



SYDNEY BOYS HIGH
MOORE PARK, SURRY HILLS

JULY 2009
TASK #3
YEAR 12

Mathematics

General Instructions:

- Reading time—5 minutes.
- Working time—2 hours.
- Write using black or blue pen.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may NOT be awarded for messy or badly arranged work.
- Answer in simplest exact form unless otherwise stated.

Total marks—89 Marks

- Attempt questions 1–6.
- The mark-value of each question is boxed in the right margin.
- Start each NEW section in a separate answer booklet.
- Hand in your answer booklets in 3 sections:
Section A(Questions 1 and 2),
Section B(Questions 3 and 4),
Section C(Questions 5 and 6),

Examiner: Mr P. Bigelow

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

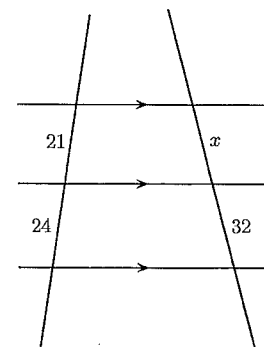
Section A

Question 1 (15 marks)

(a) Find a primitive of $1 - 2x$.

1

(b)



Find the value of x , stating reasons.

2

(c) Differentiate

(i) $\cos 4x$,

(ii) $\ln(4x + 5)$,

(iii) $\frac{\sin x}{x}$,

(iv) $\sqrt{e^x}$.

4

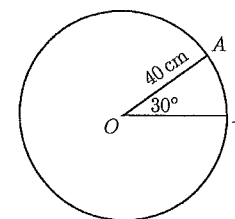
(d) Find x , correct to three significant figures, if $e^x = 4$.

1

(e) Write down the exact value of $2 \cos \pi/4$.

1

(f)



O is the centre of the circle with radius 40 cm. Find the length of the minor arc AB which subtends 30° at O .

2

(g) Sketch $y = 3 \cos 2x$ for $0 \leq x \leq 2\pi$.

2

(h) Find correct to two decimal places

2

(i) $\log_e \frac{11}{4}$,

(ii) $\tan 4^\circ$.

Question 2 (14 marks)

(a) Evaluate

(i) $\int_0^{16} \sqrt{x} dx,$

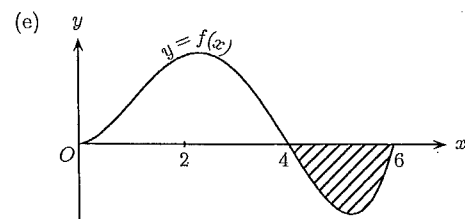
(ii) $\int_0^9 e^{\frac{x}{3}} dx,$

(iii) $\int_{\frac{1}{6}}^{\frac{1}{2}} \cos \pi x dx.$

(b) The gradient of any point on the curve $y = f(x)$ is given by $f'(x) = 2x - 5$. Given that the point $(2, -3)$ lies on the curve, find its equation.

(c) Find the equation of the normal to $y = \ln x$ at the point $(1, 0)$.

(d) If $f(x) = \frac{3}{1+3x}$, find $\int_0^1 f(x) dx.$



Given that

$$\int_0^4 f(x) dx = 15,$$

and that

$$\int_0^6 f(x) dx = 9;$$

(i) what is the area of the shaded region,

(ii) and what is the value of $\int_4^6 f(x) dx$?

(f) A doctor reports that:

“Although the patient’s blood-pressure is rising, attempts to bring it back to normal are taking effect.”

If the patient’s blood-pressure is given by B ,

what can be said about $\frac{dB}{dt}$ and $\frac{d^2B}{dt^2}$?

Marks

1

1

2

2

2

2

1

1

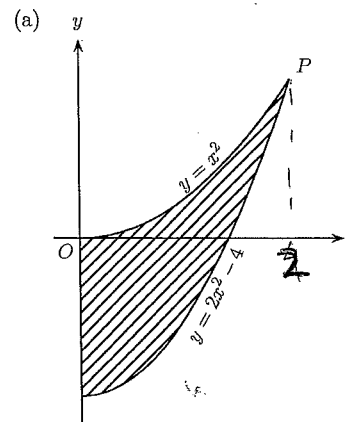
2

Section B

(Use a separate writing booklet.)

