SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2003
HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK #1

Mathematics Extension 2

General Instructions

- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- Each question is to be returned in a separate bundle.
- All necessary working should be shown in every question.

Total Marks – 60

- Attempt questions 1 – 3
- All questions are of equal value.

Examiner: *A.M.Gainford*

Question 1. (Start a new answer sheet.) (20 marks)

Marks

(a) For the complex number $z = -\sqrt{3} + i$:

(i) Write down (α) $|z|$ **2**

(β) $\arg z$.

(ii) Plot and clearly label each of the following complex numbers on the Argand diagram: **5**

(α) z

(β) $\bar{z}$

(γ) z^2

(δ) iz

(ε) $z\bar{z}$

(b) Sketch the region in the Argand diagram containing all those points z for which: **4**

$$|\arg z| < \frac{\pi}{6} \text{ and } 1 \le z\bar{z} \le 4.$$

(Take special care at boundaries and corners.)

(c) In the Argand diagram sketch the locus of z constrained such that **2**

$$\arg\left(\frac{z-2i}{z-1}\right) = \frac{\pi}{4}.$$

(d) Solve the equation $z^2 + iz + 2 = 0$. **2**

(e) (i) Show that $(x-i)$ is a factor of $P(x) = x^4 + x^3 - 11x^2 + x - 12$. **1**

(ii) Hence reduce $P(x)$ to linear factors over the complex field. **2**

(f) Calculate the complex square roots of $16 - 30i$. **2**

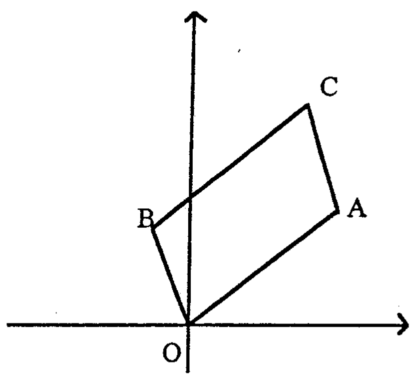
Question 2. (Start a new answer sheet.) (20 marks)

Marks

(a) (i) Show that for any two complex numbers z_1 and z_2: **1**

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right).$$

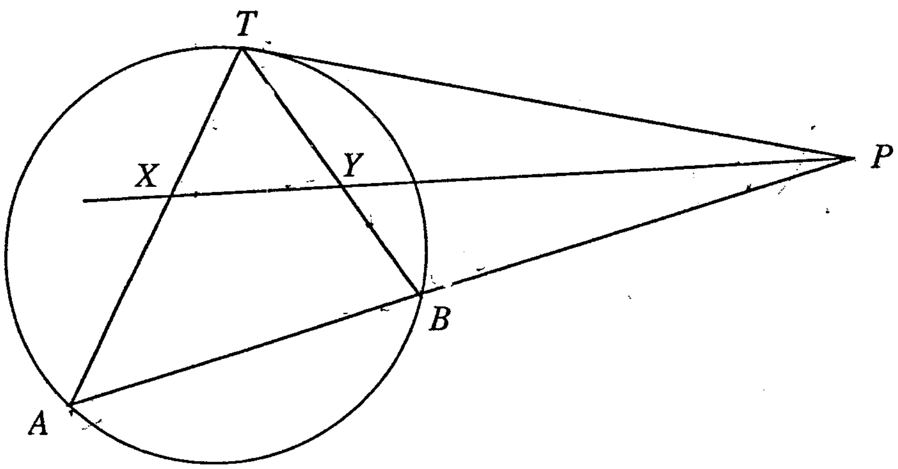
(ii) In the Argand diagram below A and B represent z_1 and z_2, and $OACB$ is a parallelogram. Use this diagram to interpret the above result geometrically. **2**



(b) (i) Find the four solutions of the equation $z^4 + 1 = 0$. **2**

(ii) Hence or otherwise write $z^4 + 1$ as a product of two quadratic factors with real coefficients. **2**

(c) **4**



The tangent at T on the circle meets a chord AB produced at P. The bisector of $\angle TPA$ meets TA and TB at X and Y respectively.

