SYDNEY BOYS HIGH
MOORE PARK, SURRY HILLS

2002
HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Mathematics Extension 2

General Instructions
- Reading time — 5 minutes
- Working time — 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks — 120
- Attempt questions 1–8
- All questions are of equal value, the mark value is shown beside each part.

Examiner: D.M.Hespe

Note: This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

Total marks – 120
Attempt Questions 1–8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (15 marks) Use a SEPARATE writing booklet.

(a) Find

(i) $\int \sin^{-1} x\, dx$ **2**

(ii) $\int \frac{x}{1+x^4}\, dx$ **2**

(iii) $\int \tan^3 x\, dx$ **2**

(b) Evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\cos x}$ using the substitution $t = \tan\frac{x}{2}$. **3**

(c) Given that $I_n = \int_1^e (\ln x)^n\, dx, \quad n = 0,\ 1,\ 2,\ \ldots,$ **3**

show that $I_n = e - nI_{n-1}$.

(d) If $x = \frac{\pi}{4} - u$,

(i) Show that $\tan x = \frac{1-\tan u}{1+\tan u}$. **1**

(ii) Hence or otherwise, show that $\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\, dx = \frac{\pi}{8}\ln 2$. **2**

Marks

Question 2 (15 marks) Use a SEPARATE writing booklet.

(a) Explain the flaw in this "proof" that $i = -i$. **2**

$$
\begin{aligned}
i &= i \\
\sqrt{-1} &= \sqrt{-1} \\
\sqrt{\frac{-1}{1}} &= \sqrt{\frac{1}{-1}} \\
\frac{\sqrt{-1}}{\sqrt{1}} &= \frac{\sqrt{1}}{\sqrt{-1}} \\
\frac{i}{1} = \frac{1}{i} &= \frac{-(-1)}{i} = \frac{-i^2}{i} = -i \\
\therefore\ i &= -i
\end{aligned}
$$

(b) $u = -3-4i$ and $v = 1-i$ are two complex numbers. Express in the form $x+iy$, where x and y are real:

(i) $\bar{u}-v$ **1**

(ii) $\dfrac{2u}{v}$ **2**

(iii) $\sqrt{u}$ **2**

(c) On an Argand diagram sketch the region defined by **2**

$$\{ z: -\tfrac{\pi}{6} \le \arg z \le \tfrac{\pi}{6} \cap |z| \le 1\}.$$

(d) (i) If a, b are the complex numbers represented by the points A and B on an Argand diagram, what geometrical properties correspond to the modulus and argument of $\dfrac{b}{a}$? **2**

(ii) Show that, if the four points representing the complex numbers z_1, z_2, z_3, and z_4 are concyclic, the fraction $\dfrac{(z_1-z_2)(z_3-z_4)}{(z_3-z_2)(z_1-z_4)}$ must be real. **4**

Marks

Question 3 (15 marks) Use a SEPARATE writing booklet.

(a) Reduce to irreducible factors over the complex field: $x^3 - 4x^2 + 7x - 6$. **3**

(b) Find the polynomial $f(x)$ of the fourth degree such that $f(0) = f(1) = 1$, $f(2) = 13$, $f(3) = 73$ and $f'(0) = 0$. **4**

(c) (i) Prove that if $P(x)$ has a root of multiplicity m, then $P'(x)$ has a root of multiplicity $m - 1$. **2**

(ii) Find the value of c if the polynomial $5x^5 - 3x^3 + c$ has a positive repeated root. **3**

(d) Let α, β, γ be the roots of the equation $x^3 + px + q = 0$, where $q \neq 0$. Find, in terms of p and q, the coefficients of the cubic equation whose roots are α^{-1}, β^{-1}, and γ^{-1}. **3**

Marks

Question 4 (15 marks) Use a SEPARATE writing booklet.

(a) Solve the simultaneous equations: **4**

$$x^2+xy+y^2 = 7,$$
$$2x^2-xy+y^2 = 28.$$

(b) Show that if $b^2 < 4ac$, the value of the function ax^2+bx+c will have the same sign as a for all real values of x. **2**

(c) (i) By considering the expression $x^2-2xy+5y^2+2x-14y+k$ as a quadratic function of x, show that it is positive for all real values of x and y if $k>10$. **4**

(ii) Show that if $k=10$, the expression may be written in the form $(x+py+q)^2+(ry+s)^2$, and hence find the simultaneous values of x and y for which the expression is zero. **4**

(iii) Deduce the minimum value of the expression for a general value of k. **1**

Marks

Question 5 (15 marks) Use a SEPARATE writing booklet.

(a) A particle P of mass m starts from rest at a point O and falls under gravity in a medium where the resistance to its motion has magnitude mkv, v being the speed of the particle and k is a constant.

(i) Draw a diagram to show the *forces* acting on the particle during this ~~downward~~ path, and hence write down the equation of motion. **2** ✓

upward.

(ii) Show that the expression for its velocity v at any time t is given by **2** ✓

$$v = \frac{g}{k}\left(1 - e^{-kt}\right).$$

(iii) Explain what is meant by the *terminal velocity* and find an expression for the terminal velocity V_T. **3** ✓

(b) A second particle Q, also of mass m, is fired vertically upwards from O with initial speed u, so that P and Q leave O simultaneously.