Marks

Question 3 (15 marks)



P is the point of intersection of $y = x^2$ and $y = 2x^2 - 4$.

(i) Find the coördinates of P .

(ii) Find the area of the shaded region.

(b) Find $\lim_{x \rightarrow 0} \frac{\sin 2x}{3x}.$

(c) Find the slope of the tangent to $y = 4 \tan 2x$ at the point where $x = \pi/6$.

(d) If $y = e^{2x} - 3$, prove that $\frac{dy}{dx} - 2y - 6 = 0$.

(e) $A(2, 3)$, $B(4, -1)$ and $C(-4, -5)$ are the vertices of a triangle.

(i) Find D and E , the midpoints of AB and AC respectively.

(ii) Show that DE is parallel to BC .

(iii) Find the lengths of DE and BC and hence show that $DE = \frac{1}{2}BC$.

1

2

1

2

2

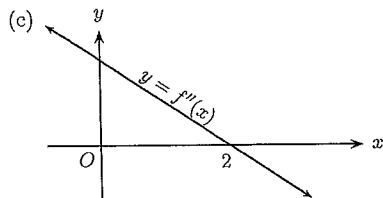
2

2

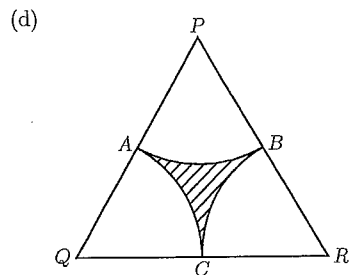
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Question 4 (15 marks)

- (a) For what values of x is $f(x) = x^3 - 3x^2 - 9x + 2$ an increasing function?
- (b) The curve $f(x) = ax^3 - 9x + b$ has a turning point at $(1, 7)$. Find a and b .



The diagram shows the graph of $y = f''(x)$. Given that $f(2) = 0$ and $f'(1) = 0$, draw a possible sketch of $y = f(x)$.



PQR is an equilateral triangle with sides 8 cm. A, B and C are the mid-points of its sides. AB, AC and BC are circular arcs, centres P, Q and R . Calculate in cm^2 , correct to two decimal places, the shaded region bounded by the arcs.

- (c) Find the values of x for which the curve $y = (x + 1)(x - 2)^2$ is concave down.
- (d) Find the volume of the solid of revolution when the area bounded by the curve $y = 2 \sec x$ and the x -axis between $x = \pi/6$ and $x = \pi/3$ is rotated about the x -axis. Answer in simplified exact form.

Marks

2

3

2

3

3

2

Section C

(Use a separate writing booklet.)

Marks

Question 5 (15 marks)

- (a) If $y = \ln \left(\frac{1-x}{1+x} \right)$, show that $\frac{dy}{dx} = \frac{-2}{1-x^2}$.

3

Hence evaluate $\int_0^{1/2} \frac{dx}{1-x^2}$.

- (b) For the curve $y = x^3 + 6x^2 + 9x + 4$:

(i) Find the coordinates of the stationary points and determine their nature.

3

(ii) Find the coordinates of any points of inflexion. (-2, 7)

2

(iii) Sketch the curve, showing essential features, including the y -intercept.

2

- (c) Solve $2 \cos x = -1$ for $0 \leq x \leq 2\pi$.

2

- (d) If $f(x) = e^{-x^2}$, find

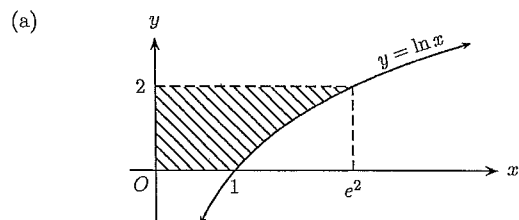
(i) $f'(x)$,

1

(ii) $f''(x)$.

