonic decreasing which means one- one function it has an inverse function.



ii) To find the inverse function we interchange x and y.

$x = y^2 + 4y$

$y^2 + 4y - x = 0$

so $y = \frac{-4 \pm \sqrt{16+4x}}{2}$

$y = \frac{-4 \pm 2\sqrt{4+x}}{2}$

$y = -2 \pm \sqrt{4+x}$

As the domain of f(x) is $x \le -2$ then range of its inverse function is $y \le -2$.

This indicates that that the inverse function is $y = -2 - \sqrt{4+x}$

QUESTION 12

a) i) Given $y = 8x^2$ then $\frac{dy}{dx} = 16x$

At P, $x = \frac{t}{2}$ so the gradient of the tangent at P is $16 \times \frac{t}{2} = 8t$

The equation of the tangent is

$y - 2t^2 = 8t(x - \frac{t}{2})$

$y = 8tx - 4t^2 + 2t^2$

$y = 8tx - 2t^2$

To find the coordinates of Q we let y= 0,
We get:

$8tx - 2t^2 = 0$

$8tx = 2t^2$

$x = \frac{t}{4}$

So Q is $(\frac{t}{4}, 0)$

Now P $(\frac{t}{2}, 2t^2)$ and Q $(\frac{t}{4}, 0)$

Hence, the coordinates of the midpoint M is

$\left(\frac{\frac{t}{2}+\frac{t}{4}}{2}, \frac{2t^2-0}{2}\right) = \left(\frac{3t}{8}, t^2\right)$

ii) To find the locus of M, we know that $x = \frac{3t}{8}$ so $t = \frac{8x}{3}$ by substituting in $y = t^2$ we get $y = \frac{64x^2}{9}$

Hence, the locus of M is $y = \frac{64x^2}{9}$

b) i) $A\cos\left(\frac{t}{2} - \theta\right) = A\cos\theta \cos\frac{t}{2} + A\sin\theta \sin\frac{t}{2}$

Let $A\cos\left(\frac{t}{2} - \theta\right) = 3\cos\frac{t}{2} + \sqrt{3}\sin\frac{t}{2}$

So $A\cos\theta = 3$ and $A\sin\theta = \sqrt{3}$

By division $\tan\theta = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$

So $\theta = \frac{\pi}{6}$

By squaring and adding

$A^2\sin^2\theta + A^2\cos^2\theta = 3 + 9$

$A^2(\sin^2\theta + \cos^2\theta) = 12$

$A^2 = 12$ so $A = 2\sqrt{3}$ as A is positive.

Hence, $2\sqrt{3}\cos\left(\frac{t}{2} - \frac{\pi}{6}\right) = 3\cos\frac{t}{2} + \sqrt{3}\sin\frac{t}{2}$

So $x = 2\sqrt{3}\cos\left(\frac{t}{2} - \frac{\pi}{6}\right) - 2$

$\dot{x} = -\sqrt{3}\sin\left(\frac{t}{2} - \frac{\pi}{6}\right)$

$\ddot{x} = -\frac{1}{2}\sqrt{3}\cos\left(\frac{t}{2} - \frac{\pi}{6}\right)$

But $\sqrt{3}\cos\left(\frac{t}{2} - \frac{\pi}{6}\right) = \frac{x+2}{2}$

$\ddot{x} = -\frac{1}{2}\left(\frac{x+2}{2}\right) = -\frac{1}{4}(x+2)$

As the acceleration is in the form

$\ddot{x} = -n^2(x - a)$

Then the motion is simple harmonic about $x = -2$.

ii) As $x = 2\sqrt{3}\cos\left(\frac{t}{2} - \frac{\pi}{6}\right) - 2$ then the amplitude is $2\sqrt{3}$.

iii) The particle reaches its maximum velocity as it passes through the centre of motion, that is when $x = -2$.

$-2 = 2\sqrt{3}\cos\left(\frac{t}{2} - \frac{\pi}{6}\right) - 2$

$0 = 2\sqrt{3}\cos\left(\frac{t}{2} - \frac{\pi}{6}\right)$

So $\cos\left(\frac{t}{2} - \frac{\pi}{6}\right) = 0$

$\frac{t}{2} - \frac{\pi}{6} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

$\frac{t}{2} = \frac{4\pi}{6}, \frac{10\pi}{6}, \frac{16\pi}{6}, \ldots$

$t = \frac{4\pi}{3}, \frac{10\pi}{3}, \frac{16\pi}{3}, \ldots$

$\dot{x} = -\sqrt{3}\sin\left(\frac{t}{2} - \frac{\pi}{6}\right)$

When $t = \frac{4\pi}{3}$, $\dot{x} = -\sqrt{3}\sin\frac{\pi}{2} = -\sqrt{3}$,
which is invalid as it is negative.