(i) Draw a diagram to show the *forces* acting on the particle during this downward path, and hence write down the equation of motion. **2** ✓

(ii) Find an expression for the time t when Q comes to rest. **3** ✓

(c) Show that, at the instant Q comes to rest, the velocity of P is given by: **3**

$$v = \frac{V_T u}{V_T + u}.$$

Marks

Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Show that if x is real, the expression $\dfrac{(x-2)^2}{x-1}$ cannot take any value between -4 and 0. **2**

(ii) Sketch the graph of the expression. **3**

(iii) Show that the equation $\dfrac{(x-2)^2}{x-1} = \dfrac{k}{x}$ has three real roots if k is positive, but only one real root if k is negative. **2**

(b) For $z = r(\cos\theta + i\sin\theta)$, find r and the smallest positive value of θ which satisfy the equation $2z^3 = 9 + 3\sqrt{3}\,i$. **2**

(c) Using the method of shells find the volume of the solid formed when the region bounded by the curve $y = x^2 + 1$ and the x-axis between $x = 0$ and $x = 2$ is rotated about the y-axis. **3**

(d) Explain why, if $\lim\limits_{n\to\infty}\left(\sqrt{n^2+n} - n\right) = \lim\limits_{n\to\infty}\left(n \times \sqrt{1+\frac{1}{n}} - n\right)$, then the limit is not zero, but a half. **3**

Marks

Question 7 (15 marks) Use a SEPARATE writing booklet.

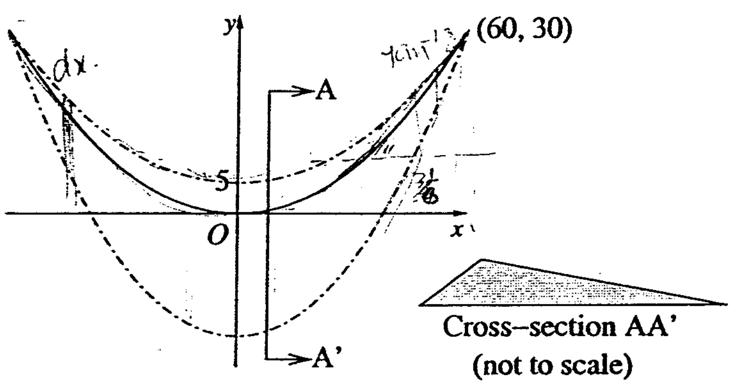
(a) Find the rational roots of $x^4+2x^3-17x^2-18x+32=0$ using the substitution $y=x^2+x$, or otherwise. **2**

(b) (i) Prove that the medians of a triangle are concurrent at a point which is a point of trisection of each median. [A *median* of a triangle is a line from a vertex to the mid point of the opposite side.] **3**

(ii) If the medians of triangle ABC meet at G, and AG is produced to K so that $AG = GK$, prove that the triangle BGK is similar to the triangle whose sides are equal in length to the three medians. **3**

(iii) Also show that the area of the triangle whose sides are equal in length to the medians is ¾ of the area of triangle ABC. **2**

(c) **5**



Barcan sand dunes are parabolic in plan view and are triangular in cross section with the inner face having an angle of repose of $\tan^{-1}\frac{3}{4}$ to the horizontal and the outer face at $\tan^{-1}\frac{1}{6}$ to the horizontal. The figure above shows one such dune (dimensions are in metres). Calculate the volume of sand.

Marks

Question 8 (15 marks) Use a SEPARATE writing booklet.

(a) (i) A circular disc, centre A, of radius a, rolls without slipping along the axis of x. Initially the point P on the edge of the disc is at the origin of coordinates. Prove that, when the radius AP has turned through an angle θ, the coordinates of P are: $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$. 3

(ii) The length, ℓ, of a curve, $y = f(x)$, is given by 3

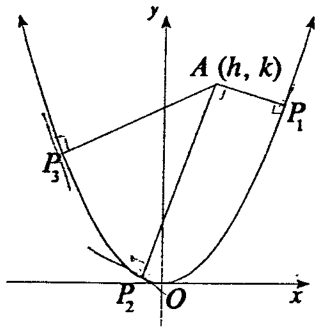
$$\ell = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

When P is again in contact with the axis of x, prove that the length of its path is $8a$.

(b) Sum the series, n being a positive integer, 4

$$1 + x\cos x + x^2\cos 2x + x^3\cos 3x + \cdots + x^n\cos nx.$$

(c) (i)



Prove that, in general, three normals can be drawn from any point to a parabola. 3

(ii) Also show that if P_1, P_2, and P_3 have coordinates (x_1, y_1), (x_2, y_2), and (x_3, y_3) respectively, then $x_1 + x_2 + x_3 = 0$. 2

End of paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a1>0, \quad -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x>0$