(i) Give the reason why $\angle PTB = \angle TAB$.

(ii) Prove $TX = TY$.

(iii) Prove $\dfrac{TX}{XA} = \dfrac{TP}{PA}$.

(d) (i) Prove that if the polynomial $P(x)$ has a root of multiplicity m, then the derived polynomial $P'(x)$ has a root of multiplicity $m-1$. **3**

(ii) Find the value of k so that the equation $5x^5 - 3x^3 + k = 0$ has a positive repeated root.

(e) The origin and the points z, $\frac{1}{z}$, and $z+\frac{1}{z}$ are joined to form a quadrilateral. **3**
Write down conditions for z so that the quadrilateral will be

(i) a rhombus

(ii) a square

(f) Containers are coded by different arrangements of coloured dots in a row. The colours used are red, white, and blue. **3**

In an arrangement, at most three of the dots are red, at most two of the dots are white, and at most one is blue.

(i) Find the number of different codes possible if six dots are used.

(ii) On some containers only five dots are used. Find the number of different codes possible in this case. Justify your answer.

Question 3. (Start a new answer sheet.) (20 marks)

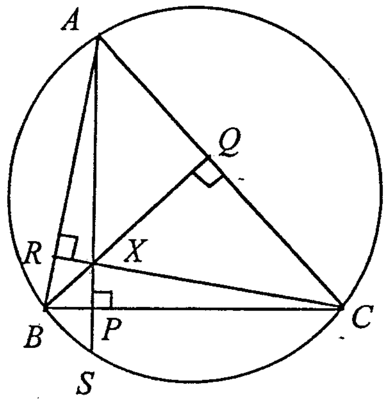
Marks

(a) (i) Use De Moivre's theorem to show that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$. **3**

(ii) Hence solve the equation $16x^4 - 16x^2 + 1 = 0$ and deduce the exact values of $\cos\frac{\pi}{12}$ and $\cos\frac{5\pi}{12}$. **3**

(b) If α, β, γ are the roots of the equation $x^3 - 2x + 3 = 0$, find the monic cubic equation with roots $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$, and hence or otherwise state the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$. **2**

(c)



In the diagram X is the orthocentreof the ΔABC, which has altitudes AP, BQ, CR. AP produced meets the circumcircle at S. **4**

(i) Copy the diagram to your answer sheet.

(ii) Prove that $XP = PS$.

(d) Shows that $x^3 + ax + b = 0$, where a and b are real numbers, has: **4**

(i) only one real root if $a > 0$

(ii) two equal roots if $4a^3 + 27b^2 = 0$

(e) Prove by mathematical induction that **4**

$$1 + 2q + 3q^2 + \ldots + nq^{n-1} = \frac{1-(n+1)q^n + nq^{n+1}}{(1-q)^2} \text{ for all positive integral } n.$$

This is the end of the paper.

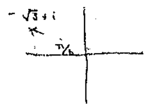
Ext 2 Solutions
Year 12, 2003, Term 1 Test

Question 1

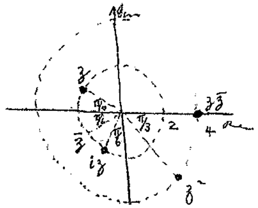
(a) $z = -\sqrt{3} + i$

(i) (α) $|z| = \sqrt{(-\sqrt{3})^2 + 1^2}$
$= \sqrt{4}$
$= 2$ ①

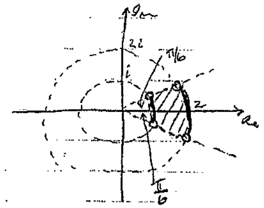
(β) $\arg z = \frac{5\pi}{6}$ ①



(ii) ⑤

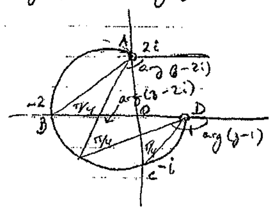


(b) ④ 1 for rays, 1 for circles, 1 for corners, 1 for boundaries.



