SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2004
HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK # 1

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes.
- Working time – 90 minutes.
- Write using black or blue pen.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may **NOT** be awarded for messy or badly arranged work.
- Hand in your answer booklets in 3 sections.
  Section A (Questions 1 - 5), Section B (Questions 6 - 9) and Section C (Questions 10 - 13).
- Start each **NEW** section in a separate answer booklet.

Total Marks - 90 Marks

- Attempt Sections A - C
- All questions are NOT of equal value.

Examiner: *E. Choy*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

Total marks – 90
Attempt Questions 1 – 13
All questions are NOT of equal value

Answer each section in a SEPARATE writing booklet. Extra writing booklets are available.

SECTION A (Use a SEPARATE writing booklet)

Question 1 (8 marks) — Marks

(a) Evaluate $(3+4i)\div(1+i)$ — 2

(b) If $(x+iy)(2+3i)=5+6i$ find x and y — 2

(c) (i) Express $\dfrac{-1-i\sqrt{3}}{1-i}$ in modulus-argument form — 2

(ii) Hence evaluate $\left(\dfrac{-1-i\sqrt{3}}{1-i}\right)^6$ — 2

Question 2 (8 marks)

If P represents the complex number z, sketch the locus of P (on *separate* diagrams) if:

(i) $|z-1|=4$ — 2

(ii) $-1\le \text{Im}(z)\le 2$ — 2

(iii) $-\dfrac{\pi}{4}\le \arg(z)\le \dfrac{2\pi}{3}$ — 2

(iv) $\arg\left(\dfrac{z-i}{z-1}\right)=\dfrac{\pi}{6}$ — 2

Question 3 (6 marks)

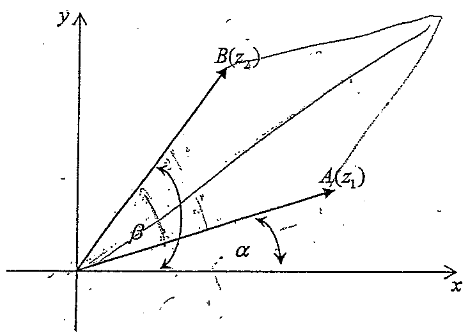
(a) If z is a non zero complex number such that $z+1/z$ is real, prove that $\text{Im}(z)=0$ or $|z|=1$ — 3

(b) Find the square roots of $-2-2i$. — 3

Leave your answer in modulus-argument form.

SECTION A (continued)

Question 4 (4 marks) Marks



In the diagram $\arg(z_1) = \alpha$ and $\arg(z_2) = \beta$. 4

If $|z_1| = |z_2|$ prove that $\arg(z_1 z_2) = \arg\left((z_1 + z_2)^2\right)$

Question 5 (4 marks)

The point A in an Argand diagram represents the complex number $3+4i$. 4

Find the complex number represented by B if ΔOAB is an equilateral triangle with B in the fourth quadrant.

O represents the complex number 0.

Leave your answer in the form $a+ib$.

SECTION B (Use a SEPARATE writing booklet)

Question 6 (7 marks) Marks

Given $P(x) = x^4 - 2x^3 + 6x^2 - 2x + 5$

(i) Find the zeros of $P(x)$ given that $1+2i$ is a zero. 3

(ii) Express $P(x)$ in factored form:

(α) over the complex field; 2

(β) over the real field. 2

Question 7 (9 marks)

If α, β and γ are the roots of the equation $x^3 - 2x + 5 = 0$, find the equation which has roots:

(i) $2/\alpha,\ 2/\beta,\ 2/\gamma$; 3

(ii) $\alpha^2, \beta^2, \gamma^2$; 3

(iii) $\alpha+\beta, \beta+\gamma, \gamma+\alpha$ 3

SECTION B (continued)

Question 8 (6 marks) **Marks**

(i) Show that $f(x)=x^n-1$ has no multiple roots, where n is an integer with $n>1$. 3