2

Question 6 (15 marks)

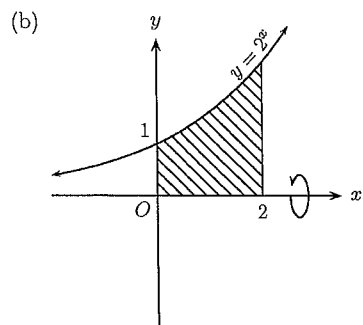


(i) Find the shaded area.

2

(ii) Hence or otherwise find $\int_1^{e^2} \ln x \, dx$.

2



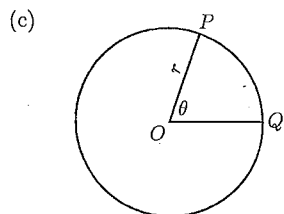
The area under the curve $y = 2^x$, between $x = 0$ and $x = 2$, is rotated about the x -axis.

(i) Show that the volume of the resulting solid is given by $\pi \int_0^2 4^x \, dx$.

2

(ii) Using Simpson's rule with five function values, find the volume of the solid correct to one decimal place.

3



The arc PQ of circle centre O and radius r cm subtends an angle of θ radians at O . The perimeter of the sector POQ is 6 cm.

(i) Show that $r = \frac{6}{\theta + 2}$.

1

(ii) Hence show that the area A cm² is given by $\frac{18\theta}{(\theta + 2)^2}$.

2

(iii) Hence find the maximum area of the sector.

3

End of Paper

Marks

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln(x + \sqrt{x^2 - a^2}), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln(x + \sqrt{x^2 + a^2})$$

NOTE: $\ln x = \log_e x$, $x > 0$

Q1. a) $\int 1 - 2x \, dx$

$= x - \frac{2x^2}{3} + C$ ✓

$\frac{86}{89} \approx 97\%$

intervals between
intercepts of transversals
on parallel lines are
in equal ratio
Intercept theorem

b) $\frac{21}{24} = \frac{x}{32}$
 $x = 28$ ✓

c) i) $y = \cos 4x$

$y' = -4 \sin 4x$ ✓

ii) $y = \ln(4x+5)$
 $y' = \frac{4}{4x+5}$ ✓

iii) $y = \frac{\sin x}{x} = \frac{u}{v}$

$v = x \quad u = \sin x$
 $v' = 1 \quad u' = \cos x$
 $y' = \frac{x \cos x - \sin x}{x^2}$ ✓

iv) $y = e^{\frac{1}{2}x} = (e^x)^{1/2}$

$y' = \frac{1}{2} e^{\frac{1}{2}x}$ ✓

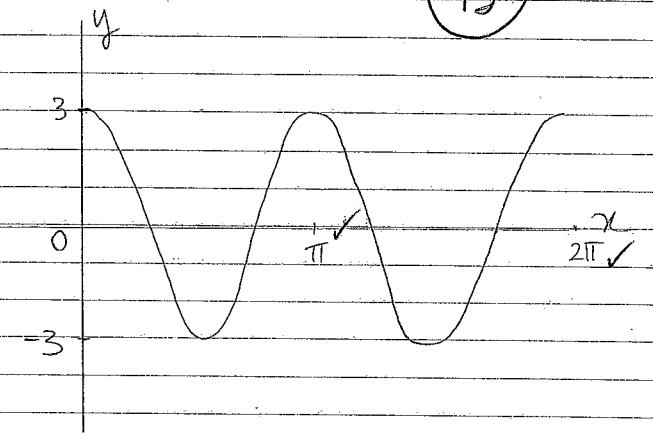
e) $e^x = 4$
then $x = \ln 4$
 $x = 1.386$
 1.39 ✓ (to 3 s.f.)

