SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2004
HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK # 2

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes.
- Working time – 2 hours.
- Write using black or blue pen.
- Board approved calculators may be used.
- All *necessary* working should be shown in every question if full marks are to be awarded.
- Marks may **NOT** be awarded for messy or badly arranged work.
- Hand in your answer booklets in **3** sections.
  Section A (Questions 1 - 3), Section B (Questions 4 - 5) and Section C (Questions 6 - 7).
- Start each **NEW** section in a separate answer booklet.

Total Marks - 75 Marks

- Attempt Sections A - C
- All questions are NOT of equal value.

Examiner: *E. Choy*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

Total marks – 78
Attempt Questions 1 – 7
All questions are NOT of equal value

Answer each section in a SEPARATE writing booklet. Extra writing booklets are available.

SECTION A (Use a SEPARATE writing booklet)

Question 1 (15 marks) — Marks

(a) Evaluate — 3

$$\int_0^3 \frac{x\,dx}{\sqrt{16+x^2}}$$

(b) By completing the square first, find — 2

$$\int \frac{dx}{x^2+6x+13}$$

(c) Use integration by parts to find — 2

$$\int xe^{-x}\,dx$$

(d) Find — 3

$$\int \cos^3\theta\,d\theta$$

(e) (i) Find real numbers A, B, and C such that — 3

$$\frac{x^2-4x-1}{(1+2x)(1+x^2)}=\frac{A}{1+2x}+\frac{Bx+C}{1+x^2}$$

(ii) Hence find $\displaystyle\int \frac{x^2-4x-1}{(1+2x)(1+x^2)}dx$ — 2

Question 2 (10 marks) **Marks**

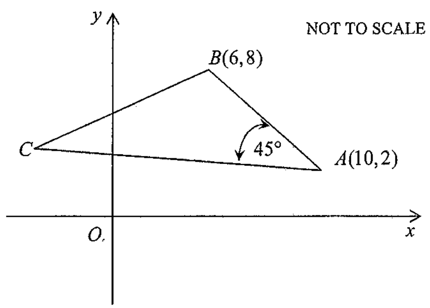
(a) On separate Argand diagrams, sketch the locus defined by:

(i) $2|z| = z + \bar{z} + 2$ **2**

(ii) $\left|z^2 - (\bar{z})^2\right| \geq 4$ **2**

(iii) $\arg(z-1) - \arg(z+1) = -\pi/3$ **2**

(b)



ΔABC is drawn in the Argand diagram above where $\angle BAC = 45°$, A and B are the points $(10, 2)$ and $(6, 8)$ respectively.
The length of side AC is twice the length of side AB.

Find:

(i) the complex number that the vector $\overrightarrow{AB}$ represents the complex number $-4 + 6i$; **2**

(ii) the complex number that the point C represents. **2**

Question 3 (12 marks)

(a) The quadratic equation $x^2 - x + K = 0$, where K is a real number, has two distinct positive real roots α and β.

(i) Show that $0 < K < \frac{1}{4}$ **1**

(ii) Show that $\alpha^2 + \beta^2 = 1 - 2K$ and deduce that $\alpha^2 + \beta^2 > \frac{1}{2}$ **2**

(iii) Show that $\frac{1}{\alpha^2} + \frac{1}{\beta^2} > 8$ **2**

Question 3 continued **Marks**

(b) (i) Show, using De Moivre's Theorem, that $z = \omega$, where $\omega = \sqrt{2} + i\sqrt{2}$ satisfies $z^4 = -16$. **3**
Hence write down, in the form $x + iy$ where x and y are real, all the other solutions of $z^4 = -16$.

