SYDNEY BOYS' HIGH SCHOOL

MOORE PARK, SURRY HILLS

Year 12 HSC Task # 3
June 1998

MATHEMATICS

3 UNIT ADDITIONAL

Time allowed *2 Hours*

Examiner: P.R. Bigelow

DIRECTIONS TO CANDIDATES

- *ALL* questions may be attempted.
- All necessary working should be shown in every question. Full marks may not be awarded for careless or badly arranged work.
- Approved calculators may be used.
- Hand up your answers in 3 separate sections: Section A (Questions 1 & 2), Section B (Questions 3 & 4) and Section C (Questions 5 & 6).
- Start a new booklet for EACH section. Indicate your name, class and teacher on each new booklet.
- If required, additional booklets may be obtained from the Examination Supervisor upon request

SECTION A

Question 1. (Start a new booklet) **Marks**

(a) Evaluate (exactly) 6

(i) $\tan^{-1}\frac{1}{\sqrt{3}}$

(ii) $\lim_{x\to 0}\frac{\tan 3x}{4x}$

(iii) $\cos(\tan^{-1}\frac{5}{12})$

(b) If $x = \cos 4t - \sin 4t$ show that $\frac{d^2x}{dt^2} = -16x$ 2

(c) Find $\int \frac{dx}{1+3x}$ 2

(d) Sketch the graph of $f(x) = 2\cos^{-1}\frac{x}{3}$, clearly indicating domain and range. 2

Marks

Question 2.

(a) Evaluate **6**

(i) $\int_0^1 \frac{dx}{\sqrt{4-x^2}}$

(ii) $\int_0^{\frac{\pi}{2}} \cos^2 x \sin x \, dx$ [Let $u = \cos x$]

(iii) $\int_{\frac{1}{4}}^{\frac{\sqrt{3}}{4}} \frac{1}{1+16x^2} \, dx$

(b) Find the first derivative of each of **4**

(i) $\tan(x^2)$

(ii) $\frac{e^{-x}}{x^2}$

(c) Find the exact value of $f''(1)$ if $f(x) = e^{-x^2}$ **2**

SECTION B

Question 3. (Start a new booklet) **Marks**

(a) The arc of the curve $y = \cos 3x$, between the lines $x = 0$ and $x = \frac{\pi}{6}$ is rotated about the x axis. Find the volume of the solid formed. **4**

(b) The area of an ink blot, which is always circular in shape, is increasing at the rate of 4 cm^2/s. Find the rate of increase of the radius in cm/s, when the radius is 3 cm. **3**

(c) (i) Show that $\frac{d^2x}{dt^2} = \frac{d}{dx}(\frac{1}{2}v^2)$ **5**

(ii) A particle is moving in a straight line, so that its acceleration, $\ddot{x}$, is given by $\ddot{x} = -100x$, where x is its displacement from a fixed position.

If $\dot{x} = 10$ when $x = 2$, find the maximum speed of the particle.

Marks

Question 4.

(a) Consider the function $y = x\ln x - 1, x > 0$ — 9

(i) Find its stationary point and determine its nature.

(ii) With a first approximation of $x = 2$, use Newton's Method once to find an approximation to the x intercept.

(iii) Show that the curve is always concave upwards.

(iv) Sketch the curve, showing all its main features.

(b) The blood pressure, P, in the aorta of the body changes between beats with respect to time, t, according to the equation: — 3

$$\frac{dP}{dt} = -kP$$

Show that this equation is satisfied by the function $P = P_0e^{-kt}$, where P_0 and k are constants.

SECTION C

Marks

Question 5. (Start a new booklet)

(a) A particle moves along the x axis with acceleration $8e^{-4t}$ at time t Its velocity at time t is denoted by v. **4**

(i) If $v = 2$ when $t = 0$, show that $v = 4 - 2e^{-4t}$.

(ii) Find the value of t when $v = 3$.