f) $2 \cos \frac{\pi}{4}$
 $= 2 \times \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$

f) $ms = 70 \times \frac{\pi}{6}$
 $= \frac{70\pi}{6} \text{ cm} = \frac{20\pi}{3}$ ✓

g) $T = \frac{2\pi}{a}$
 $a = 3$

$\frac{14}{15}$



h) i) 1.01 ✓
ii) 1.16 ✓

Q2. i) $\int_0^{16} x^{1/2} \, dx$
 $= \left[\frac{2x^{3/2}}{3} \right]_0^{16}$ ✓
 $= \frac{2}{3} \times 16^{3/2} - 0$
 $= \frac{128}{3}$ ✓

ii) $\int_0^9 e^{\frac{x}{3}} \, dx$
 $= \left[3e^{\frac{x}{3}} \right]_0^9$ ✓
 $= 3e^3 - 3e^0$
 $= 3e^3 - 3 = 3(e^3 - 1)$ ✓

$$2d. f(x) = \frac{3}{1+3x}$$

$$\int_0^1 \frac{3}{1+3x} dx$$

$$= \ln(1+3x) \Big|_0^1$$

$$= \ln(1+3) - \ln(1)$$

$$= \ln 4 - \ln 1$$

$$= \ln 4 = 2 \ln 2$$

$$\frac{14}{14}$$

2e. i) $6u^2$ ✓

ii) -6 ✓

2f. $\frac{dB}{dt} > 0$ ✓ increase slope

$\frac{d^2B}{dt^2} < 0$ ✓ concave down

Q3. i) $y = x^2$

$$y = 2x^2 - 4$$

Solve sim., equate

$$x^2 = 2x^2 - 4$$

$$x^2 = 4$$

$$x = 2$$
 ✓

$$y = 4$$

• P(2, 4) ✓

ii)

$$A = \int_0^2 x^2 - 2x^2 + 4 dx$$

$$= \int_0^2 -x^2 + 4 dx$$

3 a iii) $\int_{\frac{1}{6}}^{1/2} \cos \pi x dx$

$$= \left[\frac{\sin \pi x}{\pi} \right]_{1/6}^{1/2}$$

$$= \frac{1}{\pi} \left(\sin \frac{\pi}{2} - \sin \frac{\pi}{6} \right)$$

$$= \frac{1}{\pi} \left(1 - \frac{1}{2} \right)$$

$$= \frac{1}{2\pi}$$
 ✓

b) $f'(x) = 2x - 5$

$$f(x) = \int 2x - 5 dx$$

$$= \frac{2x^2}{2} - 5x + C$$

$$= x^2 - 5x + C$$
 ✓

sub (2, -3)

where $y = f(x)$

$$-3 = 2^2 - 10 + C$$

$$-3 = -6 + C$$
 ✓

$$3 = C$$

$$y = x^2 - 5x + 3$$
 ✓

c) $y = \ln x$

$$y' = \frac{1}{x}$$

$$m_{\text{tangent}} = 1$$

$$m_{\text{normal}} = -1$$
 ✓

equation normal

$$y - 0 = -1(x - 1)$$
 ✓

$$y = -x + 1$$

$$= \left[\frac{-x^3}{3} + 4x \right]_0^2 \checkmark$$

$$= -\frac{8}{3} + 8 + 0 - 0$$

$$= \frac{16}{3} \checkmark$$

$$3b) \lim_{x \rightarrow 0} \frac{\sin 2x}{3x}$$

$$= \lim_{2x \rightarrow 0} \frac{\sin 2x}{2x} \times \frac{2}{3} \checkmark$$

$$= \frac{2}{3} \checkmark$$

$$c) y = 4 + \tan 2x$$

$$y' = \sec^2 2x \checkmark$$

$$m = 8 \sec^2 \frac{\pi}{3}$$

$$= 8 \left(\frac{1}{\cos^2 \frac{\pi}{3}} \right)^2$$

$$= 32 \quad \text{slope of tang. is } 32$$

$$y = 4 + \tan \frac{\pi}{3} = 4\sqrt{3}$$

\therefore eqn.