(ii) Hence write $z^4 + 16$ as a product of two quadratic factors with real coefficients. **2**

(iii) Show that $\omega + \frac{\omega^3}{4} + \frac{\omega^5}{16} + \frac{\omega^7}{64} = 0$ **2**

SECTION B (Use a SEPARATE writing booklet)

Question 4 (8 marks) **Marks**

(a) Evaluate

(i) $\int_0^a x\sqrt{a-x}\,dx$ 2

(ii) $\int_0^1 \frac{\sin^{-1} x}{\sqrt{1+x}}\,dx$ 2

(b) (i) Using the substitution $t=\tan\frac{x}{2}$ show that 2

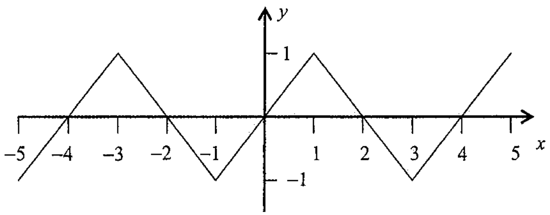
$$\int_0^{\frac{\pi}{2}} \frac{dx}{1+\cos x+\sin x} = \ln 2$$

(ii) Hence, by substituting $u=\frac{\pi}{2}-x$ evaluate 2

$$\int_0^{\frac{\pi}{2}} \frac{x\,dx}{1+\cos x+\sin x}$$

Question 5 (11 marks)

(a)



The diagram is a sketch of the function $y=h(x)$ for $-5\le x\le 5$.
On separate diagrams sketch each of the following:

(i) $y=h(x+1)$ 1

(ii) $y=\frac{1}{h(x)}$ 2

(iii) $y=h(|x|)$ 1

(iv) $y=\sqrt{h(x)}$ 2

(v) $y=h(\sqrt{x})$ 2

(b) Sketch the curve $9y^2=x(x-3)^2$ showing clearly the coordinates of any turning point. 3

SECTION C (Use a SEPARATE writing booklet)

Question 6 (9 marks) **Marks**

A firework missile of mass $0\cdot 2$ kg is projected vertically upwards from rest by means of a force that decreases uniformly in 2 seconds from $2g$ newtons to zero and thereafter ceases. Assume no air resistance and that g is the acceleration due to gravity.

(i) If the missile has an acceleration of a m/s^2 at time t seconds, show that 3

$$a=\begin{cases} g(9-5t) & t\le 2 \\ -g & t>2 \end{cases}$$

[Hint: Draw a diagram showing the forces on the missile.]

(ii) Hence find:

(α) the maximum speed of the missile; 3

(β) the maximum height reached by the missile. 3

Question 7 (10 marks) **Marks**

A particle of mass 1 kg is projected from a point O with a velocity u m/s along a smooth horizontal table in a medium whose resistance is Rv^2 newtons when the particle has velocity v m/s. R is a constant, with $R > 0$.

(i) Show that the equation of motion governing the particle is given by **1**

$$\ddot{x} = -Rv^2$$

where x is the horizontal distance travelled from O.

(ii) Hence show that the velocity, v m/s, after t seconds is given by **3**

$$t = \frac{1}{R}\left(\frac{1}{v} - \frac{1}{u}\right)$$

An equal particle is projected from O simultaneously with the first particle, but vertically upwards with velocity u m/s in the SAME medium.

(iii) Show that the equation of motion governing the second particle is given by **1**

$$\ddot{y} = -(g + Rv^2)$$

where g m/s^2 is the acceleration due to gravity and y represents the vertical distance from O where the particle has a velocity of v m/s.

(iv) Hence show that the velocity V m/s of the first particle when the second one is momentarily at rest is given by **5**

$$\frac{1}{V} = \frac{1}{u} + \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right), \text{ where } Ra^2 = g$$

THIS IS THE END OF THE PAPER

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\ x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ x > 0$

