SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2005
HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK #1

Mathematics Extension 2

General Instructions

- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- Each question is to be returned in a separate bundle.
- All necessary working should be shown in every question.

Total Marks – 85

- Attempt questions 1 – 3
- All questions are not of equal value.

Examiner: *C. Kourtesis*

Question 1. (Start a new answer sheet.) (31 marks)

Marks

(a) Given that $w=\sqrt{3}+i$, express the following in the form $a+ib$ where a and b are real. **4**

(i) $-iw$

(ii) w^2

(ii) w^{-1}

(b) If $z=1-i$ find: **4**

(i) $|z|$ and $\arg z$

(ii) z^8 in exact form

(c) Consider the equation **3**

$$z^2+kz+(4-i)=0$$

Find the complex number k given that $2i$ is a root of the equation.

(d) If $z=x+iy$ prove that **3**

$$z+\frac{|z|^2}{z}=2\,\mathrm{Re}(z)$$

(e) Sketch the locus of z satisfying **4**

(i) $|z+2i|=2$

(ii) $\mathrm{Re}(z^2)=0$

(f) (i) Plot on the Argand diagram all complex numbers that are roots of $z^5=1$. **4**

(ii) Express z^5-1 as a product of real linear and quadratic factors.

(g) (i) By solving the equation $z^3+1=0$ find the three cube roots of -1. 9

(ii) Let ω be a cube root of -1, where ω is not real.
Show that $\omega^2+1=\omega$

(iii) Hence simplify $(1-\omega)^{12}$.

(iv) Find a quadratic equation with real coefficients whose roots are ω^2 and $-\omega$.

Question 2. (Start a new answer sheet.) (31 marks)

Marks

(a) Given that $\text{cis}\,\theta = \cos\theta + i\sin\theta$ find in exact form **3**

$$\text{cis}\frac{\pi}{12}\,\text{cis}\frac{\pi}{6}$$

(b) The equation $x^3 + Ax + B = 0$ (A, B real) has three real roots $\alpha,\ \beta$ and γ. **9**

(i) Evaluate $\alpha^{-1} + \beta^{-1} + \gamma^{-1}$ and $\alpha^2 + \beta^2 + \gamma^2$ in terms of A and B.

(ii) Prove that $A < 0$.

(iii) Find the cubic polynomial whose roots are $\alpha^2,\ \beta^2$ and γ^2.

(c) It is given that $z = 1 + i$ is a zero of $P(z) = z^3 + pz^2 + qz + 6$ where p and q are real numbers. **4**

(i) Explain why $\bar{z}$ is also a zero of $P(z)$. (State the theorem.)

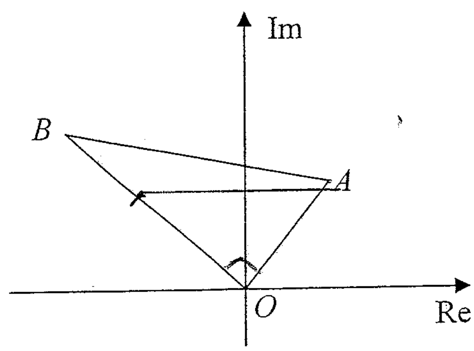
(ii) Find the values of p and q.

(d) Find the number of ways in which six women and six men can be arranged in three sets of four for tennis if: **5**

(i) there are no restrictions.

(ii) each man has a woman as a partner.

(e) In the Argand diagram the points O, A and B are the vertices of a triangle with $\angle AOB = 90°$ and $\dfrac{OB}{OA} = 2$. **6**



The vertices A and B correspond to the complex numbers z_1 and z_2 respectively.