Question 1

(a) (i) $\int \sin^{-1}x\,dx = \int \sin^{-1}x \frac{d(x)}{dx}\,dx$

$= x\sin^{-1}x - \int x \cdot \frac{1}{\sqrt{1-x^2}}\,dx$ 1

$= x\sin^{-1}x + \sqrt{(1-x^2)} + C$ 1

(ii) $I = \int \frac{x}{1+x^4}\,dx$

let $u = x^2 \Rightarrow du = 2x\,dx$

$\therefore I = \frac{1}{2}\int \frac{du}{1+u^2}$ 1

$= \frac{1}{2}\tan^{-1}u + c$

$= \frac{1}{2}\tan^{-1}(x^2) + c$ 1

(iii) $I = \int \tan^3 x\,dx = \int \tan x \cdot \tan^2 x\,dx$

ie $I = \int \tan x(\sec^2 x - 1)\,dx$

$= \int \tan x \sec^2 x\,dx - \int \tan x\,dx$ 1

$= \frac{1}{2}\tan^2 x - \log_e(\cos x)^{-1}$ 1

$= \frac{1}{2}\tan^2 x + \log_e(\cos x)$

(b) $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\cos x}$

$t = \tan\frac{x}{2}$

$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$\frac{dx}{dt} = \frac{2}{\sec^2\frac{x}{2}}$

$\frac{dx}{dt} = \frac{2}{1+t^2}$

When $x=0, t=0$

When $x=\frac{\pi}{2}, t=1$

$= \int_0^1 \frac{2\,dt}{\left(1+\frac{1-t^2}{1+t^2}\right)(1+t^2)}$

$= 2\int_0^1 \frac{dt}{2}$ 1

$= [t]_0^1$

$= 1$ ½

(c) $I_n = \int_1^e (\ln x)^n\,dx$

$= \int_1^e (\ln x)^n \frac{d(x)}{dx}\,dx$ (½)

$= [x\ln x]_1^e - \int_1^e x \cdot n(\ln x)^{n-1} \cdot \frac{1}{x}\,dx$

$= e - n\int_1^e (\ln x)^{n-1}\,dx$

$\therefore I_n = e - nI_{n-1}$

(d)

(i) $\tan\left(\frac{\pi}{4} - u\right) = \frac{\tan\frac{\pi}{4} - \tan u}{1 + \tan\frac{\pi}{4}\tan u}$

$= \frac{1 - \tan u}{1 + \tan u}$

(ii)

(d)(ii) When $x = 0,\ u = \frac{\pi}{4}$ $dx = -du$

When $x = \frac{\pi}{4},\ u = 0$

$$1 + \tan x = 1 + \frac{1 - \tan u}{1 + \tan u}$$

$$= \frac{2}{1 + \tan u}$$

$$\therefore I = \int_{\pi/4}^{0} \ln\left[\frac{2}{1+\tan u}\right] \cdot -du \quad \tfrac{1}{2}$$

$$= \int_{0}^{\pi/4} \ln\left[\frac{2}{1+\tan u}\right] du$$

ie $$= \int_0^{\pi/4} \ln 2 \, du - \int_0^{\pi/4} \ln(1+\tan u)\, du \quad \tfrac{1}{2}$$

ie $$\int_0^{\pi/4} \ln(1+\tan u)\, du = \int_0^{\pi/4} \ln 2 \, du - \int_0^{\pi/4} \ln(1+\tan u)\, du \quad \tfrac{1}{2}$$

$$\therefore \int_0^{\pi/4} \ln(1+\tan u)\, du = \ln 2 \int_0^{\pi/4} du$$

$$= \ln 2 \left[u\right]_0^{\pi/4} \quad \tfrac{1}{2}$$

$$2I = \ln 2 \cdot \frac{\pi}{4}$$

$$I = \frac{\pi}{8} \ln 2$$

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Question 2

(a) The real number law

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

does not necessarily hold for complex numbers (other than real ones).

∴ Line four is incorrect. (2)

(b) $u = -3-4i$; $v = 1-i$

(i) $\bar{u} - v = -3+4i-(1-i)$
$= -4+5i$ (1)

(ii) $\frac{2u}{v} = 1-7i$ (calculator)

OR $\frac{2u}{v} = \frac{2(-3-4i)}{1-i} \times \frac{1+i}{1+i}$

$= \frac{2(-3-4i)(1+i)}{2}$

$= -3-3i-4i+4$

$= 1-7i$ (2)

(iii) $\sqrt{u} = 1-2i$ (calculator)

OR Let $\sqrt{u} = a+ib$

$\therefore (a+ib)^2 = -3-4i$

$a^2-b^2+2abi = -3-4i$

$\therefore a^2-b^2 = -3$; $2ab = -4$

$b = -\frac{2}{a}$

$\therefore a^2 - \left(\frac{-2}{a}\right)^2 = -3$

$(a^2+4)(a^2-1) = 0$

$\therefore a = \pm 2i$, or ± 1

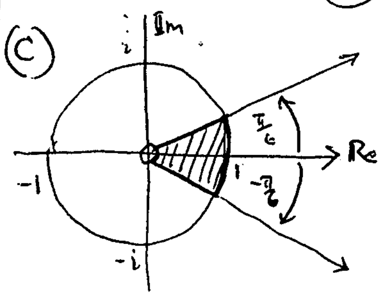
but a is real

$\therefore a = \pm 1$

$b = \mp 2$

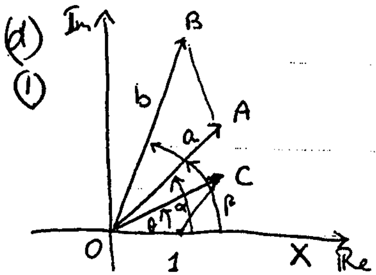
Hence $\sqrt{u} = 1-2i$ (Principal Root) (2)

(c)



(2)