(iii) Find the limiting value of the speed as t becomes large.

(b) A projectile is fired horizontally with an initial velocity u m/s from a position h metres above ground level. The acceleration due to gravity is g m/s^2 **8**

(i) Find the equations of motion for this projectile.

(ii) Find the time taken to reach the ground.

(iii) Show that the horizontal range is given by $u\sqrt{\frac{2h}{g}}$

Marks

Question 6.

(a) A metal rod, which is initially at a temperature of 10° C, is placed in a warm room. **5**

After t minutes, the temperature θ° of the rod is such that

$$\frac{d\theta}{dt} = \frac{30-\theta}{20}$$

(i) Find an expression for θ in terms of t.

(ii) Calculate the temperature of the rod after 1 hour has elapsed, giving your answer to the nearest degree.

(iii) Find the time taken for the temperature of the rod to rise to 20° C, giving your answer correct to the nearest minute.

(b) At high tide, the water level reaches the 15 metre mark on a wharf, whereas at low tide the water level drops to the 7 metre mark. Assuming the water level flows in simple harmonic motion and the high tide was at 7:50 am and the low tide was at 1:30 pm. **4**

(i) Find the amplitude and period of the motion.

(ii) Find during what time interval the level dropped below the 9 metre mark. Giving your answer to the nearest minute.

(c) What is the rate of change in the area of an equilateral triangle, A, with respect to its perimeter, P? **3**

THIS IS THE END OF THE PAPER.

SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

1998
HIGHER SCHOOL CERTIFICATE

ASSESSMENT TASK #3

Mathematics Extension 1

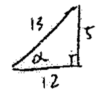
Sample Solutions

1998 Ext I Task # 3

(1) (a) (i) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$.

(ii) $\lim\limits_{x\to 0} \frac{\tan 3x}{4x} = \frac{3}{4} \lim\limits_{x\to 0} \frac{\tan 3x}{3x} = \frac{3}{4}$

(iii) let $\alpha = \tan^{-1}\frac{5}{12} \Rightarrow \tan\alpha = \frac{5}{12}$



$\therefore \cos\left(\tan^{-1}\frac{5}{12}\right) = \cos\alpha = \frac{12}{13}$

(b) $x = \cos 4t - \sin 4t$

$\dot{x} = -4\sin 4t - 4\cos 4t$

$\ddot{x} = -16\cos 4t + 16\sin 4t$

$= -16(\cos 4t - \sin 4t)$

$= -16x$

(c) $\int \frac{dx}{1+3x}$

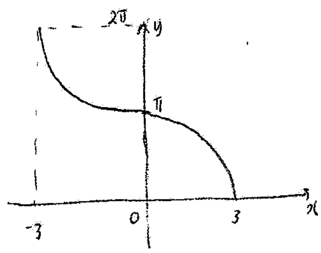
$= \frac{1}{3}\int \frac{3dx}{1+3x}$

$= \frac{1}{3}\ln|1+3x| + c$

(d) $f(x) = 2\cos^{-1}\frac{x}{3}$

D: $-1 \le \frac{x}{3} \le 1 \Rightarrow -3 \le x \le 3$

R: $0 \le \frac{y}{2} \le \pi \Rightarrow 0 \le y \le 2\pi$



(2) (a) (i) $\int_0^1 \frac{dx}{\sqrt{4-x^2}} = \sin^{-1}\left(\frac{x}{2}\right)\Big]_0^1$

$= \sin^{-1}\left(\frac{1}{2}\right)$

$= \frac{\pi}{6}$

(ii) $\int_0^{\pi/2} \cos^2 x \sin x\, dx$

$u = \cos x$

$du = -\sin x\, dx$

$x = 0, u = 1$

$x = \frac{\pi}{2}, u = 0$

$= -\int_1^0 u^2 (-du)$

$= \int_0^1 u^2 du = \frac{1}{3}u^3\Big]_0^1$

$= \frac{1}{3}$

(iii) $\int_{\frac{1}{4}}^{\frac{\sqrt{3}}{4}} \frac{1}{1+16x^2}\, dx$

$= \frac{1}{16}\int_{\frac{1}{4}}^{\frac{\sqrt{3}}{4}} \frac{1}{\frac{1}{16}+x^2}\, dx$