$$y - 4\sqrt{3} = 32(x - \frac{\pi}{6})$$

$$y - 4\sqrt{3} = 32x - \frac{32\pi}{6}$$

$$32x - y - \frac{32\pi}{6} + 4\sqrt{3} = 0$$

$$3d) y = e^{2x} - 3$$

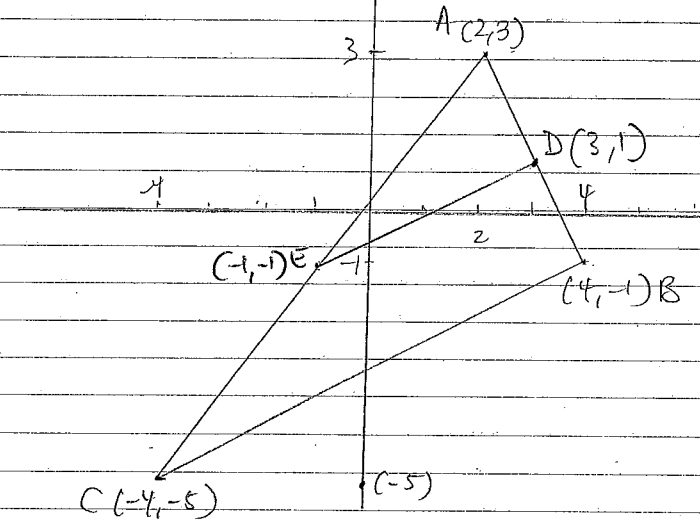
$$\frac{dy}{dx} = 2e^{2x} \checkmark$$

$$\left(\frac{15}{15} \right)$$

$$\text{LHS} = 2e^{2x} - 2e^{2x} + 6 - 6$$

$$= 0 = \text{RHS}$$

3e.



$$D(3,1) \checkmark \quad E(-1,-1) \checkmark$$

ii) intercept theorem

since E and D are both midpts.

$$\frac{CE}{EA} = 1 \checkmark$$

$$\frac{DB}{AD} = 1 \checkmark$$

$$\therefore \frac{CE}{EA} = \frac{DB}{AD} \therefore \text{intercepts in equal ratio on ED and CB}$$

$\therefore ED \parallel CB$

$$\begin{aligned} \text{iii) } DE &= \sqrt{(4)^2 + (2)^2} \\ &= \sqrt{16 + 4} \\ &= \sqrt{20} \\ &= 2\sqrt{5} \end{aligned}$$

$$\begin{aligned} BC &= \sqrt{8^2 + 4^2} \\ &= \sqrt{64 + 16} \\ &= \sqrt{80} \\ &= 4\sqrt{5} \\ &= 2 \times 2\sqrt{5} \\ &= 2 \times DE \end{aligned}$$