(c) $\arg\left(\frac{z-2i}{z-1}\right) = \frac{\pi}{4}$

$\arg(z-2i) - \arg(z-1) = \frac{\pi}{4}$ ② 1 for arc of circle, 1 for details



Using isosceles ~~triangles OAB and OCD~~
B is −2 c is $-i$

∴ centre of circle is $\left(-\frac{1}{2}, \frac{1}{2}\right)$
$r^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2$
$= \frac{5}{2}$

∴ The locus is the major arc of the circle $\left(x+\frac{1}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2 = \frac{5}{2}$ as shown

(d) $z^2 + iz + 2 = 0$

$\therefore z = \frac{-i \pm \sqrt{-1 - 4 \times 1 \times 2}}{2}$

$= \frac{-i \pm \sqrt{-9}}{2}$ ②

$= \frac{-i \pm 3i}{2}$

$= i$ or $-2i$

(e) (i) $P(i) = 1 - i + 11 + i - 12$
$= 0$ ①

∴ $x - i$ is a factor of $P(x)$

(ii) As $P(x)$ has real coefficients $x + i$ is also a factor.

$$\begin{array}{r} x^2 + x - 12 \\ x^2+1 \overline{\smash{)}\, x^4 + x^3 - 11x^2 + x - 12} \\ \underline{x^4 \qquad + x^2 \qquad\qquad} \\ x^3 - 12x^2 + x - 12 \\ \underline{x^3 \qquad\quad + x \qquad} \\ -12x^2 \quad - 12 \\ \underline{-12x^2 \quad - 12} \\ - \end{array}$$

② 1 division, 1 factoring

$\therefore P(x) = (x+i)(x-i)(x+4)(x-3)$

(f) $(a+ib)^2 = 16 - 30i$

$a^2 - b^2 + 2abi = 16 - 30i$

$a^2 - b^2 = 16$, $2ab = -30$
$b = -\frac{15}{a}$ 1 for equating

$\therefore a^2 - \frac{225}{a^2} = 16$

$\therefore a^4 - 16a^2 - 225 = 0$

$\therefore a^2 = \frac{16 \pm \sqrt{256 - 4 \times 1 \times -225}}{2}$ ②

$= \frac{16 \pm 34}{2}$

$= 25$ or ~~-9~~ 1 for solution

∴ $a = \pm 5$ $b = \mp 3$

∴ Square roots are $5 - 3i$ and $-5 + 3i$

[1] 2. (a) (i) $\text{L.H.S} = |z_1 + z_2|^2 + |z_1 - z_2|^2,$

$$= (z_1 + z_2)\overline{(z_1 + z_2)} + (z_1 - z_2)\overline{(z_1 - z_2)},$$
$$= z_1\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_1} + z_2\overline{z_2} + z_1\overline{z_1} - z_1\overline{z_2} - z_2\overline{z_1} + z_2\overline{z_2},$$
$$= 2z_1\overline{z_1} + 2z_2\overline{z_2},$$
$$= 2|z_1|^2 + 2|z_2|^2,$$
$$= 2\left(|z_1|^2 + |z_2|^2\right).$$ ✓

[2] (ii) The sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of the sides. ✓✓

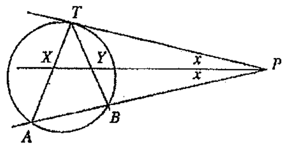
[2] (b) (i) Let $z = r\,\text{cis}\,\theta$,

then $z^4 = r^4\text{cis}\,4\theta$.

i.e., $r^4\text{cis}\,4\theta = -1,$
$$= \text{cis}\,(\pi + 2n\pi).$$
$r = 1,\quad 4\theta = \pi + 2n\pi,$
$$\theta = (2n+1)\tfrac{\pi}{4},$$
$$= \pm\tfrac{\pi}{4},\quad \pm\tfrac{3\pi}{4}.$$
$$\therefore\ z = \frac{1}{\sqrt{2}}(1 \pm i),\quad -\frac{1}{\sqrt{2}}(1 \pm i).$$ ✓✓