(d) (i)



Let $|a| = OA$, $|b| = OB$

Now $\left|\frac{b}{a}\right| = \frac{|b|}{|a|}$

and $\arg\left(\frac{b}{a}\right) = \arg b - \arg a$

ie $\angle COX = \angle BOX - \angle AOX$ (2)

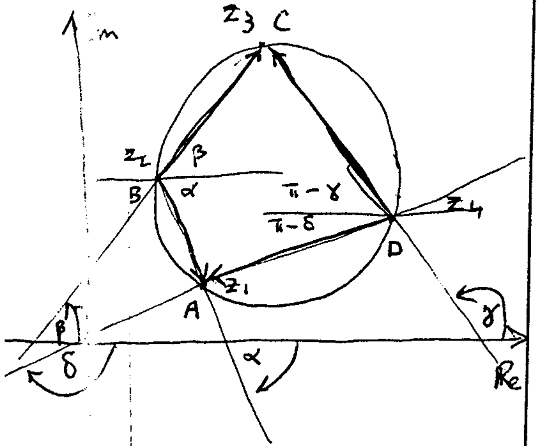
(ii) As among z_1, z_2, z_3, z_4 lie in cyclic order, consider diagram below, where

arg $(z_1 - z_2) = \alpha$

arg $(z_3 - z_2) = \beta$

arg $(z_3 - z_4) = \gamma$

arg $(z_1 - z_4) = \delta$



Now $\angle ABC = \pi - \angle ADC$

(opp angles, cyclic quad)

$\therefore \alpha + \beta = \pi - (2\pi - (\gamma + \delta))$

$= -\pi + \gamma + \delta$

$\therefore \alpha + \beta - \gamma - \delta = -\pi$

ie $\arg\left[\dfrac{(z_1 - z_2)(z_3 - z_4)}{(z_3 - z_2)(z_1 - z_4)}\right] = -\pi$

4 $\therefore$ The no. is real ($\arg z = k\pi$)

Note If the z's are in a different order, an argument can be based on the theorem of equality of angles at the circumference standing on the same arc.

Alternatively,

In the diagram at left,

let $\angle ABC = \theta$

$\therefore \angle ADC = \pi - \theta$ (opp angles of cyclic quad).

let $|z_2 - z_1| = r$; $|z_4 - z_1| = s$

$|z_4 - z_3| = t$; $|z_2 - z_3| = u$

Now $z_2 - z_1 = \frac{r}{s}\,\text{cis}\,\theta\,(z_4 - z_1)$

$\therefore \dfrac{z_2 - z_1}{z_4 - z_1} = \dfrac{r}{s}\,\text{cis}\,\theta$

Similarly

$\dfrac{z_4 - z_3}{z_2 - z_3} = \dfrac{t}{u}\,\text{cis}(\pi - \theta)$

$\therefore \dfrac{z_2 - z_1}{z_4 - z_1} \times \dfrac{z_4 - z_3}{z_2 - z_3} = \dfrac{rt}{su}\,\text{cis}(\pi - \theta)\,\text{cis}\,\theta$

$= \dfrac{rt}{su}\,\text{cis}\,\pi$

RHS is a Real number ($\text{cis}\,\pi =$

QED

(c) (i) The normal to the curve $x^2 = 4ay$ at the point defined by

$x = 2at, y = at^2$

$\therefore \dfrac{dy}{dx} = t$

$m = -\dfrac{1}{t}$

$y - at^2 = -\dfrac{1}{t}(x - 2at)$

$yt - at^3 = -x + 2at$

$x + ty = 2at + at^3$

$A(h,k)$ lies on the normal $\Rightarrow h + kt = 2at + at^3$

$\therefore at^3 + (2a - k)t - h = 0$

The above equation gives the parameter t for a point where its normal passes through $A(h,k)$

$at^3 + (2a - k)t - h = 0$

It does have three solutions, so there are in general 3 normals (even if they are *imaginary* normals)

(ii) The points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$ would correspond to parameters t_1, t_2 & t_3 respectively.

t_1, t_2 & t_3 represent the three roots of $at^3 + (2a - k)t - h = 0$

So $t_1 + t_2 + t_3 = 0$ (sum of roots)

$\therefore 2at_1 + 2at_2 + 2at_3 = 0$

$\therefore x_1 + x_2 + x_3 = 0 \qquad (\because 2at_i = x_i, i = 1,2,3)$

(b)

Let $z = x\,cis\,\theta$, for real x

$$1+z+z^2+z^3+\ldots+z^n=\frac{z^{n+1}-1}{z-1}$$

$\text{Re}(1+z+z^2+z^3+\ldots+z^n)=1+x\cos\theta+x^2\cos2\theta+\ldots+x^n\cos n\theta$
(De Moivre's Theorem)

$$\therefore 1+x\cos\theta+x^2\cos2\theta+\ldots+x^n\cos n\theta=\text{Re}\left(\frac{z^{n+1}-1}{z-1}\right)$$