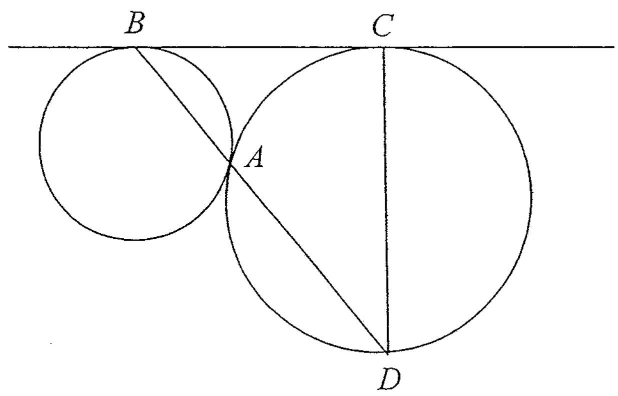
Show that:

(i) $2z_1 + iz_2 = 0$

(ii) the equation of the circle with AB as diameter and passing through O is given by

$$\left|z - z_1\left(\tfrac{1}{2} + i\right)\right| = \tfrac{\sqrt{5}}{2}\left|z_1\right|.$$

(f) **4**



The two circles touch at A and a common external tangent touches them at B and C. BA produced meets the larger circle at D.

Prove that CD is a diameter.

Question 3. (Start a new answer sheet.) (23 marks)

Marks

(a) In how many ways can three different trophies be awarded to five golfers if a golfer may receive at most two trophies? **3**

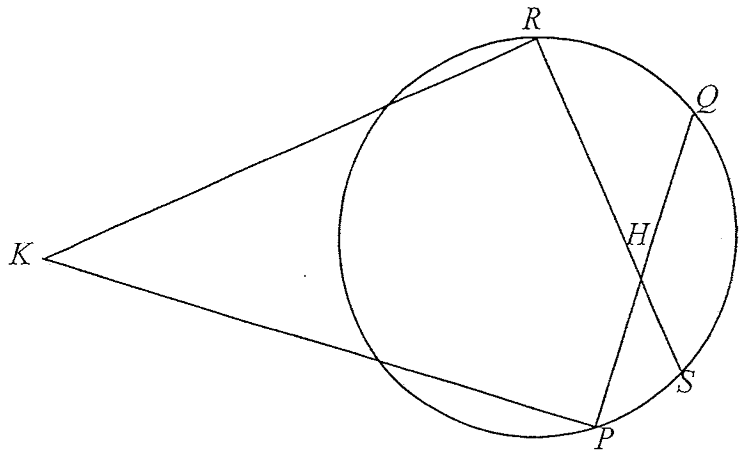
(b) Sketch the region in the Argand diagram consisting of all points z satisfying **4**

$$|\arg z| < \frac{\pi}{4} \text{ and } z + \overline{z} < 4 \text{ and } |z| > 2 .$$

(c) (i) Prove that $(1 + i\tan\theta)^n + (1 - i\tan\theta)^n = \dfrac{2\cos n\theta}{\cos^n\theta}$, where n is a positive integer. **6**

(ii) Hence or otherwise show that $(1+z)^4 + (1-z)^4 = 0$ has roots $\pm i\tan\frac{\pi}{8}$ and $\pm i\tan\frac{3\pi}{8}$

(d)



In the diagram above PQ and RS are two chords intersecting at H, and $\angle KPQ = \angle KRS = 90°$. **6**

(i) Copy the diagram onto your answer sheet, indicating the above information.

(ii) Prove that (α) $\angle PKH = \angle PQS$.

(β) KH produced is perpendicular to QS.

(e) If α is a real root of the equation $x^3 + ux + v = 0$ prove that the other two roots are real if $4u + 3\alpha^2 \leq 0$. **4**

End of paper.

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Mathematics Extension 2

Sample Solutions

Question	Marker
1	PSP
2	DH
3	PRB

Question 1

(a) $w=\sqrt{3}+i$

(i) $-iw=-i\left(\sqrt{3}+i\right)=1-i\sqrt{3}$

(ii) $w^2=\left(\sqrt{3}+i\right)^2=2+i2\sqrt{3}$

(iii) $w^{-1}=\dfrac{\overline{w}}{|w|^2}=\dfrac{\sqrt{3}-i}{4}=\dfrac{\sqrt{3}}{4}-i\left(\dfrac{1}{4}\right)$