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Sample Solutions

Section	Marker
A	Mr Hespe
B	Mr Parker
C	Mr Kourtesis

[3] 1. (a) Method 1:

$$\begin{aligned} I &= \int_0^3 \frac{x\,dx}{\sqrt{16+x^2}}, \\ &= \tfrac{1}{2}\int_{16}^{25} u^{-\frac{1}{2}}\,du, \\ &= \tfrac{1}{2}\left[2u^{\frac{1}{2}}\right]_{16}^{25}, \\ &= 5-4, \\ &= 1. \end{aligned}$$

put $u = 16 + x^2$, $du = 2x\,dx$; when $x = 0$, $u = 16$; $x = 3$, $u = 25$

Method 2:

$$\begin{aligned} I &= \int_4^5 \frac{u\,du}{u}, \\ &= u\Big]_4^5, \\ &= 1. \end{aligned}$$

put $u^2 = 16 + x^2$, $2u\,du = 2x\,dx$; when $x = 0$, $u = 4$; $x = 3$, $u = 5$

Method 3:

$$\begin{aligned} I &= \int_0^{\tan^{-1}\frac{3}{4}} \frac{4\tan\theta . 4\sec^2\theta\,d\theta}{4\sec\theta}, \\ &= 4\sec\theta\Big]_0^{\tan^{-1}\frac{3}{4}}, \\ &= 4\left\{\tfrac{5}{4} - 1\right\}, \\ &= 1. \end{aligned}$$

put $x = 4\tan\theta$, $dx = 4\sec^2\theta\,d\theta$; when $x = 0$, $\theta = 0$; $x = 3$, $\theta = \tan^{-1}\frac{3}{4}$

Method 4:

$$\begin{aligned} I &= \tfrac{1}{2}\int_0^9 \frac{du}{\sqrt{16+u}}, \\ &= \left[\tfrac{1}{2} \times 2 \times \sqrt{16+u}\,\right]_0^9, \\ &= 5-4, \\ &= 1. \end{aligned}$$

put $u = x^2$, $du = 2x\,dx$; when $x = 0$, $u = 0$; $x = 3$, $u = 9$

Method 5:

$$\begin{aligned} I &= \tfrac{1}{2}\int_0^3 \frac{d(x^2)}{\sqrt{16+x^2}}, \\ &= \left[\tfrac{1}{2} \times 2 \times \sqrt{16+x^2}\,\right]_0^3, \\ &= 5-4, \\ &= 1. \end{aligned}$$

[2] (b) $$\begin{aligned} \mathrm{I} &= \int \frac{dx}{(x^2+6x+9)+13-9}, \\ &= \int \frac{dx}{(x+3)^2+4}, \\ &= \tfrac{1}{2}\tan^{-1}\left(\frac{x+3}{2}\right)+\mathrm{C}. \end{aligned}$$

[2] (c) $$\begin{aligned} \mathrm{I} &= \int xe^{-x}dx, & u &= x, & v' &= e^{-x} \\ & & u' &= 1, & v &= -e^{-x} \\ &= -xe^{-x}+\int e^{-x}dx, \\ &= -xe^{-x}-e^{-x}+\mathrm{C}. \end{aligned}$$

[3] (d) Method 1:

$$\begin{aligned} \mathrm{I} &= \int \cos^2\theta.\cos\theta d\theta, & \text{put } \sin\theta &= u \\ & & \cos\theta d\theta &= du \\ &= \int (1-\sin^2\theta).\cos\theta d\theta, \\ &= \int (1-u^2)du, \\ &= u-\tfrac{1}{3}u^3+\mathrm{C}, \\ &= \sin\theta-\tfrac{1}{3}\sin^3\theta+\mathrm{C}. \end{aligned}$$

Method 2:

$$\begin{aligned} \mathrm{I} &= \int \cos^2\theta.\cos\theta d\theta, \\ &= \int (1-\sin^2\theta).d\sin\theta, \\ &= \sin\theta-\tfrac{1}{3}\sin^3\theta+\mathrm{C}. \end{aligned}$$