So

$$\frac{z^{n+1}-1}{z-1}=\frac{x^{n+1}cis(n+1)\theta-1}{x\,cis\,\theta-1}=\frac{x^{n+1}\cos(n+1)\theta-1+ix^{n+1}\sin(n+1)\theta}{x\cos\theta-1+ix\sin\theta}$$

$$=\frac{x^{n+1}\cos(n+1)\theta-1+ix^{n+1}\sin(n+1)\theta}{x\cos\theta-1+ix\sin\theta}\times\frac{x\cos\theta-1-ix\sin\theta}{x\cos\theta-1-ix\sin\theta}$$

$$=\frac{a+ib}{(x\cos\theta-1)^2+x^2\sin^2\theta}=\frac{a+ib}{x^2\cos^2\theta-2x\cos\theta+1+x^2\sin^2\theta}$$

$$=\frac{a+ib}{x^2-2x\cos\theta+1}$$

$$\text{So Re}\left(\frac{z^{n+1}-1}{z-1}\right)=\text{Re}\left(\frac{a+ib}{x^2-2x\cos\theta+1}\right)=\frac{a}{x^2-2x\cos\theta+1}$$

From above

$$\frac{z^{n+1}-1}{z-1}=\frac{a+ib}{x^2-2x\cos\theta+1}$$

So $a=(x\cos\theta-1)\left[x^{n+1}\cos(n+1)\theta-1\right]+x^{n+1}\sin(n+1)\theta(x\sin\theta)$

$=x^{n+2}\cos\theta\cos(n+1)\theta-x^{n+1}\cos(n+1)\theta-(x\cos\theta-1)+x^{n+2}\sin(n+1)\theta\sin\theta$

$=1-x\cos\theta+x^{n+2}\cos\theta\cos(n+1)\theta+x^{n+2}\sin(n+1)\theta\sin\theta-x^{n+1}\cos(n+1)\theta$

$=1-x\cos\theta+x^{n+2}\cos n\theta-x^{n+1}\cos(n+1)\theta$

$$\therefore 1+x\cos\theta+x^2\cos2\theta+\ldots+x^n\cos n\theta=\frac{1-x\cos\theta+x^{n+2}\cos n\theta-x^{n+1}\cos(n+1)\theta}{1-2x\cos\theta+x^2}$$

QUESTION 3

(a) $P(x) = x^3-4x^2+7x-6$.

now $P(2) = 8-16+14-6$
$= 0$.

$\therefore$ $x-2$ is a factor.

	1	-4	7	-6
2		2	-4	6
	1	-2	3	0

$\therefore P(x) = (x-2)(x^2-2x+3)$

$= (x-2)((x-1)^2-(\sqrt{2}i)^2)$

$\boxed{= (x-2)(x-1-\sqrt{2}i)(x-1+\sqrt{2}i)}$ (3)

(b) Let $f(x) = ax^4+bx^3+cx^2+dx+e$.

now $f'(x) = 4ax^3+3bx^2+2cx+d$.

now $f'(0) = 0$

$\therefore \boxed{d=0}$

$f(0) = 1$

$\therefore \boxed{e=1}$

$f(1) = 1$

$\therefore \boxed{a+b+c=0}$ — A

$f(2) = 13$

$\therefore \boxed{4a+2b+c=0}$ — B

$f(3) = 73$

$\therefore \boxed{9a+3b+c=8}$ — C

Solving (A), (B) & (C)

$\therefore \boxed{f(x) = x^4-x^2+1.}$ (4)

(c) (i) $P(x) = (x-\alpha)^m Q(x)$.

$P'(x) = m(x-\alpha)^{m-1}Q(x) + (x-\alpha)^m \cdot Q'(x)$

$= (x-\alpha)^{m-1}[mQ(x) + (x-\alpha)Q'(x)]$

$= (x-\alpha)^{m-1} \cdot R(x)$. Q.E.D

(ii) Let $P(x) = 5x^5-3x^3+c = 0$.

Since pos. repeated root.

$P'(x) = 25x^4-9x^2 = 0$.

$\therefore x=0$ or $x = \pm\frac{3}{5}$

Clearly $x = \frac{3}{5}$.

$\therefore 5(\frac{3}{5})^5 - 3(\frac{3}{5})^3 + c = 0$

$c = \frac{81}{125} - \frac{24}{6}$

$\boxed{c = \frac{16}{6}}$

(d) Given $x^3+px+q=0$ — (A)

let $y=\frac{1}{x} \Rightarrow x=\frac{1}{y}$ in (A)

$(\frac{1}{y})^3 + p\cdot\frac{1}{y} + q = 0$

$1 + py^2 + qy^3 = 0$

OR

$\boxed{qx^3+px^2+1=0}$

$\therefore$ co-efficients are $q, p, 0, 1$.

QUESTION FOUR

(a) $x^2+xy+y^2=7$ — (1)

$2x^2-xy+y^2=28$ — (2)

× (1) × 4

$4x^2+4xy+4y^2=28$ — (3)

(3) − (2)

$3x^2+5xy+3y^2=0$

$(2x+3y)(x+y)=0$

∴ $y=-x$ or $-\frac{2x}{3}$ ✓2

Sub. in (1)

If $y=-x$

$x^2-x^2+x^2=7$

$x=\pm\sqrt{7}$

∴ $y=\mp\sqrt{7}$

If $y=-\frac{2x}{3}$

$x^2-\frac{2x^2}{3}+\frac{4x^2}{9}=7$

$9x^2-6x^2+4x^2=63$

$7x^2=63$

$x^2=9$

$x=\pm 3$

$y=\mp 2$

∴ $\boxed{(\sqrt{7},-\sqrt{7}),(-\sqrt{7},\sqrt{7}),(3,-2)\ (-3,2)}$ ✓4

(b) If $b^2<4ac \Rightarrow$ no roots, quadratic will be same sign for all x.