(b) $z=1-i$

(i) $|z|=\sqrt{2},\arg(z)=-\dfrac{\pi}{4}$

(ii) $z^8=\left(\sqrt{2}\operatorname{cis}\left(-\dfrac{\pi}{4}\right)\right)^8=16\operatorname{cis}\left(-\dfrac{8\pi}{4}\right)=16\operatorname{cis}(-2\pi)=16$

(c) $p(z)=z^2+kz+(4-i)$

$p(2i)=0\Rightarrow(2i)^2+k(2i)+4-i=0$

$\therefore -4+2ki+4-i=0$

$\therefore 2ki=i$

$\therefore k=\dfrac{1}{2}$

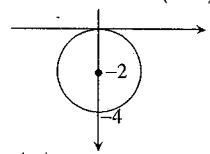
(d) $z=x+iy$

$\because \dfrac{1}{z}=\dfrac{\overline{z}}{|z|^2}$

$\therefore z+\dfrac{|z|^2}{z}=z+\overline{z}=2\operatorname{Re}z$

(e) (i) $x^2+(y+2)^2=4$

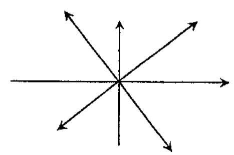
A circle with centre $(0,-2)$ ie $-2i$ and radius 2



(ii) $\operatorname{Re}(z^2)=x^2-y^2=0$

$\therefore x^2=y^2$

$\therefore y=\pm x$



(f) (i)

$$z^5 = 1\times \text{cis}(0)$$
$$= \text{cis}(0+2k\pi), k\in\mathbb{Z}$$
$$= \text{cis}(2k\pi)$$
$$z = \left[\text{cis}(2k\pi)\right]^{1/5}$$
$$= \text{cis}\left(\frac{2k\pi}{5}\right) \quad \text{(deMoivre's Theorem)}$$

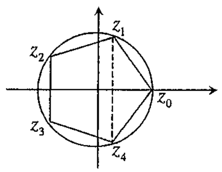
$k=0$: $z_0 = \text{cis}(0) = 1$

$k=1$: $z_1 = \text{cis}\left(\frac{2\pi}{5}\right)$

$k=2$: $z_2 = \text{cis}\left(\frac{4\pi}{5}\right)$ $\quad \boxed{|z_k| = 1}$

$k=-1$: $z_3 = \text{cis}\left(-\frac{2\pi}{5}\right)$

$k=-2$: $z_4 = \text{cis}\left(-\frac{4\pi}{5}\right)$



The 5 roots must form a regular pentagon inscribed in a unit circle.
As well:
z_1 and z_4 are conjugates
z_2 and z_3 are conjugates

(ii)

$$(z-\alpha)(z-\bar{\alpha}) = z^2 - 2\,\text{Re}(\alpha)z + |\alpha|^2$$
$$z^5 - 1 = (z-z_0)(z-z_1)(z-z_2)(z-z_3)(z-z_4)$$
$$= (z-1)(z-z_1)(z-\bar{z}_1)(z-z_2)(z-\bar{z}_2)$$
$$= (z-1)\left(z^2-(2\,\text{Re}\,z_1)z+|z_1|^2\right)\left(z^2-(2\,\text{Re}\,z_2)z+|z_2|^2\right)z$$
$$= (z-1)\left(z^2 - 2z\cos\frac{2\pi}{5}+1\right)\left(z^2-2z\cos\frac{4\pi}{5}+1\right)$$

(g) (i)

$$z^3 = -1$$
$$= 1\times\text{cis}(\pi)$$
$$= \text{cis}(\pi+2k\pi), k\in\mathbb{Z}$$
$$= \text{cis}(2k+1)\pi$$
$$z = \left[\text{cis}(2k+1)\pi\right]^{1/3}$$
$$= \text{cis}(2k+1)\frac{\pi}{3} \quad \text{(deMoivre's Theorem)}$$

$k=0$: $z = \text{cis}\frac{\pi}{3} = \frac{1}{2}+\frac{\sqrt{3}}{2}i$

$k=1$: $z = \text{cis}\frac{3\pi}{3} = -1$

$k=-1$: $\text{cis}\left(-\frac{\pi}{3}\right) = \frac{1}{2}-\frac{\sqrt{3}}{2}i$