[2] (ii) $z^4 = \left(z + \tfrac{1}{\sqrt{2}} + \tfrac{i}{\sqrt{2}}\right)\left(z + \tfrac{1}{\sqrt{2}} - \tfrac{i}{\sqrt{2}}\right)\left(z - \tfrac{1}{\sqrt{2}} + \tfrac{i}{\sqrt{2}}\right)\left(z - \tfrac{1}{\sqrt{2}} - \tfrac{i}{\sqrt{2}}\right),$
$$= \left(z^2 + \tfrac{2z}{\sqrt{2}} + \tfrac{1}{2} + \tfrac{1}{2}\right)\left(z^2 - \tfrac{2z}{\sqrt{2}} + \tfrac{1}{2} + \tfrac{1}{2}\right),$$
$$= \left(z^2 + \sqrt{2}z + 1\right)\left(z^2 - \sqrt{2}z + 1\right).$$ ✓✓

(c)



[1] (i) $P\hat{T}B = T\hat{A}B$ (Angle between tangent and chord is equal to the angle in the alternate segment.) ✓

[2] (ii) $T\hat{P}Y = X\hat{P}A$ (PX bisects $T\hat{P}A$),

$P\hat{T}Y = X\hat{A}P$ (as in (i)),

$\therefore \Delta TPY \,|||\, \Delta AXP$ (equiangular). ✓

$T\hat{Y}P = A\hat{X}P$ (corresp. angles in similar Δ's),

$\therefore T\hat{Y}X = T\hat{X}Y$ (supplementary to equal angles),

$\therefore \Delta TXY$ is isosceles (base angles equal).

$\therefore TX = TY$ (opposite equal angles of isosceles Δ). ✓

[1] (iii) $\dfrac{TY}{XA} = \dfrac{TP}{PA}$ (corresp. sides of similar Δ's),

but $TX = TY$,
$$\therefore \frac{TX}{XA} = \frac{TP}{PA}.$$ ✓

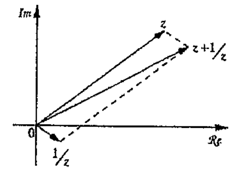
[1] (d) (i) Let $P(x) = (x - \alpha)^m.Q(x),$
$$P'(x) = m(x-\alpha)^{m-1}.Q(x) + (x-\alpha)^m.Q'(x),$$
$$= (x-\alpha)^{m-1}\left(m.Q(x) + (x-\alpha).Q'(x)\right),$$
$$= (x-\alpha)^{m-1}.R(x).$$ ✓

i.e., if $P(x)$ has a root of multiplicity m, then $P'(x)$ has a root of multiplicity $m-1$.

[2] (ii) $P(x) = 5x^5 - 3x^3 + k.$
$$P'(x) = 25x^4 - 9x^2,$$
$$= x^2(25x^2 - 9),$$
$$= x^2(5x-3)(5x+3).$$
$\therefore$ The repeated positive root is $x = \frac{3}{5}$. ✓

So $k = 3\left(\frac{3}{5}\right)^3 - 5\left(\frac{3}{5}\right)^5,$
$$= \frac{405 - 243}{625},$$
$$= \frac{162}{625}.$$ ✓

[1] (e) (i) For a rhombus, $|z| = \left|\dfrac{1}{z}\right|.$
$$z\overline{z} = \frac{1}{z\overline{z}},$$
$$x^2 + y^2 = \frac{1}{x^2+y^2},$$
$$x^4 + 2x^2y^2 + y^4 = 1,$$
$$(x^2+y^2)^2 = 1,$$
$$x^2 + y^2 = 1,\quad (x^2+y^2 \neq -1),$$
i.e., $|z| = 1.$ ✓