(ii) If the roots of $x^n-1=0$ are $1,\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_{n-1}$ show that 3

$$(1-\alpha_1)(1-\alpha_2)(1-\alpha_3)\cdots(1-\alpha_{n-1})=n$$

Question 9 (9 marks)

Consider $\omega=\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}$

(i) Prove that $\omega^5=1$ and $1+\omega+\omega^2+\omega^3+\omega^4=0$ 3

(ii) Prove that $z=\omega+1/\omega$ is a root of $z^2+z-1=0$ 3

(iii) Hence prove that $\cos\frac{2\pi}{5}=\frac{\sqrt{5}-1}{4}$ 3

SECTION C (Use a SEPARATE writing booklet)

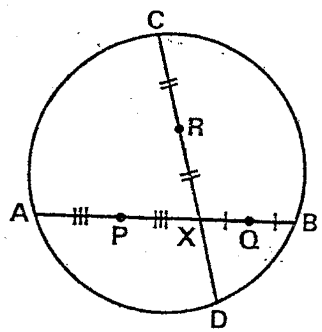
Question 10 (6 marks) **Marks**

Given that $y=x^3-3px+q$ where $p,q\in\mathbb{R}$

(i) Find the coordinates of the stationary points (in terms of p and q) of $y=f(x)$. 3

(ii) Hence, find the relationship between p and q for $f(x)=x^3-3px+q$ to have 3 distinct real roots. 3

Question 11 (6 marks)



6

AB and *CD* are two chords of a circle intersecting at a point *X*. *P*, *Q* and *R* are the midpoints of *AX*, *XB* and *CX* respectively.

Prove that the circle *PQR* also bisects *DX*.

SECTION C (continued)

Question 12 (11 marks)

A and B are two points in an Argand diagram presenting the complex numbers $z_1 = -1$ and $z_2 = \cos\theta + i\sin\theta$ respectively, where $\frac{\pi}{2} < \theta < \pi$.
C is the point representing the complex number $z_3 = z_1 + z_2$.

(i) Sketch the quadrilateral $OACB$ in an Argand diagram, where O is the point representing the complex number 0. **3**

Mark an angle in the diagram which is equal to θ.

(ii) Let $z_4 = z_2 - z_1$

(α) Show that $\frac{z_4}{z_3} = i\left(\frac{\sin\theta}{\cos\theta - 1}\right)$. Hence find $\arg\left(\frac{z_4}{z_3}\right)$. **4**

(β) Using (α) show that the diagonals of the quadrilateral $OACB$ are perpendicular to each other. **4**

Question 13 (6 marks)

How many ways are there to place nine different rings on the four fingers of your right hand (excluding the thumb) if:

(i) the order of the rings on a finger does not matter? **3**

(ii) the order of the rings on a finger is considered? **3**

THIS IS THE END OF THE PAPER

Sydney Boys' High School

Name: Standard Answers

Maths Class: ______ Teacher: D.M.H.

Paper: Ext 2, Assessment #1

2004.

Section: A

Sheet No. 1 of ______ for this Section:

Q.No	Tick	Mark
1	✓	8
2	✓	8
3	✓	6
4	✓	4
5	✓	4
6		30
7		
8		
9		
10		

1(a) $\frac{3+4i}{1+i} \times \frac{1-i}{1-i} = \frac{3-3i+4i+4}{1+1}$

$= \frac{7}{2} + \frac{i}{2}$

(b) Method 1: $(x+iy)(2+3i) = 2x + 3ix + 2iy - 3y$

Equating real & imaginary parts,

$2x - 3y = 5$ — ①

$3x + 2y = 6$ — ②

①×2: $4x - 6y = 10$ — ③

②×3: $9x + 6y = 18$ — ④

③+④: $13x = 28$

$x = 28/13$

$2y = 6 - 3(28/13)$

$y = -3/13$

Method 2: $x + iy = \frac{5+6i}{2+3i} \times \frac{2-3i}{2-3i}$

$= \frac{10-15i+12i+18}{4+9}$

$= \frac{28-3i}{13}$

$\therefore x = \frac{28}{...}\quad y = \frac{-3}{...}$

1 (c) (i) $-1-i\sqrt{3} = 2\,\text{cis}\left(-\frac{2\pi}{3}\right)$ ✓

$1-i = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$

$\therefore \dfrac{-1-i\sqrt{3}}{1-i} = \sqrt{2}\,\text{cis}\left(\dfrac{-5\pi}{12}\right)$ ✓