$= \frac{1}{16} \times 4\tan^{-1}(4x)\Big]_{\frac{1}{4}}^{\frac{\sqrt{3}}{4}}$

$= \frac{1}{4}\left[\tan^{-1}\sqrt{3} - \tan^{-1}(1)\right] = \frac{1}{4}\left[\frac{\pi}{3} - \frac{\pi}{4}\right] = \frac{\pi}{48}$

2 (b) (i) $\frac{d(\tan(x^2))}{dx} = 2x \sec^2(x^2)$

(ii) $\frac{d\left(\frac{e^{-x}}{x^2}\right)}{dx} = \frac{x^2(-e^{-x}) - e^{-x}(2x)}{x^4}$

$= \frac{-xe^{-x}(x+2)}{x^4}$

$= \frac{-e^{-x}(x+2)}{x^3}$

(e) $f(x) = e^{-x^2}$

$f'(x) = -2xe^{-x^2}$

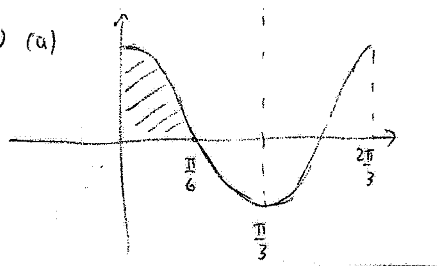
$f''(x) = e^{-x^2}(-2) + (-2x)(-2xe^{-x^2})$

$= -2e^{-x^2}(1-2x^2)$

$f''(1) = -2e^{-1}(-1)$

$= \frac{2}{e}$

(3) (a)



$V = \pi \int_0^{\pi/6} \cos^2 3x \, dx$

$= \frac{\pi}{2} \int_0^{\pi/6} 2\cos^2 3x \, dx$

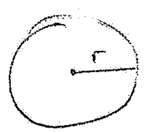
$= \frac{\pi}{2} \int_0^{\pi/6} (1 + \cos 6x) \, dx$

$= \frac{\pi}{2}\left[x + \frac{1}{6}\sin 6x\right]_0^{\pi/6}$

$= \frac{\pi}{2}\left(\frac{\pi}{6} + \frac{1}{6}\sin\pi\right)$

$= \frac{\pi^2}{12}$ c.u.

b)



$A = \pi r^2$

$\frac{dA}{dt} = 4$

$\therefore \frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$

$(r=3)$ $4 = 2\pi(3)\frac{dr}{dt}$

$\frac{dr}{dt} = \frac{2}{3\pi}$ cm/s.

(3) (c) (i) $RHS = \frac{d(\frac{1}{2}v^2)}{dx}$

$= \frac{d(\frac{1}{2}v^2)}{dv} \times \frac{dv}{dx}$

$= v\frac{dv}{dx}$

$= \frac{dx}{dt} \cdot \frac{dv}{dx}$

$= \frac{dv}{dt}$

$= a$

$= \ddot{x}$

$= LHS$

(ii) $\ddot{x} = -100x$

$\therefore \frac{d(\frac{1}{2}v^2)}{dx} = -100x$

$\therefore \frac{1}{2}v^2 = -50x^2 + C$

$(v = 10, x = 2)$

$v^2 = -100x^2 + k$

$100 = -100 \times 4 + k$

$\therefore k = 500$

$\therefore v^2 = 500 - 100x^2$

$= 100(5 - x^2)$

max speed when $\ddot{x} = 0 \Rightarrow x = 0$

$\therefore v^2 = 500$

$\therefore |v| = \sqrt{500} = 10\sqrt{5}$ units/sec.