[2] (ii) For a square, $\arg z - \arg\dfrac{1}{z} = \pm\dfrac{\pi}{2},$ ✓
$$\text{or}\quad 2(\arg z) = \frac{\pi}{2} + n\pi,$$
$$\arg z = \pm\frac{\pi}{4}\ \text{or}\ \pm\frac{3\pi}{4}.$$
$\therefore$ for a square, $|z| = 1$, $\arg z = \pm\dfrac{\pi}{4}$ or $\pm\dfrac{3\pi}{4}$. ✓

[1] (f) (i) With 6 dots, we can have only 3 red, 2 white, and 1 blue.

$\therefore$ Number of codes $= \dfrac{6!}{3!2!},$
$$= \frac{6.5.4}{2.1},$$
$$= 60.$$ ✓

[2] (ii) Extra ways with 5 dots are: ✓

3 red + 2 white $= \dfrac{5!}{3!2!} = \dfrac{5.4}{2.1} = 10.$

3 red + 1 white + 1 blue $= \dfrac{5!}{3!} = 20.$

2 red + 2 white + 1 blue $= \dfrac{5!}{2!2!} = \dfrac{5.4.3}{2.1} = 30.$

$\therefore$ The number of codes with only 5 dots is 60. ✓

Question 3

(a) (i) $(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$ (A) (D.M.T.)

Also $(\cos\theta + i\sin\theta)^4 = \cos^4\theta + 4\cos^3\theta . i\sin\theta + 6\cos^2\theta (i\sin\theta)^2 + 4\cos\theta (i\sin\theta)^3 + (i\sin\theta)^4$

$= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta$. — (B)

Equating real parts in (A) & (B)

$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$.

$= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$

$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$

$\therefore \boxed{\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1.}$ — (C)

(ii) Now $16x^4 - 16x^2 + 1 = 0$

$\Rightarrow 8x^4 - 8x^2 + 1 = \frac{1}{2}$

In (C) let $\cos 4\theta = \frac{1}{2}$.

$4\theta = 2k\pi \pm \frac{\pi}{3}$

$\Rightarrow 4\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$

$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}$.

Hence $x = \cos\frac{\pi}{12}, \cos\frac{5\pi}{12}, \cos\frac{7\pi}{12}, \cos\frac{11\pi}{12}$

$= \pm\cos\frac{\pi}{12}, \pm\cos\frac{5\pi}{12}$.

Now $S_2 = -\cos^2\frac{\pi}{12} - \cos^2\frac{5\pi}{12} = -1$. and $S_4 = \cos^2\frac{\pi}{12}.\cos^2\frac{5\pi}{12} = \frac{1}{16}$.

$\therefore S_2 = \cos^2\frac{\pi}{12} + \cos^2\frac{5\pi}{12} = 1$

Form a quadratic $A^2 - S_2 A + S_4 = 0$

$A^2 - A + \frac{1}{16} = 0$

$A = \frac{1 \pm \sqrt{1-\frac{1}{4}}}{2}$

$= \frac{2 \pm \sqrt{3}}{4}$

$\therefore \boxed{\cos\frac{\pi}{12} = \frac{1}{2}\sqrt{2+\sqrt{3}}}$ & $\boxed{\cos\frac{5\pi}{12} = \frac{1}{2}\sqrt{2-\sqrt{3}}.}$

NB ($\cos\frac{\pi}{12} > \cos\frac{5\pi}{12} > 0$)

(b) given $x^3 - 2x + 3 = 0$ (A)

Let $X = \frac{1}{x}$ or $x = \frac{1}{X}$ in (A)

$(\frac{1}{X})^3 - 2(\frac{1}{X}) + 3 = 0$

$1 - 2X^2 + 3X^3 = 0$ OR $\boxed{3X^3 - 2X^2 + 1 = 0.}$ (B) OR $\boxed{x^3 - \frac{2}{3}x^2 + \frac{1}{3} = 0}$

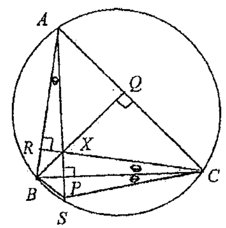
Hence $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = -\frac{b}{a}$ in (B)

$= --\frac{2}{3}$

$= \boxed{\frac{2}{3}}$

Both full marks.