Method 3:

$$\begin{aligned} \mathrm{I} &= \int \cos^2\theta.\cos\theta d\theta, & u &= \cos^2\theta, & v' &= \cos\theta \\ & & u' &= -2\sin\theta\cos\theta, & v &= \sin\theta \\ &= \cos^2\theta\sin\theta+2\int \sin^2\theta\cos\theta d\theta, \\ &= \cos^2\theta\sin\theta+2\int \cos\theta(1-\cos^2\theta)d\theta, \\ &= \cos^2\theta\sin\theta+2\sin\theta-2\int \cos^3\theta d\theta, \\ 3\mathrm{I} &= \cos^2\theta\sin\theta+2\sin\theta+c, \\ \mathrm{I} &= \tfrac{1}{3}\{\cos^3\theta\sin\theta+2\sin\theta\}+\mathrm{C}. \end{aligned}$$

[3] (e) (i) Method 1:

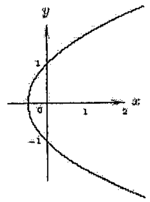
$$\begin{aligned} x^2-4x-1 &= A(1+x^2)+(Bx+C)(1+2x). \\ \text{Put } x &= -\tfrac{1}{2}, \\ \tfrac{1}{4}+2-1 &= A(1\tfrac{1}{4}), \\ A &= 1. \\ \text{Also, } x^2-4x-1 &= x^2(A+2B)+x(B+2C)+(A+C), \\ \text{so } A+2B &= 1, \\ 1+2B &= 1, \\ 2B &= 0, \\ B &= 0. \\ \text{And } B+2C &= -4, \\ 2C &= -4, \\ C &= -2. \end{aligned}$$

Method 2:

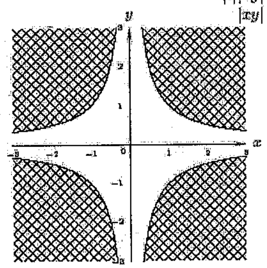
$$\begin{aligned} \frac{x^2-4x-1}{(1+2x)(1+x^2)} &= \frac{A}{1+2x}+\frac{Bx+C}{x^2+1}. \\ \lim_{x\to-\frac{1}{2}}\left\{\frac{x^2-4x-1}{1+x^2}\right\} &= \lim_{x\to-\frac{1}{2}}\left\{A+\left(\frac{Bx+C}{1+x^2}\right)\times(1+2x)\right\}, \\ \frac{\frac{1}{4}+2-1}{1+\frac{1}{4}} &= A, \\ A &= 1. \\ \lim_{x\to i}\left\{\frac{x^2-4x-1}{1+2x}\right\} &= \lim_{x\to i}\left\{\left(\frac{A}{1+2x}\right)\times(1+x^2)+Bx+C\right\}, \\ \frac{-1-4i-1}{1+2i} &= Bi+C, \\ -2-4i &= Bi+C-2B+2iC, \\ \therefore\ C-2B &= -2, \\ B+2C &= -4, \\ 2C-4B &= -4, \\ 5B &= 0, \\ B &= 0, \\ C &= -2. \end{aligned}$$

[2] (ii) $$\begin{aligned} \mathrm{I} &= \tfrac{1}{2}\int \frac{2xdx}{1+2x}-2\int \frac{dx}{1+x^2}, \\ &= \tfrac{1}{2}\ln(1+2x)-2\tan^{-1}x+\mathrm{C}. \end{aligned}$$

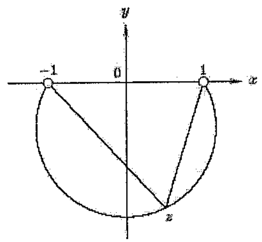
[2] 2. (a) (i) $2\sqrt{x^2+y^2} = x+iy+x-iy+2,$
$$x^2+y^2 = x^2+2x+1,$$
$$y^2 = 4\left(\tfrac{1}{2}\right)\left(x+\tfrac{1}{2}\right).$$



[2] (ii)
$$\left|(x+iy)^2-(x-iy)^2\right| \geq 4,$$
$$\left|x^2+2ixy-y^2-(x^2-2ixy-y^2)\right| \geq 4,$$
$$|ixy| \geq 1,$$
$$|i||xy| \geq 1,$$
$$|xy| \geq 1.$$



[2] (iii)



[2] (b) (i) $\overrightarrow{BA} = (10-6)+i(2-8),$
$$= 4-6i.$$

Note that the question was in error: what was meant was $\overrightarrow{AB}$. Both answers were accepted, $4-6i$ or $-4+6i$.