(ii)

$$z^3+1 = (z+1)(z^2-z+1)$$
$$\omega^3 = -1, \omega\neq -1$$
$$\therefore \omega^3+1 = (\omega+1)(\omega^2-\omega+1) = 0$$
$$\therefore \omega^2-\omega+1 = 0 \quad (\because \omega\neq -1)$$
$$\therefore \omega^2+1 = \omega$$

(iii)

$$(1-\omega)^{12} = (-\omega^2)^{12} \quad \text{(from (ii))}$$
$$= (\omega^3)^8$$
$$= (-1)^8$$
$$= 1$$

(iv)

$$(z-\omega^2)(z+\omega) = 0$$
$$z^2 + (\omega-\omega^2)z - \omega^3 = 0$$
$$\therefore z^2 + (1)z - (-1) = 0 \quad \text{(from (ii))}$$
$$\therefore z^2+z+1 = 0$$

OR more simply since $z^3+1 = (z+1)(z^2-z+1)$
and the three roots of -1 are so that $z^2-z+1=0$ must have roots $\omega, -\omega^2$.
So let $y=-z$ and $y^2+y+1=0$ MUST have roots $-\omega, \omega^2$.

Question 2

(a) Method 1: [3]

$$\text{cis}\frac{\pi}{12}\,\text{cis}\frac{\pi}{6} = \text{cis}\left(\frac{\pi}{12}+\frac{\pi}{6}\right),\ \text{by de Moivre's theorem}$$
$$= \text{cis}\frac{\pi}{4},$$
$$= \frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}.$$

Method 2:

$$\text{cis}\frac{\pi}{12}\,\text{cis}\frac{\pi}{6} = \left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right),$$
$$= \cos\frac{\pi}{12}\cos\frac{\pi}{6}-\sin\frac{\pi}{12}\sin\frac{\pi}{6}+i\left(\sin\frac{\pi}{12}\cos\frac{\pi}{6}+\cos\frac{\pi}{12}\sin\frac{\pi}{6}\right),$$
$$= \cos\frac{\pi}{4}+i\sin\frac{\pi}{4},$$
$$= \frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}.$$

(b) i. [3]

$$\alpha+\beta+\gamma = 0,$$
$$\alpha\beta+\alpha\gamma+\beta\gamma = A,$$
$$\alpha\beta\gamma = -B.$$

Now, $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}$,
$$= -\frac{A}{B}.$$

Also, $(\alpha+\beta+\gamma)^2 = \alpha^2+\beta^2+\gamma^2+2(\alpha\beta+\alpha\gamma+\beta\gamma)$.
$$\therefore\ \alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma),$$
$$= 0-2A,$$
$$= -2A.$$

ii. Method 1: [2]

$A = -\frac{1}{2}(\alpha^2+\beta^2+\gamma^2)$.
But $\alpha^2+\beta^2+\gamma^2 > 0$ if $\alpha \neq \beta \neq \gamma$.
$\therefore\ A < 0$.

Method 2:
$P'(x) = 3x^2 + A$.
If $A > 0$ then $P(x)$ is monotonic increasing so there can be only one real root. But there are 3 real roots so $A < 0$.

iii. Put $X = x^2$. [4]

$$\therefore\ x = \sqrt{X}.$$
$$X\sqrt{X}+A\sqrt{X}+B = 0,$$
$$\sqrt{X}(X+A) = -B,$$
$$X(X^2+2XA+A^2) = B^2.$$

New equation is $x^3+2Ax^2+A^2x-B^2 = 0$.