$$\begin{aligned} BC &= 2DE \\ DE &= \frac{1}{2} BC \end{aligned}$$

Q4. increasing $f'(x) > 0$

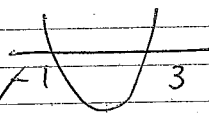
$$f'(x) = 3x^2 - 6x - 9$$

$$\text{solve } 3x^2 - 6x - 9 > 0$$

$$x^2 - 2x - 3 > 0$$

$$(x-3)(x+1) > 0$$

$$x < -1, x > 3$$



$$b) f(x) = ax^3 - 9x + b$$

turning pt $f'(x) = 0$

$$f'(x) = 3ax^2 - 9 = 0$$

$$\text{solve } 3ax^2 - 9 = 0$$

$$ax^2 - 3 = 0$$

$$x^2 = \frac{3}{a}$$

$$x = \pm \sqrt{\frac{3}{a}}$$

since $x = 1$

$$\text{and } \frac{3}{a} > 0$$

$$x \neq -\sqrt{\frac{3}{a}}$$

$$\begin{aligned} x &= \sqrt{\frac{3}{a}} \\ &= 1 \end{aligned}$$

$$\sqrt{\frac{3}{a}} = 1$$

$$\frac{3}{a} = 1$$

$$3 = a$$

$$\text{Sub in to } y = 3x^3 - 9x + b$$

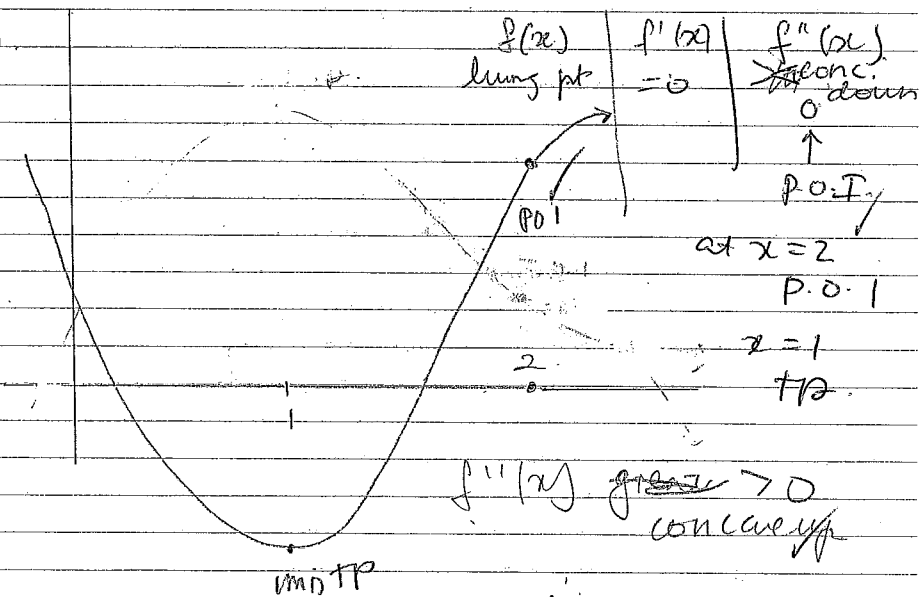
sub $(1, 7)$ to find b

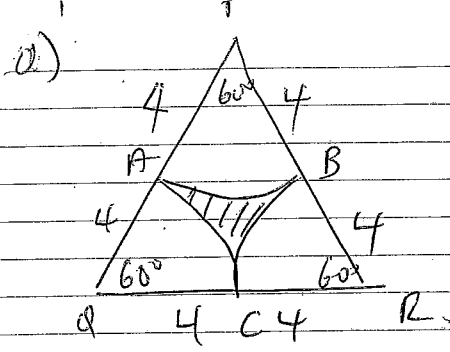
$$7 = 3 - 9 + b$$

$$7 = -6 + b$$

$$b = 13$$

4c)





$$\begin{aligned}
 A(\text{PQR}) &= \frac{1}{2} 8^2 \sin 60^\circ \\
 &= \frac{16}{2} \times \sqrt{3} \quad \checkmark \\
 &= 16\sqrt{3} \text{ u}^2 \\
 A \text{ of circles} &= 2 \times \frac{1}{2} \pi r^2 \\
 &= \frac{1}{2} \pi \times 4^2 \\
 &= 8\pi \text{ u}^2 \quad \checkmark \\
 A(\text{shaded}) &= 16\sqrt{3} - 8\pi \text{ u}^2 \\
 &= 2.58 \text{ u}^2 \quad \checkmark
 \end{aligned}$$

e) ∇ conc. down. $y'' < 0$.

$$\begin{aligned}
 y &= x(x+1)(x^2-4x+4) \quad \checkmark \\
 &= x^3 - 4x^2 + 4x + x^3 - 4x^2 + 4x \\
 &= x^3 - 3x^2 + 4x \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 y' &= 3x^2 - 6x \\
 y'' &= 6x - 6 \\
 &= 6(x-1) \quad \checkmark \\
 \text{same } 6(x-1) < 0 \\
 x < 1 \quad \checkmark
 \end{aligned}$$

$\frac{15}{15}$

f) $v = \pi \int_a^b y^2 dx$

$$\begin{aligned}
 &= 4\pi \int_{\pi/6}^{\pi/3} \sec^2 x \, dx \quad \checkmark \\
 &= 4\pi \left[\tan x \right]_{\pi/6}^{\pi/3} \\
 &= 4\pi \left(\tan \frac{\pi}{3} - \tan \frac{\pi}{6} \right) \\
 &= 4\pi \left(\sqrt{3} - \frac{1}{\sqrt{3}} \right) \quad \checkmark
 \end{aligned}$$