When $t = \frac{10\pi}{3}$, $\dot{x} = -\sqrt{3}\sin\frac{3\pi}{2} = \sqrt{3}$,
which is valid as it is positive.

Hence, the first time the particle reaches its maximum velocity in the positive direction is when $t = \frac{10\pi}{3}$.

c) i) $f(x) = (e^{2x} - 1)(e^x - 1)$.

$= e^{3x} - e^{2x} - e^x + 1$

So $f'(x) = 3e^{3x} - 2e^{2x} - e^x$

$= e^x(3e^{2x} - 2e^x - 1)$

$= e^x(3e^{2x} - 3e^x + e^x - 1)$

$= e^x(3e^x(e^x - 1) + 1(e^x - 1))$

$= e^x(3e^x + 1)(e^x - 1)$

Let $f'(x) = 0$ to find the possible stationary turning points. we get:

$e^x(3e^x + 1)(e^x - 1) = 0$

but $e^x > 0$ for all x

Hence, the only solution possible is when $e^x = 1$ that is when $x = 0$ and

$f(0) = (1-1)(1-1) = 0$

So (0, 0) is a turning point.

Now, $f''(x) = 9e^{3x} - 4e^{2x} - e^x$

So $f''(0) = 9 - 4 - 1 = 4 > 0$

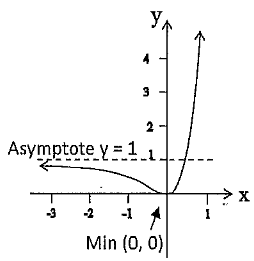
Hence, there is a minimum turning point at (0, 0)

ii) $f(x) = e^{3x} - e^{2x} - e^x + 1$

as $x \to -\infty$, $e^x \to 0$, so $e^{3x} - e^{2x} - e^x \to 0$

Hence, $f(x) \to 1$

iii)



QUESTION 13

a) $2\tan^{-1}x - \cos^{-1}y = \frac{\pi}{2}$ (1)

$\tan^{-1}x + 3\cos^{-1}y = \frac{5\pi}{6}$ (2)

Multiplying (1) by 3 and then add to (2),we get:

$7\tan^{-1}x = \frac{3\pi}{2} + \frac{5\pi}{6}$

$7\tan^{-1}x = \frac{7\pi}{3}$

$\tan^{-1}x = \frac{\pi}{3}$ Hence, $x = \sqrt{3}$

By Substituting in (1), we get:

$\frac{2\pi}{3} - \cos^{-1}y = \frac{\pi}{2}$

$\cos^{-1}y = \frac{\pi}{6}$ Hence, $y = \frac{\sqrt{3}}{2}$

b) Prove that $(n+3)! > 4^{n+1}$ for $n \ge 1$

For n = 1

LHS = $(1+3)! = 4! = 24$

RHS = $4^2 = 16$

Since LHS > RHS then the statement is true for n=1

Assume the statement is true for n = k that is $(k+3)! > 4^{k+1}$

Our aim is to prove it true for n = k+1 that is $(k+4)! > 4^{k+2}$

Starting from the assumption and multiplying both sides by (k + 4), we get:

$(k+3)!\,(k+4) > (k+4) \times 4^{k+1}$

$(k+4)! > (k+4) \times 4^{k+1}$

$> k \times 4^{k+1} + 4 \times 4^{k+1}$

$> k \times 4^{k+1} + 4^{k+2}$

$> 4^{k+2}$

Note: The sum of two positive is greater than only one of them.

Hence, if the statement is true for n= k then it is also true $n = k + 1$.

Now the statement is true for n=1 and by mathematical induction it is true for $n = 2$, 3 and so on.

Hence the statement is true for all integers $n \ge 1$.

c) i) If $T = A + Be^{kt}$

$\frac{dT}{dt} = kBe^{kt}$ But $T - A = Be^{kt}$

Hence, $\frac{dT}{dt} = k(T - A)$

ii) Given $A = 18$ then $T = 18 + Be^{kt}$

When $t = 9$, $T = 82$ When $t = 12$, $T = 50$

So $82 = 18 + Be^{9k}$ So $50 = 18 + Be^{12k}$

$64 = Be^{9k}$ (1) $32 = Be^{12k}$ (2)

By dividing (2) by (1), we get:

$\frac{32}{64} = e^{12k} \div e^{9k}$

$\frac{1}{2} = e^{3k}$

$\ln\left(\frac{1}{2}\right) = 3k$

$k = -\frac{\ln 2}{3}$

By substituting in (1), we get:

$64 = Be^{9 \times -\frac{\ln 2}{3}}$

$64 = Be^{-3\ln 2}$

$B = 512$

Now, $T = 18 + 512e^{kt}$

To find the initial temperature we let $t = 0$, we get:

$T = 18 + 512e^{0} = 530°C$

d) i) Probability(win more than half the games)

$= P(\text{win 4games}) + P(\text{win 5games}) + P(\text{win 6 games})$

$= {}^6C_4\left(\frac{4}{5}\right)^4\left(\frac{1}{5}\right)^2 + {}^6C_5\left(\frac{4}{5}\right)^5\left(\frac{1}{5}\right)^1 + {}^6C_6\left(\frac{4}{5}\right)^6\left(\frac{1}{5}\right)^0$

$= 0.24576 + 0.393216 + 0.262144$

$= 0.90112$

ii) P (win bonus for 12 months) $= 0.90112^{12}$

$= 0.286676\ldots$

$= 0.287$ (3 d.p.)

iii) P (win at least 2 months)

$= P(\text{win2}) + P(\text{win 3}) + P(\text{win 4}) + P(\text{win 5}) + P(\text{win 6})$

$= 1 - [\,P(\text{win 1}) + P(\text{win 0})\,]$

$= 1 - [\,{}^{12}C_1\, 0.90112^1 \times 0.09888^{11} + {}^{12}C_{12}\, 0.09888^{12}\,]$

$= 1 - [\,9.553377 \times 10^{-11} + 8.735776 \times 10^{13}\,]$

$= 1 - 9.640734 \times 10^{-11}$

$= 0.9999\ldots$

QUESTION 14

a) i) Vertical motion / Horizontal motion

$\ddot{y} = -g$ $\quad\quad \ddot{x} = 0$

$\dot{y} = -gt + v\sin\alpha$ $\quad\quad \dot{x} = v\cos\alpha$

$y = -\frac{1}{2}gt^2 + vt\sin\alpha + b$ $\quad\quad x = vt\cos\alpha$

So $t = \frac{x}{v\cos\alpha}$

Hence, the equation of the path is

$y = -\frac{1}{2}g\left(\frac{x}{v\cos\alpha}\right)^2 + v\left(\frac{x}{v\cos\alpha}\right)\sin\alpha + b$

$y = -\frac{gx^2}{2v^2}\sec^2\alpha + x\tan\alpha + b$

$y = -\frac{gx^2}{2v^2}(1 + \tan^2\alpha) + x\tan\alpha + b$

ii) From the diagram, $\tan\alpha = \frac{b}{a}$

By substituting in the equation of the path, we get:

$y = \frac{gx^2}{2v^2}\left(1 + \frac{b^2}{a^2}\right) + x \times \frac{b}{a} + b$

$y = \frac{gx^2}{2v^2}\left(\frac{a^2}{a^2} + \frac{b^2}{a^2}\right) + x \times \frac{b}{a} + b$

$y = \frac{gx^2}{2v^2}\left(\frac{a^2+b^2}{a^2}\right) + x \times \frac{b}{a} + b$

Now, B(a, 0) lies on the equation of the path.

By substituting the coordinates of B, we get:

$0 = -\frac{ga^2}{2v^2}\left(\frac{a^2+b^2}{a^2}\right) + a \times \frac{b}{a} + b$

$0 = -\frac{g}{2v^2}(a^2 + b^2) + b + b$

But from the diagram, $AB^2 = a^2 + b^2$ then

$0 = -\frac{g}{2v^2}(AB^2) + 2b$

that is $\frac{g}{2v^2}(AB^2) = 2b$

$AB^2 = \frac{4bv^2}{g}$

$AB = 2v\sqrt{\frac{b}{g}}$

b) i) $y = \frac{4}{\pi}\sin^{-1}x$ then $\frac{\pi y}{4} = \sin^{-1}x$

So $x = \sin\left(\frac{\pi y}{4}\right)$

Now, $V = \pi\int_0^h 8y\, dy - \pi\int_0^h \sin^2\left(\frac{\pi y}{4}\right) dy$

$V = \pi\int_0^h 8y - \sin^2\left(\frac{\pi y}{4}\right) dy$

$V = \pi\int_0^h 8y - \frac{1}{2} + \frac{1}{2}\cos\left(\frac{\pi y}{2}\right) dy$

$V = \pi\left[4y^2 - \frac{y}{2} + \frac{1}{2} \times \frac{2}{\pi}\sin\left(\frac{\pi y}{2}\right)\right]_0^h$

$V = \pi\left[4h^2 - \frac{h}{2} + \frac{1}{\pi}\sin\left(\frac{\pi h}{2}\right) - 0\right]$

$V = 4\pi h^2 - \frac{\pi h}{2} + \sin\left(\frac{\pi h}{2}\right)$

ii) $V = 4\pi h^2 - \frac{\pi h}{2} + \sin\left(\frac{\pi h}{2}\right)$

$\frac{dV}{dh} = 8\pi h - \frac{\pi}{2} + \frac{\pi}{2}\cos\left(\frac{\pi}{2}h\right)$