(4) $y = x\ln x - 1 \quad , x > 0$

(i) $y' = x \times \frac{1}{x} + \ln x = 0$

$\therefore \ln x = -1 \Rightarrow x = e^{-1}$

$y'' = \frac{1}{x}$ at $x = e^{-1}$, $y'' > 0$ $\therefore$ min.

$x = e^{-1}$ $\quad y = e^{-1}\ln e^{-1} - 1$

$= -e^{-1} - 1$

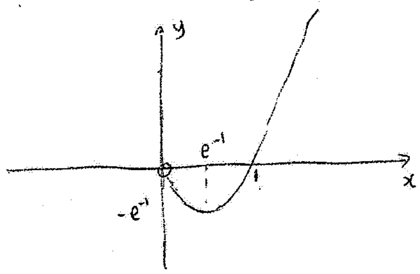
$\therefore (e^{-1}, -e^{-1} - 1)$

(ii) $x_0 = 2$ $\quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{2\ln 2 - 1}{1 + \ln 2} \doteqdot 1.8$

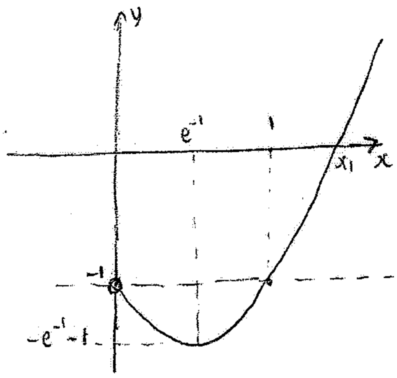
(iii) $y'' = \frac{1}{x} > 0$ for $x > 0$ $\therefore$ concave up.

4(a)(iv) sketch $y = x\ln x - 1$

I: First sketch $y = x\ln x$



II: shift curve down 1 unit



(b) $\frac{dP}{dt} = -kP$

$\therefore P = P_0 e^{-kt}$

LHS $= \frac{dP}{dt} = -k(P_0 e^{-kt})$

$= -kP$

$=$ RHS

(5) (a) $a = 8e^{-4t}$

(i) $\therefore \frac{dv}{dt} = -2e^{-4t} + c$

$\therefore v = -2e^{-4t} + c$

$(t=0,\ v=2)$

$\therefore 2 = -2e^{0} + c$

$\therefore c = 4$

$\therefore v = 4 - 2e^{-4t}$

(ii) $3 = 4 - 2e^{-4t}$

$\therefore -1 = -2e^{-4t}$

$\therefore e^{-4t} = \frac{1}{2}$

$\therefore -4t = \ln(\frac{1}{2}) = -\ln 2$

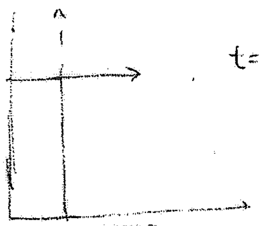
$\therefore t = \frac{1}{4}\ln 2$

(iii) as $t \to \infty$ $e^{-4t} \to 0$

$\therefore v \to 4$

(b)

(i)



$t=0,\ x=0,\ \dot{x}=u$

$y=h,\ \dot{y}=0$

$\boxed{\ddot{x} = 0}$

$\therefore \dot{x} = c$

$\therefore \boxed{\dot{x} = u}$

$\therefore x = ut + k$

$\therefore \boxed{x = ut}$ $(t=0, x=0)$

$\boxed{\ddot{y} = -g}$

$\therefore \dot{y} = -gt + c_1$

$\boxed{\dot{y} = -gt}$

$\therefore y = -\frac{g}{2}t^2 + k_1$

$\therefore \boxed{y = h - \frac{g}{2}t^2}$ $(t=0, y=h)$

(ii) $y=0 \Rightarrow h - \frac{g}{2}t^2 = 0$

$\therefore \frac{g}{2}t^2 = h$

$t^2 = \frac{2h}{g}$

$\therefore t = \sqrt{\frac{2h}{g}}$ secs.