(c) (i)



(ii) Aim: To prove that $XP = PS$.

Construction: Join BS and CS.

Proof: $\angle ARC = \angle APC = 90°$ (data).

$\therefore$ ARPC is a cyclic quadrilateral (two equal angles on same side of interval $\Rightarrow$ pts are concyclic).

$\therefore \angle RAP = \angle RCP = \theta$ (angles in the same segment standing on same arc are equal)

Also $\angle BAS = \angle BCS = \theta$ (" " ")

$\therefore$ In $\triangle XCS$

$\angle XCP = \angle SCP = \theta$ (shown)

$\angle XPC = \angle SPC = 90°$ (data)

PC is common

$\therefore \triangle XPC \equiv \triangle SPC$ (AAS) $\therefore \underline{XP = PS}$ (corresponding sides of congruent triangles)

(3)

(d) (i) $x^3+ax+b=0$ where $a,b \in R$, $a>0$.

Let $P(x) = x^3+ax+b$ — (A)

$P'(x) = 3x^2+a$ — (B)

Clearly $P'(x) > 0$ for all x.

$\therefore$ no stationary pts $\Rightarrow$ 1 real root.

(ii) For 2 equal roots.

$P'(x) = P(x) = 0$ for some x.

now in (B) if $3x^2+a=0$

$x^2 = \frac{-a}{3}$ NB $(a<0)$

& $x = \pm\sqrt{\frac{-a}{3}}$

Sub in (A) $-\frac{a}{3} \times \pm\sqrt{\frac{-a}{3}} + a \times \pm\sqrt{\frac{-a}{3}} + b = 0$

$\therefore \pm \frac{2}{3} a \sqrt{\frac{-a}{3}} = -b$

squaring $-\frac{4}{27}a^3 = b^2$

$\therefore \boxed{4a^3+27b^2=0}$

(4)

(e) Let the statement be

$$S(n) = 1+2q+3q^2+\dots+nq^{n-1} = \frac{1-(n+1)q^n+nq^{n+1}}{(1-q)^2}$$

STEP I. Prove $S(1)$ is true.

LHS = 1 RHS = $\frac{1-2q+q^2}{(1-q)^2} = 1$ $\therefore S(1)$ is true.

STEP II. Assume $S(k)$ is true

i.e. $1+2q+3q^2+\dots+kq^{k-1} = \frac{1-(k+1)q^k+kq^{k+1}}{(1-q)^2}$

STEP III R.T.P. $S(k+1)$ is true

i.e. $1+2q+3q^2+\dots+(k+1)q^k = 1-(k+2)q^{k+1}+(k+1)q^{k+2}$.

$$LHS = \frac{1-(k+1)q^k+kq^{k+1}}{(1-q)^2} + (k+1)q^k$$

$$= \frac{1-(k+1)q^k+kq^{k+1}+(k+1)(1-q)^2q^k}{(1-q)^2}$$

$$= \frac{1-(k+1)q^k+kq^{k+1}+(1-2q+q^2)(k+1)q^k}{(1-q)^2}$$

$$= \frac{1-(k+1)q^k+kq^{k+1}+(k+1)q^k-2(k+1)q^{k+1}+(k+1)q^{k+2}}{(1-q)^2}$$

$$= \frac{1+q^{k+1}(k-2k-2)+(k+1)q^{k+2}}{(1-q)^2}$$

$$= \frac{1-(k+2)q^{k+1}+(k+1)q^{k+2}}{(1-q)^2}$$

= RHS.

CONCLUSION

We have shown that $S(n)$ is true for $n=1$ $\therefore$ by the Principle of Mathematical Induction $S(n)$ is true for all positive integers.