also $b^2>0$ ∴ a and c will be the same sign.

if $c<0$ then $a<0$

if $c>0$ then $a>0$ Q.E.D. 2

(c) (i) $x^2-2x(y-1)+5y^2-14y+k$

$\Delta_1=4(y-1)^2-4(5y^2-14y+k)$

$=4((y-1)^2-5y^2+14y-k)$

$=4(y^2-2y+1-5y^2+14y-k)$

$=4(-4y^2+12y+1-k)$ ✓2

for $\Delta_1<0$

$-4y^2+12y+1-k<0$

This requires $\Delta_2=144+16(1-k)<0$

$16(1-k)<-144$

$1-k<-9$

$k>10$ ✓4

(ii) If $k=10$

$x^2-2xy+5y^2+2x-14y+10$ (A)

$\equiv(x+py+q)^2+(ry+s)^2$

$=x^2+p^2y^2+q^2+2xpy+2pqy+2xq+r^2y^2+2rsy+s^2$

equating

$p^2+r^2=5 \Rightarrow \boxed{r=\pm 2}$

$2p=-2 \Rightarrow \boxed{p=-1}$

$2pq+2rs=-14 \Rightarrow \boxed{s=\mp 3}$ ✓2

$2q=2 \Rightarrow \boxed{q=1}$

$q^2+r^2=10.$

∴ (A) becomes

$(x-y+1)^2+(2y-3)^2.$ ✓4

which is zero if $2y=3 \Rightarrow \boxed{y=\frac{3}{2}}$

& $x-y+1=0 \Rightarrow \boxed{x=\frac{1}{2}}$

(iii) as $k \Rightarrow$ ~~∞ there is no~~ ~~minimum~~ minimum is $(k-10)$ 1

$$I=\int_0^{2\pi}\frac{\sqrt{2}}{\sqrt{1-\cos\theta}}a(1-\cos\theta)\,d\theta$$

$$=a\sqrt{2}\int_0^{2\pi}\sqrt{1-\cos\theta}\,d\theta$$ ✓

$\cos 2A=1-2\sin^2 A$

$2\sin^2 A=1-\cos 2A$

$1-\cos\theta=2\sin^2\frac{\theta}{2}$

$$=a\sqrt{2}\int_0^{2\pi}\sqrt{2}\sin\tfrac{\theta}{2}\,d\theta$$

For $0\le\theta\le 2\pi$ $\sin\frac{\theta}{2}\ge 0$

$$=2a\int_0^{2\pi}\sin\tfrac{\theta}{2}\,d\theta$$

$$=4a\left[-\cos\tfrac{\theta}{2}\right]_0^{2\pi}$$

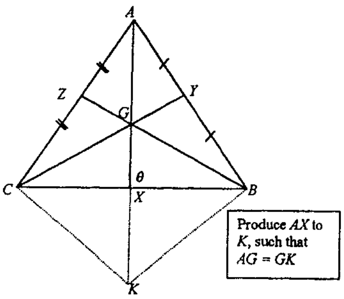
$$=4a[1-(-1)]$$

$$=8a$$ ✓

$x^4+2x^3-17x^2-18x+32=0$
$y=x^2+x \Rightarrow y^2=x^4+2x^3+x^2$
$x^4+2x^3-17x^2-18x+32=(x^4+2x^3+x^2)-18(x^2+x)+32$
$x^4+2x^3-17x^2-18x+32=0 \Leftrightarrow y^2-18y+32=0$
$(y-2)(y-16)=0$
$y=2,16$
$x^2+x=2 \Rightarrow x^2+x-2=0 \Rightarrow (x+2)(x-1)=0 \Rightarrow x=1,-2$
$x^2+x=16 \Rightarrow x^2+x-16=0$ (no rational solutions)

The rational solutions are $x=-2,1$

(b)



(i) BZ and CY are medians.
WE NEED TO PROVE that AX (passing through G) is a median.

$AZ : ZC = AG : GK = 1 : 1$, so by the midpoint theorem for triangles $ZG \parallel GK$

Similarly $CY \parallel BK$.

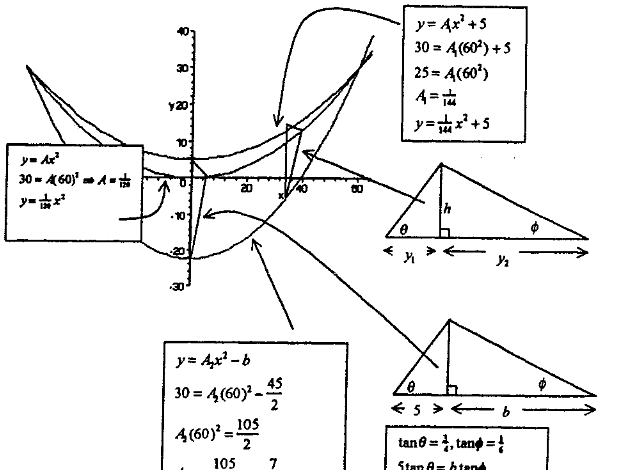
So $CYBK$ is a parallelogram and $CX = BX$ (diagonals bisect one another).
So AX is a median.
Similarly $XK = GX$, so $GX = \frac{1}{2}GK = \frac{1}{2}AX$, so $AG : GX = 2 : 1$.
Thus G is a point of trisection of AX **QED**

(b) (ii) $BG : BZ = 2 : 3$
From above $GK : AX = 2 : 3$
$\because BK \parallel CG$ & $BK = 2\times GY$ (midpoint theorem for triangles)
So $BK : CY = 2 : 3$

Hence ΔBGK is similar to a triangle with lengths equal to the medians, since the sides are equiproportional.