(iii) $R = u \times \sqrt{\frac{2h}{g}}$

$= u\sqrt{\frac{2h}{g}}$ m

(6) (a) $\frac{d\theta}{dt} = \frac{30-\theta}{20} = -\frac{1}{20}(\theta - 30)$

(i) $\therefore \theta = 30 + \theta_0 e^{-\frac{1}{20}t}$ (Newton's Law of cooling)

$t=0,\ \theta = 10 \Rightarrow \theta_0 = -20$

$\therefore \theta = 30 - 20e^{-\frac{1}{20}t}$

(ii) $t = 60$ mins, $\theta = ?$

$\therefore \theta = 30 - 20e^{-3}$

$\doteqdot 29^\circ$

(iii) $\theta = 20,\ t = ?$

$\therefore 20 = 30 - 20e^{-\frac{1}{20}t}$

$\therefore -10 = -20e^{-1/20 t}$

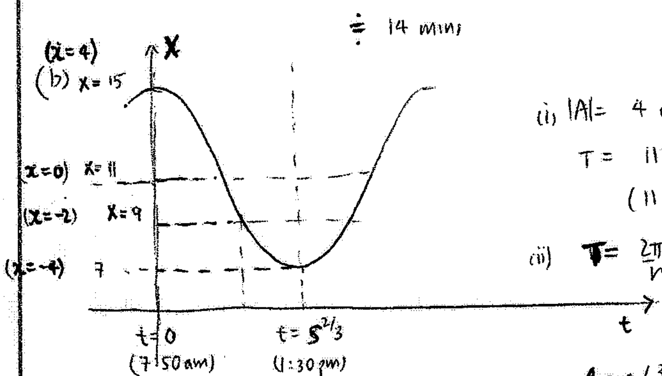
$\therefore e^{-1/20 t} = \frac{1}{2}$

$\therefore -\frac{1}{20}t = \ln(\frac{1}{2}) = -\ln 2$

$\therefore t = 20\ln 2$ mins.

$\doteqdot 14$ mins

(b)



(i) $|A| = 4$ metres

$T = 11\frac{1}{3}$ hours $= \frac{34}{3}$ hours

(11 hrs 20 mins)

(ii) $T = \frac{2\pi}{n}$ $\therefore n = \frac{2\pi}{T} = \frac{2\pi \times 3}{34}$

$= \frac{3\pi}{17}$

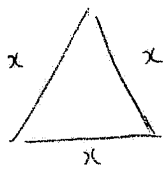
$x = 4\cos\left(\frac{3\pi}{17}t\right)$

$\therefore -2 = 4\cos\left(\frac{3\pi}{17}t\right)$

$\therefore \frac{3\pi}{17}t = \frac{2\pi}{3}, \frac{4\pi}{3}$

$t = \frac{34}{9}, \frac{68}{9}$ hours later $\therefore$ **11:37 am to 3:23 pm**

(6) (c)



$P = 3x$

$A = \frac{1}{2} x^2 \sin 60°$

$= \frac{x^2\sqrt{3}}{4}$

method 1: $x = \frac{P}{3}$

$A = \left(\frac{P}{3}\right)^2 \times \frac{\sqrt{3}}{4}$

$= \frac{\sqrt{3}P^2}{36}$

$\boxed{\frac{dA}{dP} = \frac{\sqrt{3}}{36} P}$

method 2:

$\frac{dA}{dP} = \frac{dA}{dx} \times \frac{dx}{dP}$

$= \frac{2x\sqrt{3}}{4} \times \frac{1}{3}$

$= \frac{x\sqrt{3}}{12} = \frac{3x\sqrt{3}}{36}$

$= \boxed{\frac{\sqrt{3}P}{36}}$