$\frac{-2}{3}+\frac{1}{4} = \frac{-8+3}{12}$

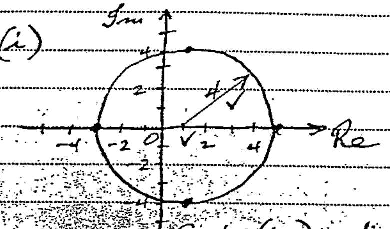
½ for $\frac{-1+\sqrt{3}}{2} - i\left(\frac{1+\sqrt{3}}{2}\right)$

(ii) $\left(\dfrac{-1-i\sqrt{3}}{1-i}\right)^6 = 8\,\text{cis}\left(\dfrac{-30\pi}{12}\right)$ ✓

$= 8\,\text{cis}\left(-\frac{5\pi}{2}\right)$

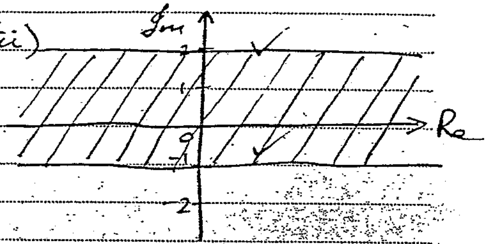
$= -8i$ ✓

2 (i)

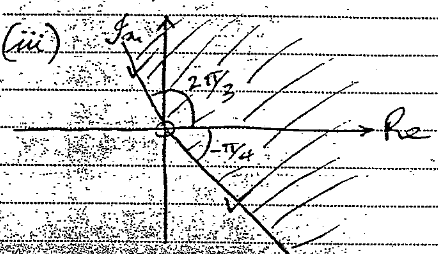


Centre (1,0), radius 4

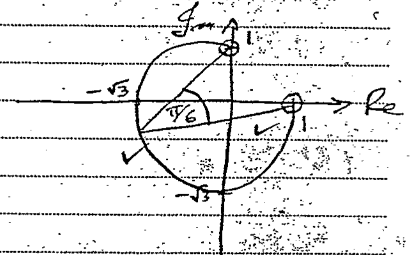
(ii)



(iii)



(iv)



3(a) Let $z = a+ib$, then

$z+\frac{1}{z} = a+ib + \dfrac{1}{a+ib} \times \dfrac{a-ib}{a-ib}$

$= a+ib + \dfrac{a-ib}{a^2+b^2}$ ✓

As $z+\frac{1}{z}$ is real, $b - \dfrac{b}{a^2+b^2} = 0$

ie $b\left(1-\dfrac{1}{a^2+b^2}\right) = 0$

so $b = 0$ or $a^2+b^2 = 1$ ✓

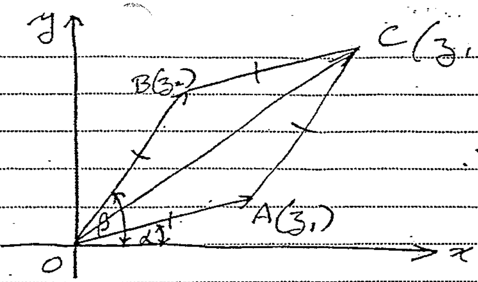
$\therefore \text{Im}(z) = 0$ or $|z| = 1$ ✓