(iii) Call the triangle with the lengths equal to the medians Φ
From (ii) ΔBGK is similar to Φ with sides in the ratio $2 : 3$, so the ratio of the areas is $4 : 9$
So since $CYBK$ is a parallelogram, $\Delta BGX = \frac{1}{2}\Delta BGK$
$\Delta AXB = \frac{1}{2}AX \times XB\sin\theta$
$\Delta BXG = \frac{1}{2}GX \times XB\sin\theta = \frac{1}{3}\times(\frac{1}{2}AX \times XB\sin\theta)$ $\quad(\because GX = \frac{1}{3}AX)$
$\Delta BXG = \frac{1}{3}\Delta AXB = \frac{1}{6}\Delta ABC$ $\quad(\because \Delta AXB = \frac{1}{2}\Delta ABC)$
$\Delta BXG = \frac{1}{2}\Delta BGK = \frac{2}{9}\Phi$
$\therefore \frac{2}{9}\Phi = \frac{1}{6}\Delta ABC$
$\frac{\Phi}{\Delta ABC} = \frac{1}{6}\times\frac{9}{2} = \frac{3}{4}$
QED

(c)



Area of the cross sectional triangle $\frac{1}{2}h(y_1+y_2)$
$y_1 = \frac{1}{144}x^2+5-\frac{1}{120}x^2 = -\frac{1}{720}x^2+5$
$y_2 = \frac{1}{120}x^2-(\frac{7}{480}x^2-\frac{45}{2}) = -\frac{1}{160}x^2+\frac{45}{2}$

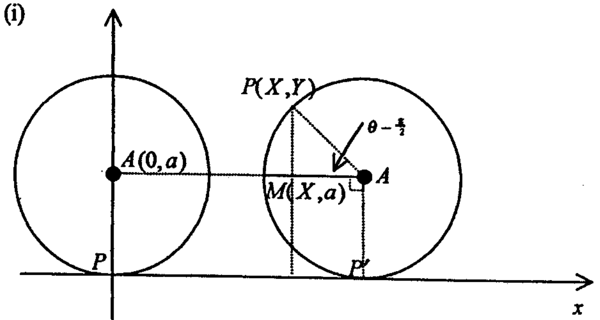
$y_1+y_2 = \frac{1}{144}x^2+5-(\frac{7}{480}x^2-\frac{45}{2})$
$\quad = -\frac{11}{1440}x^2+\frac{55}{2}$
$\tan\theta = \frac{h}{y_1} \Rightarrow h = y_1\tan\theta = y_1\times\frac{3}{4} = \frac{3y_1}{4}$
$h = \frac{3(-\frac{1}{720}x^2+5)}{4} = -\frac{1}{960}x^2+\frac{15}{4}$
Area $= \frac{1}{2}\left(-\frac{1}{960}x^2+\frac{15}{4}\right)\left(-\frac{11}{1440}x^2+\frac{55}{2}\right) = \frac{11}{2764800}x^4-\frac{11}{384}x^2+\frac{825}{16}$

$\Delta V \cong \left(\frac{11}{2764800}x^4-\frac{11}{384}x^2+\frac{825}{16}\right)\Delta x$
$V = \int_{-60}^{60}\left(\frac{11}{2764800}x^4-\frac{11}{384}x^2+\frac{825}{16}\right)dx$
$\quad = 2\int_0^{60}\left(\frac{11}{2764800}x^4-\frac{11}{384}x^2+\frac{825}{16}\right)dx$
$\quad = 2\times\left[\frac{11}{13824000}x^5-\frac{11}{1152}x^3+\frac{825}{16}x\right]_0^{60}$
$\quad = 3300$

So the volume of the Barcan dune is 3300 cubic units

(8)

(a) (i)



On the x-axis $PP' = a\theta$ since there is no slipping ✓

Let $\angle PAP' = \theta \Rightarrow \angle PAM = \theta - \frac{\pi}{2}$

$Y = a + a\sin(\theta - \frac{\pi}{2}) = a - a\sin(\frac{\pi}{2} - \theta) = a(1-\cos\theta)$ ✓

$X = a\theta - a\cos(\frac{\pi}{2} - \theta) = a\theta - a\cos(\theta - \frac{\pi}{2}) = a(\theta - \sin\theta)$ ✓

$\sin(90^\circ - A) = \cos A$
$\cos(90^\circ - A) = \sin A$
$\sin(-A) = -\sin A$
$\cos(-A) = \cos A$

(ii)