(c) i. If $a+ib$ is a complex zero of the polynomial $P(x)$ of degree $n \geq 1$ with real coefficients, then $a-ib$ is also a zero of $P(x)$. [1]

ii. Let the roots be α, $1+i$, $1-i$, then [3]

$$z^3+pz^2+qz+6 = (z-\alpha)(z-1-i)(z-1+i),$$
$$= (z-\alpha)(z^2-2z+2),$$
$$= z^3-(\alpha+2)z^2+(2\alpha+2)z-2\alpha.$$

Equating coefficients gives $\alpha = -3$.
$$p = -(-3+2),$$
$$= 1.$$
$$q = -6+2,$$
$$= -4.$$

(d) i. There are ${}^{12}C_4$ ways of getting the first group and ${}^{8}C_4$ ways of getting the second group leaving the third group. As the group order does not matter, we have $\frac{{}^{12}C_4 \times {}^{8}C_4}{3!} = 5775$. [2]

ii. There are ${}^{6}C_2 \times {}^{6}C_2$ ways of getting the first and ${}^{4}C_2 \times {}^{4}C_2$ ways of getting the second group, leaving the third group. As before, the group order does not matter, so we have $\frac{({}^{6}C_2 \times {}^{4}C_2)^2}{3!} = 1350$. [3]
Note that we are not asked to arrange the people within the groups, only to form the groups.

(e) i. Method 1: [2]

$$z_2 = 2iz_1\ \text{(Twice the length and rotated anti-clockwise by } 90^\circ),$$
$$iz_2 = -2z_1,$$
$$\therefore\ 2z_1+2iz_2 = 0.$$

Method 2:

Let $z_1 = a+ib$,
$$z_2 = 2i(a+ib),$$
$$= 2ai-2b.$$
$$\therefore\ 2z_1 = 2a+2bi,$$
$$iz_2 = -2a-2bi.$$

So $2z_1+iz_2 = 0$.

ii. Method 1: [4]

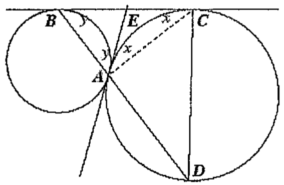
$$\text{Centre} = \frac{z_1+z_2}{2},$$
$$= \frac{z_1}{2}-\frac{2z_1}{2i}\times\frac{i}{i},$$
$$= z_1\left(\tfrac{1}{2}+i\right).$$

$$\text{Radius} = \tfrac{1}{2}|z_1 - z_2|,$$
$$= \tfrac{1}{2}|z_1 - 2z_1 i|,$$
$$= \tfrac{1}{2}|z_1|\,|1 - 2i|,$$
$$= \tfrac{1}{2}|z_1|\sqrt{1^2 + 2^2},$$
$$= \tfrac{\sqrt{5}}{2}|z_1|.$$
$$\therefore \quad \left|z - z_1\left(\tfrac{1}{2} + i\right)\right| = \tfrac{\sqrt{5}}{2}|z_1|$$

Method 2:

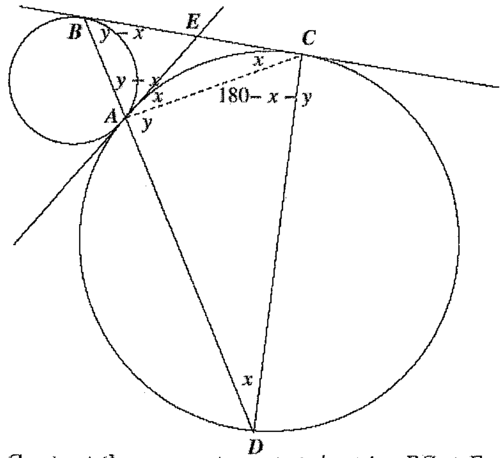
$$\text{Centre} = \frac{a-2b}{2} + \frac{i}{2}(b+2a),$$
$$= \frac{a+ib}{2} + \frac{2ai-2b}{2},$$
$$= \frac{z_1}{2} + \frac{z_2}{2},$$
$$= \frac{z_1}{2} - \frac{2z_1}{2i} \times \frac{i}{i},$$
$$= z_1\left(\tfrac{1}{2} + i\right).$$
$$\text{Radius}^2 = \left(\frac{a-2b}{2}\right)^2 + \left(\frac{b+2a}{2}\right)^2,$$
$$= \frac{a^2 - 4ab + 4b^2 + b^2 + 4ab + 4a^2}{4},$$
$$= \frac{5a^2 + 5b^2}{4}.$$
$$\text{Radius} = \tfrac{\sqrt{5}}{2}\sqrt{a^2+b^2},$$
$$= \tfrac{\sqrt{5}}{2}|z_1|$$
$$\therefore \quad \left|z - z_1\left(\tfrac{1}{2} + i\right)\right| = \tfrac{\sqrt{5}}{2}|z_1|$$