(b) $-2-2i = \sqrt{8}\,\text{cis}\left(-\frac{3\pi}{4}+2n\pi\right)$ ✓

$\therefore \sqrt{-2-2i} = 8^{1/4}\,\text{cis}\left(-\frac{3\pi}{8}+n\pi\right)$

$= 8^{1/4}\,\text{cis}\left(-\frac{3\pi}{8}\right)$ ✓, $8^{1/4}\,\text{cis}\left(\frac{5\pi}{8}\right)$ ✓

4



OACB is a rhombus as $|z_1| = |z_2|$ ✓

$\therefore \arg C = \dfrac{\alpha+\beta}{2} = \arg(z_1+z_2)$

$\therefore \arg(z_1+z_2)^2 = 2\left(\dfrac{\alpha+\beta}{2}\right)$ ✓

$= \alpha+\beta$

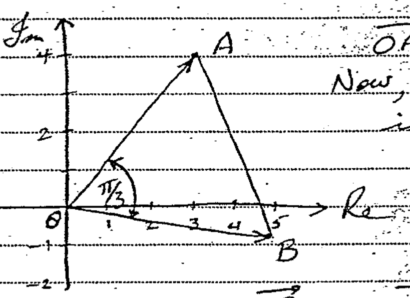
(using deMoivre's thm.)

Now, $\arg(z_1z_2) = \arg z_1 + \arg z_2$

$= \alpha+\beta$ ✓

$\therefore \arg(z_1z_2) = \arg(z_1+z_2)^2$ ✓

5



$\vec{OA} = 3+4i$

Now, unit vector with arg $-\pi/3$ is $\text{cis}\left(-\frac{\pi}{3}\right) = \frac{1}{2} - \frac{i\sqrt{3}}{2}$ ✓

$\vec{OB} = \vec{OA} \times \text{cis}\left(-\frac{\pi}{3}\right)$ ✓

$= (3+4i)\times\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)$ ✓

$= \frac{1}{2}\left(3 - 3i\sqrt{3} + 4i + 4\sqrt{3}\right)$

$= \dfrac{3+4\sqrt{3}}{2} + \dfrac{i}{2}\left(4-3\sqrt{3}\right)$ ✓

$\approx 4.9641 - 0.5981i$

Question 6

$P(x)=x^4-2x^3+6x^2-2x+5$

(i) Since $x=1+2i$ is a zero AND the coefficients are real then $x=1-2i$ is also a zero (*conjugate root theorem*).
So $[x-(1+2i)][x-(1-2i)]=x^2-2x+5$ is a factor of $P(x)$.

So $P(x)=x^4-2x^3+6x^2-2x+5=(x^2-2x+5)(x^2+bx+1)$

Collecting terms of degree 1 and comparing coefficients we get
$-2+5b=-2\Rightarrow b=0$

[Or by applying long division methods ie
$(x^4-2x^3+6x^2-2x+5)\div(x^2-2x+5)$ etc....]

Thus $P(x)=x^4-2x^3+6x^2-2x+5=(x^2-2x+5)(x^2+1)$, so the zeros of $P(x)$ are $x=1+2i,1-2i,\pm i$

(ii) (α) $P(x)=x^4-2x^3+6x^2-2x+5=(x-(1+2i))(x-(1-2i))(x-i)(x+i)$

(β) $P(x)=x^4-2x^3+6x^2-2x+5=(x^2-2x+5)(x^2+1)$

Question 7

$x^3-2x+5=0$

(i) Apply the transformation $y=\frac{2}{x}\Rightarrow x=\frac{2}{y}$

So $\left(\frac{2}{y}\right)^3-2\frac{2}{y}+5=0\Rightarrow\frac{8}{y^3}-\frac{4}{y}+5=0\Rightarrow 5y^3-4y^2+8=0$
So the equation with roots $2/\alpha,\ 2/\beta,\ 2/\gamma$ is $\boxed{5x^3-4x^2+8=0}$

(ii) Apply the transformation $y=x^2$
NB $x^3-2x+5=0\Rightarrow x(x^2-2)=-5$
So square both sides and we get $x^2(x^2-2)^2=25\Rightarrow y(y-2)^2=25$
So the equation with roots $\alpha^2,\beta^2,\gamma^2$ is given by
$\boxed{x(x-2)^2=25 \text{ OR } x^3-4x^2+4x-25=0}$