Now, $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$

But $\frac{dV}{dt} = \frac{9\pi}{2}$ then

$\frac{9\pi}{2} = \left[8\pi h - \frac{\pi}{2} + \frac{\pi}{2}\cos\left(\frac{\pi}{2}h\right)\right] \times \frac{dh}{dt}$

When $h = 1$, we get:

$\frac{9\pi}{2} = \left[8\pi - \frac{\pi}{2} + \frac{\pi}{2}\cos\left(\frac{\pi}{2}\right)\right] \times \frac{dh}{dt}$

$\frac{9\pi}{2} = \frac{17\pi}{2} \times \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{9}{17}$ m s^{-1}

iii) $A = \pi(x^2_{\text{parabola}} - x^2_{\text{inverse}})$

$A = \pi\left(8h - \sin^2\left(\frac{\pi h}{4}\right)\right)$

$= \pi\left[8h - \frac{1}{2} + \frac{1}{2}\cos\left(\frac{\pi h}{2}\right)\right]$

$\frac{dA}{dh} = \pi\left[8 - \frac{\pi}{4}\sin\left(\frac{\pi h}{2}\right)\right]$

$\frac{dA}{dt} = \frac{dA}{dh} \times \frac{dh}{dt}$

$\frac{dA}{dt} = \pi\left[8 - \frac{\pi}{4}\sin\left(\frac{\pi h}{2}\right)\right] \times \frac{9}{17}$

When $h = 1$,

$\frac{dA}{dt} = \pi\left[8 - \frac{\pi}{4}\right] \times \frac{9}{17}$

$= \frac{9\pi}{68}(32 - \pi)$ m^{2}s^{-1}

c) $(1 + x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_n x^n$

$x^2(1 + x)^n = {}^nC_0 x^2 + {}^nC_1 x^3 + \ldots + {}^nC_n x^{n+2}$

Integrating both sides with respect to x, we get:

$\int x^2(1 + x)^n dx = \int({}^nC_0 + {}^nC_1 x + \ldots + {}^nC_n x^n)\, dx$

LHS $= \int x^2(1 + x)^n dx$

Let $u = 1 + x$ then $du = dx$

LHS $= \int x^2(1 + x)^n\, dx$

$= \int (u - 1)^2 u^n\, du$

$= \int (u^{n+2} - 2u^{n+1} + u^n)\, du$

$= \frac{1}{n+3}u^{n+3} - \frac{2}{n+2}u^{n+2} + \frac{1}{n+1}u^{n+1} + c$

$= \frac{1}{n+3}(x + 1)^{n+3} - \frac{2}{n+2}(x + 1)^{n+2} + \frac{1}{n+1}(x + 1)^{n+1}$

RHS $= \int \binom{n}{0} x^2 + \binom{n}{1} x^3 + + \ldots + \binom{n}{n} x^{n+2}\, dx$

$= \frac{1}{3}\binom{n}{0}x^3 + \frac{1}{4}\binom{n}{1}x^4 + \ldots + \frac{1}{n+3}\binom{n}{n}x^{n+3} + C$

Now, by considering that the fact that LHS = RHS and by Substituting $x = 0$ into both of them, we get:

$\frac{1}{n+3} - \frac{2}{n+2} + \frac{1}{n+1} = C$

$\frac{(n+1)(n+2) - 2(n+3)(n+1) + (n+2)(n+3)}{(n+3)(n+2)(n+1)} = C$

$\frac{(n^2+3n+2) - 2(n^2+4n+3) + (n^2+5n+6)}{(n+3)(n+2)(n+1)} = C$

$\frac{2}{(n+3)(n+2)(n+1)} = C$

$\frac{2(n!)}{(n+3)!} = C$

$\frac{1}{n+3}(x + 1)^{n+3} - \frac{2}{n+2}(x + 1)^{n+2} + \frac{1}{n+1}(x + 1)^{n+1}$

$= \frac{1}{3}\binom{n}{0}x^3 + \frac{1}{4}\binom{n}{1}x^4 + \ldots + \frac{1}{n+3}\binom{n}{n}x^{n+3} + \frac{2(n!)}{(n+3)!}$

By substituting $x = -1$, we get:

$0 = -\frac{1}{3}\binom{n}{0} + \frac{1}{4}\binom{n}{1} + \cdots + \frac{(-1)^{n+3}}{n+3}\binom{n}{n} + \frac{2(n!)}{(n+3)!}$

Hence, $\frac{1}{3}\binom{n}{0} - \frac{1}{4}\binom{n}{1} + \ldots - \frac{(-1)^{n+3}}{n+3}\binom{n}{n} = \frac{2(n!)}{(n+3)!}$