(f) Method 1:



Construct the common tangent at A cutting BC at E.
Join AC.
Let $A\hat{C}E = x$, $E\hat{B}A = y$.
$EC = EA = EB$ (equal tangents from external point),
$E\hat{C}A = E\hat{A}C = x$ (equal angles of isosceles $\triangle$),
$E\hat{B}A = B\hat{A}E = y$ (equal angles of isosceles $\triangle$),
$2x + 2y = 180°$ (angle sum of $\triangle ABC$),
$x + y = 90° = B\hat{A}E$,
$\therefore$ $C\hat{A}D = 90°$ (supplementary to $B\hat{A}E$),
$\therefore$ CD is a diameter (angle in a semi-circle is a right angle).

Method 2:



Construct the common tangent at A cutting BC at E.
Join AC.
Let $A\hat{D}C = x$, $C\hat{A}D = y$.
$A\hat{C}D = 180° - x - y$ (angle sum of $\triangle$),
$E\hat{C}A = x$ (angle in alternate segment),
$D\hat{B}C = y - x$ (angle sum of $\triangle$).
$EC = EA = EB$ (equal tangents from external point),
$E\hat{C}A = E\hat{A}C = x$ (equal angles of isosceles $\triangle$),
$E\hat{B}A = B\hat{A}E = y - x$ (equal angles of isosceles $\triangle$),
$B\hat{A}D = 2y = 180°$ (supplementary angles),
$\therefore$ $y = 90°$
$B\hat{C}D = 180° - y = 90°$.
$\therefore$ CD is a diameter (radius $\perp$ tangent at the point of tangency).

Question 3

(a) **Method 1:**

Case 1: 3 different golfers receive prizes

$\binom{5}{3}$ picks the golfers and then the prizes can be awarded in 3! ways

ie $\binom{5}{3} \times 3! = 60$ ways.

Case 2: 1 golfer receives two prizes

Pick the golfer to receive the prize in $\binom{5}{1}$ ways and his prizes in $\binom{3}{2}$ ways.

Then the remaining prize can go to one of the 4 others

ie $\binom{5}{1} \times \binom{3}{2} \times \binom{4}{1} = 60$ ways

Total = 60 + 60 =120

Method 2:

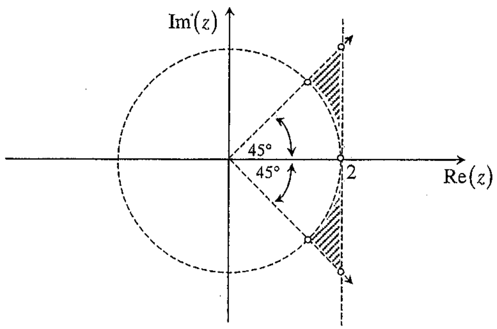
There are $5^3 = 125$ ways of dividing up the prizes with no restrictions.

There are 5 ways in which a golfer can get all the prizes.

So there are $125 - 5 = 120$ ways in dividing up the prizes so that a golfer gets no more than 2 prizes.