$x = a(\theta - \sin\theta)$ $\quad$ $y = a(1-\cos\theta)$

$\frac{dx}{d\theta} = a(1-\cos\theta)$ $\quad$ $\frac{dy}{d\theta} = a\sin\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta}\bigg/\frac{dx}{d\theta} = a\sin\theta \big/ a(1-\cos\theta) = \frac{\sin\theta}{1-\cos\theta}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\left(\frac{\sin\theta}{1-\cos\theta}\right)^2} = \sqrt{\frac{(1-\cos\theta)^2+\sin^2\theta}{(1-\cos\theta)^2}}$$

$$= \sqrt{\frac{1-2\cos\theta+\cos^2\theta+\sin^2\theta}{(1-\cos\theta)^2}}$$

$$= \sqrt{\frac{2-2\cos\theta}{(1-\cos\theta)^2}} = \sqrt{\frac{2(1-\cos\theta)}{(1-\cos\theta)^2}} = \frac{\sqrt{2}}{\sqrt{1-\cos\theta}}$$ ✓

We need to use the following substitution

$x = a(\theta - \sin\theta)$

$dx = a(1-\cos\theta)d\theta$

$0 \le \theta \le 2\pi$

Ex. 2 4/6

Q5

i) $m\ddot{x} = mg - mkv$

$\ddot{x} = g - kv$

ii) $\frac{dv}{dt} = g - kv$

$\frac{dt}{dv} = \frac{1}{g-kv}$

$t = -\frac{1}{k}\ln(g-kv) + c$

$v=0, t=0, \quad c = \frac{1}{k}\ln g$

$-kt = \ln\left[\frac{g-kv}{g}\right]$

$e^{-kt} = \frac{g-kv}{g}$

$g - kv = ge^{-kt}$

$v = \frac{g}{k}(1-e^{-kt})$

iii) terminal velocity – net accel is zero
max velocity as $t\to\infty$
$V_t = \frac{g}{k}$

iv) $mkv \downarrow \quad mg \downarrow \quad a = -mg - mkv$

$\frac{dv}{dt} = -g - kv$

$dt = \frac{-1}{g+kv}\,dv$

$t = -\frac{1}{k}\ln(g+kv) + c$

$v=u, t=0 \quad c = \frac{1}{k}\ln(g+ku)$

$t = \frac{1}{k}\ln\left[\frac{g+ku}{g+kv}\right]$

Q comes to rest $v=0$

$t = -\frac{1}{k}\ln\left(\frac{g}{g+ku}\right)$

or $\frac{1}{k}\ln\left(\frac{g+ku}{g}\right)$

(c) $V = \frac{g}{k}(1-e^{-kt})$

$t = \frac{1}{k}\ln\left(\frac{g+ku}{g}\right)$

$= \frac{g}{k}\left(1-e^{\ln\frac{g}{g+ku}}\right)$

$= \frac{g}{k}\left(1-\frac{g}{g+ku}\right)$

$= \frac{g}{k}\left(\frac{ku}{g+ku}\right)$

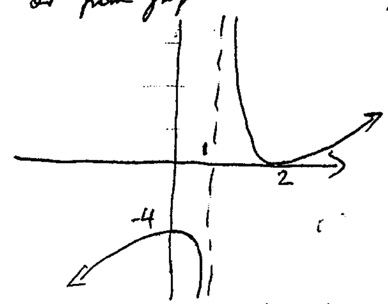
$V = \frac{gu}{g+ku}$

$V = \frac{\frac{g}{k}u}{\frac{g}{k}+u}$

$V = \frac{V_T u}{V_T + ku}$

Q6 let $y = \frac{(x-2)^2}{x-1}$
Solve as a quadratic in x for Δ.

or from graph



observe y does not exist between 0 & 4

(ii) $\frac{(x-2)^2}{(x-1)} = \frac{k}{x}$.
Draw $y = \frac{k}{x}$ and $y = -\frac{k}{x}$ on above graph $y = \pm\frac{k}{x}$ will

out of these places where $y = \frac{k}{x}$ will cut at one

b) $2z^3$
$= 2r^3(\cos 3\theta + i\sin\ldots$
$= 2r^3\cos 3\theta = 9$
$2r^3\sin 3\theta = 3\sqrt{3}$
$\tan 3\theta = \frac{3\sqrt{3}}{9}$
$= \frac{\sqrt{3}}{3}$ or …
$3\theta = \frac{\pi}{6}$
$\theta = \frac{\pi}{18}$ smallest value
…
$2r^3\sin\frac{\pi}{6} = 3\sqrt{3}$
$r^3 = 3\sqrt{3}$
$r = \sqrt[3]{3\sqrt{3}}$
$r = (3^{3/2})^{1/3}$
$r = 3^{\frac{1}{2}}$ or $\sqrt{\ldots}$

c) $V = \pi\int_0^2 x(x^2+1)\ldots$
12π cu…

d) $\sqrt{n^2+1} - n \times \frac{\sqrt{n^2+n}\ldots}{\sqrt{n^2+n}\ldots}$
$= \frac{n^2+n-n^2}{\sqrt{n^2+n}+n}$
$= \frac{n}{\sqrt{n^2+n}+n}$
$= \frac{1}{\frac{1}{n}\sqrt{n^2+n}+\ldots}$
$= \frac{1}{\frac{1}{n}\sqrt{1+\frac{1}{n}}\ldots}$
$= \frac{1}{\sqrt{1+\frac{1}{n}}+1}$
as $n\to\infty$
$\lim = \frac{1}{1+1} = \ldots$