(iii) We have to use the fact that $\alpha+\beta+\gamma=0\Rightarrow\alpha+\beta=-\gamma$ and so on. So use the transformation $y=-x\Rightarrow x=-y$
So we get $(-y)^3-2(-y)+5=0\Rightarrow -y^3+2y+5=0$.
So the equation with roots $\alpha+\beta,\beta+\gamma,\gamma+\alpha$ is $\boxed{x^3-2x-5=0}$

Question 8

(a) $f(x)=x^n-1\Rightarrow f'(x)=nx^{n-1}$
Suppose α is a multiple root then $f(\alpha)=f'(\alpha)=0\Rightarrow n\alpha^{n-1}=0\Rightarrow\alpha=0$
BUT $f(0)\neq 0$, so $f(x)$ cannot have any multiple roots.

(b) $x^n-1=(x-1)(x-\alpha_1)(x-\alpha_2)...(x-\alpha_{n-1})$

By long division; series; or other means $x^n-1=(x-1)(1+x+x^2+...+x^{n-1})$
So $x^n-1=(x-1)(x-\alpha_1)(x-\alpha_2)...(x-\alpha_{n-1})=(x-1)(1+x+x^2+...+x^{n-1})$
So $(x-\alpha_1)(x-\alpha_2)...(x-\alpha_{n-1})=1+x+x^2+...+x^{n-1}$, sub $x=1$ into both sides
$\therefore(1-\alpha_1)(1-\alpha_2)...(1-\alpha_{n-1})=(1+1+1^2+...+1^{n-1})=n$ QED

Question 9

$\omega=\text{cis}\frac{2\pi}{5}\neq 1$

(i) $\omega^5=\text{cis}\left(5\times\frac{2\pi}{5}\right)=\text{cis}2\pi=1$. [de Moivre's Theorem – DMT]
$\omega^5-1=(\omega-1)(1+\omega+\omega^2+\omega^3+\omega^4)=0$
$\because\omega\neq 1\Rightarrow 1+\omega+\omega^2+\omega^3+\omega^4=0$

OR $1+\omega+\omega^2+\omega^3+\omega^4=\frac{\omega^5-1}{\omega-1}=\frac{1-1}{\omega-1}=0$ [using geometric series]

(ii) $1+\omega+\omega^2+\omega^3+\omega^4=0\Rightarrow\frac{1}{\omega^2}+\frac{1}{\omega}+1+\omega+\omega^2=0$ $[\div\omega^2]$

Let $z=\omega+\frac{1}{\omega}\Rightarrow(z^2-2)+1+z=0\Rightarrow z^2+z-1=0$

$[\because\frac{1}{\omega^2}+\omega^2=\left(\omega+\frac{1}{\omega}\right)^2-2]$

2004 Extension 2 Task #1 Solutions

Section B

(ii) **Alternative solution**

Examine z^2+z-1 when $z=\omega+\frac{1}{\omega}$

$$z^2+z-1$$
$$=\left(\omega+\frac{1}{\omega}\right)^2+\omega+\frac{1}{\omega}-1$$
$$=\omega^2+2+\frac{1}{\omega^2}+\omega+\frac{1}{\omega}-1$$
$$=\frac{\omega^4+\omega^3+\omega^2+\omega+1}{\omega^2}$$
$$=\frac{0}{\omega^2}$$
$$=0$$

So $z=\omega+\frac{1}{\omega}$ is a solution of $z^2+z-1=0$

(iii) $|\omega|=1\Rightarrow\frac{1}{\omega}=\overline{\omega}\Rightarrow\omega+\frac{1}{\omega}=2\text{Re}\,\omega=2\cos\frac{2\pi}{5}$