(b) $|\arg z| < \frac{\pi}{4} \Rightarrow -\frac{\pi}{4} < \arg z < \frac{\pi}{4}$

$z + \bar{z} < 4 \Rightarrow x < 2$



(c) (i) $\text{LHS} = (1 + i\tan\theta)^n + (1 - i\tan\theta)^n$

$= \left(1 + i\frac{\sin\theta}{\cos\theta}\right)^n + \left(1 - i\frac{\sin\theta}{\cos\theta}\right)^n$

$= \left(\frac{\cos\theta + i\sin\theta}{\cos\theta}\right)^n + \left(\frac{\cos\theta - i\sin\theta}{\cos\theta}\right)^n$

$= \frac{[\text{cis}\theta]^n + [\text{cis}(-\theta)]^n}{\cos^n\theta}$

$= \frac{\text{cis}n\theta + \text{cis}(-n\theta)}{\cos^n\theta}$ (deMoivre's Theorem)

$= \frac{2\cos n\theta}{\cos^n\theta}$ $(z + \bar{z} = 2\text{Re}\,z)$

= RHS

(ii) $(1+z)^4 + (1-z)^4 = \frac{2\cos 4\theta}{\cos^4\theta}$ where $z = i\tan\theta$ from (i)

$(1+z)^4 + (1-z)^4 = 0 \Leftrightarrow \frac{2\cos 4\theta}{\cos^4\theta} = 0$

$\therefore \cos 4\theta = 0$

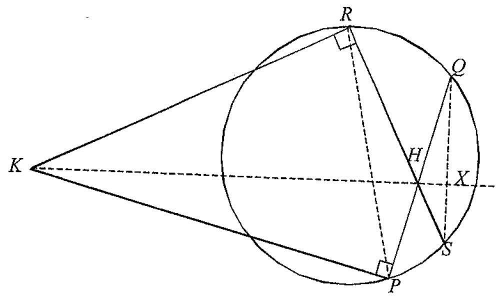
$\therefore 4\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$

$\therefore \theta = \pm\frac{\pi}{8}, \pm\frac{3\pi}{8}$

$\because z = i\tan\theta \Rightarrow z = i\tan\left(\pm\frac{\pi}{8}\right), i\tan\left(\pm\frac{3\pi}{8}\right)$

$\therefore z = \pm i\tan\left(\frac{\pi}{8}\right), \pm i\tan\left(\frac{3\pi}{8}\right)$ $[\because \tan(-x) = -\tan(x)]$

(d) (i)



Join QS and produce KH to intersect with QS at X.
Join RP

(ii) (α) $PKRH$ is a cyclic quadrilateral (opposite angles are supplementary)

$\angle PKH = \angle PRH$ (angles in the same segment)

$\angle PRH = \angle PQS$ (angles in the same segment)

$\therefore \angle PKH = \angle PQS$

(β) $\angle PHK + \angle PKH = 90°$ ($\because \angle KPH = 90°$)

$\angle QHX = \angle PHK$ (vertically opposite angles)

$\therefore \angle QHX + \angle PQS = 90°$ ($\because \angle PKH = \angle PQS$)

$\therefore \angle QXH = 90°$ (angle sum of Δ)

$\therefore KH$ (produced) $\perp QS$

(e) If α is a real root of the equation $x^3 + ux + v = 0$ then $\alpha^3 + u\alpha + v = 0$

Now $x^3 + ux + v = (x-\alpha)(x^2 + Ax + B)$

$$\begin{array}{l} \quad\quad x^2 + \alpha x + (u+\alpha^2) \\ (x-\alpha)\overline{)x^3 + 0x^2 + ux + v} \\ \quad\quad x^3 - \alpha x^2 \\ (x-\alpha)\overline{)0 + \alpha x^2 + ux} \\ \quad\quad \alpha x^2 - \alpha^2 x \\ (x-\alpha)\overline{)0 + (u+\alpha^2)x + v} \\ \quad\quad \underline{(u+\alpha^2)x - (u+\alpha^2)\alpha} \\ \quad\quad\quad 0 \end{array}$$

$\because v + (u+\alpha^2)\alpha = 0$

$\therefore x^3 + ux + v = (x-\alpha)\left[x^2 + \alpha x + (u+\alpha^2)\right]$

With $x^2 + \alpha x + (u+\alpha^2) = 0$ to have real roots then

$\Delta = \alpha^2 - 4(u+\alpha^2) = -3\alpha^2 - 4u \geq 0$

$\therefore 3\alpha^2 + 4u \leq 0$