Q5. $y = \ln(1-x) - \ln(1+x)$

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{-1}{1-x} - \frac{1}{1+x} \\
 &= \frac{-1-x-1+x}{(1-x)(1+x)} \quad \checkmark \\
 &= \frac{-2}{1-x^2} \quad \checkmark \\
 \int_0^{1/2} \frac{1}{1-x^2} dx \\
 &= -\frac{1}{2} \int_0^{1/2} \frac{-2}{1-x^2} dx \\
 &= -\frac{1}{2} \left[\ln \frac{1-x}{1+x} \right]_0^{1/2} \\
 &= -\frac{1}{2} \left[\ln \frac{1/2}{3/2} - \ln 1 \right] \\
 &= -\frac{1}{2} \left[\ln \frac{1}{3} \right] \\
 &= -\frac{1}{2} (\ln 1 - \ln 3) \\
 &= -\frac{1}{2} (-\ln 3) \\
 &= \frac{\ln 3}{2} \quad \checkmark
 \end{aligned}$$

5b. stat pts. $y' = 0$

$$y = x^3 + 6x^2 + 9x - 4$$
$$y' = 3x^2 + 12x + 9$$
$$y'' = 6x + 12$$

$y' = 0$ for stat pts.

$$x^2 + 4x + 3 = 0$$

$$(x+3)(x+1) = 0$$
$$x = -1, -3 \checkmark$$

$$y = 0, 4 \checkmark$$

$$(-1, 0) \checkmark \quad (-3, 4) \checkmark$$

test nature.

$$y'' = -6 + 12$$
$$= 6 \checkmark$$

min TP $(-1, 0)$ ✓

test nat:

$$y'' = -12 + 12$$
$$= -6 \checkmark$$

max TP $(-3, 4)$ ✓

ii) solve $y'' = 0$ for p.o.i.

$$6x = -12$$

$$x = -2$$

$$y = 2$$

$(-2, 2) \checkmark$

check concavity changes

$$y'' = 6x + 12 = 6(x+2)$$

x	-3	-2	-1
y''	-6	0	6

concur. \wedge \cup ✓

\therefore concav changes

$\therefore (-2, 2)$ is p.o.i.