$z^2+z-1=0\Rightarrow z=\frac{-1\pm\sqrt{5}}{2}$

$\because \cos\frac{2\pi}{5}>0\Rightarrow 2\cos\frac{2\pi}{5}=\frac{-1+\sqrt{5}}{2}$

$\therefore \cos\frac{2\pi}{5}=\frac{-1+\sqrt{5}}{4}$

[NB the other solution of the quadratic is $2\cos\frac{4\pi}{5}$]

SECTION C

Q10 $y=x^3-3px+q$

(i) $y'=3x^2-3p=0$ $\therefore$ st. pts when $x=\pm\sqrt{p}$

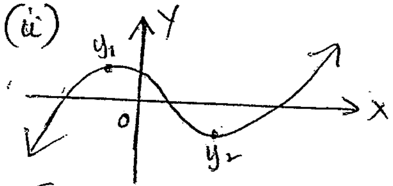
When $x=\sqrt{p}$, $y=-2p^{3/2}+q$

When $x=-\sqrt{p}$, $y=2p^{3/2}+q$

$\therefore$ St. pts $(\sqrt{p},\ q-2p^{3/2})$

$(-\sqrt{p},\ q+2p^{3/2})$

(ii)



For 3 distinct real roots

$y_1y_2<0\Rightarrow(q-2p^{3/2})(q+2p^{3/2})<0$

$\therefore\ q^2<4p^3$

or $|q|<2p\sqrt{p}$

Q11 Let E be the point where circle PQR meets DX.

Now $CX.DX=AX.BX$ (product of chords of circle)

$\therefore (2RX).DX=(2PX)(2QX)$

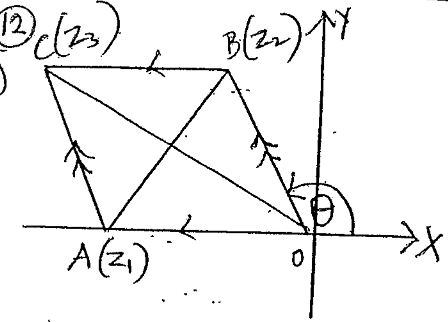
$\Rightarrow (RX)\frac{1}{2}(DX)=(PX)(QX)$

However $(RX)(EX)=(PX)(QX)$

$(EX)=\frac{1}{2}(DX)$

Q12

(i)



(ii) $z_4=z_2-z_1$

$=(\cos\theta+1)+i\sin\theta$

(α) $z_3=z_1+z_2=(-1+\cos\theta)+i\sin\theta$

$\therefore \frac{z_4}{z_3}=\frac{(\cos\theta+1)+i\sin\theta}{(\cos\theta-1)+i\sin\theta}\cdot\frac{(\cos\theta-1)-i\sin\theta}{(\cos\theta-1)-i\sin\theta}$

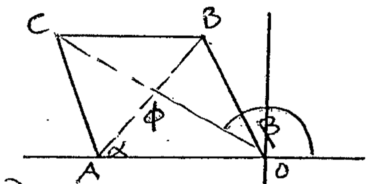
$=\frac{-2i\sin\theta}{2(1-\cos\theta)}$ on simplifying

$=-i\left[\frac{\sin\theta}{1-\cos\theta}\right]$

and $\arg\left(\frac{z_4}{z_3}\right)=-\frac{\pi}{2}$ since

$\frac{\sin\theta}{1-\cos\theta}<0$ for $\frac{\pi}{2}<\theta<\pi$

(β)



$\arg\left(\frac{z_4}{z_3}\right)=\arg z_4-\arg z_3=-\frac{\pi}{2}$

$\Rightarrow \arg z_3-\arg z_4=\frac{\pi}{2}$

$\arg z_3=\beta$, $\arg z_4=\alpha$

$\arg z_3-\arg z_4=\beta-\alpha=\phi$

$\therefore \phi=\frac{\pi}{2}$

Diagonals are $\perp$

or careful expl using × by i

Q13 (i) 4^9 (ii) $\frac{12!}{3!\,9!}\times 9!=\frac{12!}{}$