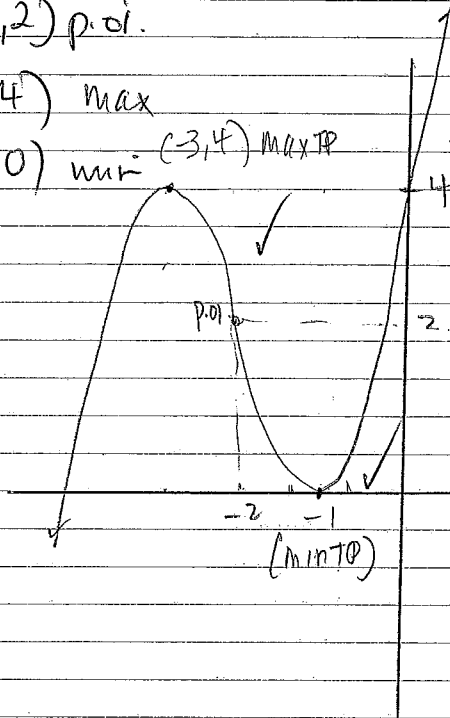
y int $x = 0$. $y = 4$ ✓

$$(0, 4)$$

$$(-2, 2) \text{ p.o.i.}$$

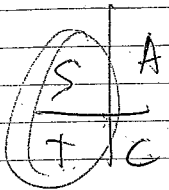
$$(-3, 4) \text{ max}$$

$$(-1, 0) \text{ min} \quad (3, 4) \text{ max TP}$$



5c) $2 \cos x = -1$

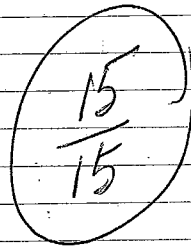
$\cos x = -1/2$



$x = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{2\pi}{3}$

5d) $f(x) = e^{-x^2}$

$f'(x) = e^{-x^2} \times -2x = -2xe^{-x^2}$



$\int u'v = uv - \int uv'$

$v = e^{-x^2}, u = -2x$
 $v' = -2xe^{-x^2}, u' = -2$

$f''(x) = -2e^{-x^2} + 4x^2e^{-x^2}$

~~$= -2e^{-x^2}$~~

$= -2e^{-x^2}(1 - 2x^2)$

Q6. non shaded
 ~~$A = \dots$~~

ai) $y = \log_e x$

$\log_e x = y$
 $x = e^y$

$A = \int_0^2 e^y dy$

$= [e^y]_0^2$
 $= e^2 - 1$

aii) $\int_{e^2}^e \ln x dx$
 $= -2e^2 - e^2 + 1$
 $= e^2 + 1$

6a 6b

i) $V = \pi \int_a^b y^2 dx$

$= \pi \int_0^2 4x^2 dx$

ii) $A = \frac{h}{3} (\text{first part} + \text{FOPE})$

y^2	0	0.5	1	1.5	2
y	1	$2^{0.5}$	2	$2^{1.5}$	4
w	1	4	2	4	1

$h = 0.5$

$\frac{h}{3} = \frac{1}{6}$

$A = \frac{1}{6} (5 + 4(2^{0.5} + 2^{1.5}) + 4)$
 $= \frac{1}{6} (9 + 4(2^{0.5} + 2^{1.5}))$
 $= 4.3 \text{ u}^2$

$V = \pi \int_0^2 4x^2 dx$

c) let $P = 6 \text{ cm}$

i) $P = 2r + r\theta$
 $= r(2 + \theta)$
 $6 = r(2 + 2)$
 $r = \frac{6}{4} = 1.5$

$= 4.3\pi = 13.5 \text{ u}^3$

ii) ~~Area~~

6c ii)

$$A = \pi r^2$$

$$= \pi \left(\frac{6}{\theta+2} \right)^2$$

$$= \frac{36\pi}{(\theta+2)^2}$$

$$= \frac{36\pi}{\theta(\theta+2)^2}$$

~~$$A = \frac{\pi r^2}{\frac{36\pi}{\theta(\theta+2)^2}}$$~~

$$A = \frac{\theta}{2\pi} \times \pi r^2$$

$$= \frac{\theta}{2\pi} \times \pi \left(\frac{36}{\theta+2} \right)^2$$

$$= \frac{36 \cdot 18\theta}{(\theta+2)^2}$$

~~iii) $A = \frac{18\theta}{(\theta+2)^2}$~~

~~find A' $u = 18\theta$ $v = (\theta+2)^2$

$$A' = 32\theta(\theta+2) + 18(\theta+2)^2$$~~

~~solve $= 2(\theta+2)(16\theta + 9(\theta+2))$

so when $A' = 0$

$$= 2(\theta+2)(16\theta + 9\theta + 18)$$

$$= 2(\theta+2)(25\theta + 18)$$~~

~~$$A' = 0$$

$$\theta = -2, \frac{-18}{25}$$~~

$$A = \frac{18\theta}{(\theta+2)^2} = \frac{u}{v}$$

$$v = (\theta+2)^2 \quad u = 18\theta$$

$$v' = 2(\theta+2) \quad u' = 18$$

$$A' = \frac{18(\theta+2)^2 - 36\theta(\theta+2)}{(\theta+2)^4}$$

$$= \frac{18(\theta+2)(\theta+2-2\theta)}{(\theta+2)^4}$$

$$= \frac{18(2-\theta)}{(\theta+2)^3}$$

$\frac{13}{15}$

Set $A' = 0$ for stat pt.
denominator $\neq 0$

$$\theta = 2$$

Test for max.

i.e.

$\theta=1$	$\theta=2$	$\theta=3$

$$A = \frac{36}{(4)^2}$$

$$= \frac{36}{16}$$

$$= 9/4 \quad u^2$$