[2] (ii) Method 1: $\overrightarrow{AC} = \overrightarrow{AB} \times 2 \times \operatorname{cis}\frac{\pi}{4},$
$$= -(4-6i)\times 2\times\left(\tfrac{1}{\sqrt{2}}+\tfrac{i}{\sqrt{2}}\right),$$
$$= -\left(4\sqrt{2}+4\sqrt{2}i-6\sqrt{2}i+6\sqrt{2}\right),$$
$$= -10\sqrt{2}+2\sqrt{2}i.$$
$$\therefore\ C = (10+2i)+\left(-10\sqrt{2}+2\sqrt{2}i\right),$$
$$= 10\left(1-\sqrt{2}\right)+2\left(1+\sqrt{2}\right)i.$$

Method2: $\overrightarrow{OC} = \overrightarrow{OA}+\overrightarrow{AB}\times 2\times\operatorname{cis}\frac{\pi}{4},$
$$= (10+2i)-(4-6i)\times 2\times\left(\tfrac{1}{\sqrt{2}}+\tfrac{i}{\sqrt{2}}\right),$$
$$= (10+2i)-\left(4\sqrt{2}+4\sqrt{2}i-6\sqrt{2}i+6\sqrt{2}\right),$$
$$= 10+2i-10\sqrt{2}+2\sqrt{2}i,$$
$$= 10\left(1-\sqrt{2}\right)+2\left(1+\sqrt{2}\right)i.$$

[1] 3. (a) (i) Roots α, β positive implies $\alpha\beta > 0$,
i.e. $\frac{c}{a} > 0$ or $K > 0$.
Also, for distinct real roots, $\Delta = 1-4K > 0,$
$$1 > 4K,$$
$$K < \tfrac{1}{4}.$$
So, $0 < K < \frac{1}{4}$.

[2] (ii)
$$\alpha+\beta = 1,$$
$$\alpha\beta = K,$$
$$\alpha^2+\beta^2 = (\alpha+\beta)^2-2\alpha\beta,$$
$$= 1-2K.$$

Method 1:
$K < \frac{1}{4}$,
$\therefore\ \alpha^2+\beta^2 > 1-2\left(\frac{1}{4}\right)$ ("greater than" as we are subtracting "less than"),
i.e. $\alpha^2+\beta^2 > \frac{1}{2}$.

Method 2:
$$2K = 1-(\alpha^2+\beta^2),$$
$$K = \frac{1-(\alpha^2+\beta^2)}{2} < \frac{1}{4},$$
$$-(\alpha^2+\beta^2) < -\frac{1}{2},$$
$$\therefore\ \alpha^2+\beta^2 > \tfrac{1}{2}.$$

[2] (iii) $\frac{1}{\alpha^2}+\frac{1}{\beta^2} = \frac{\alpha^2+\beta^2}{\alpha^2\beta^2}$,

$= \frac{1-2K}{K^2}$, (from above).

Now, also from above,

$\alpha^2+\beta^2 > \frac{1}{2}$,

$\alpha^2\beta^2 < \left(\frac{1}{4}\right)^2$,

$\frac{1}{\alpha^2\beta^2} > 16$.

So $\frac{1}{\alpha^2}+\frac{1}{\beta^2} > 16 \times \frac{1}{2}$,

$\therefore\ \frac{1}{\alpha^2}+\frac{1}{\beta^2} > 8$.

[3] (b) (i) $\omega = 2\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)$,

$= 2\text{cis}\,\frac{\pi}{4}$.

$\omega^4 = \left(2\text{cis}\,\frac{\pi}{4}\right)^4$,

$= 16\text{cis}\,\pi$,

$= -16+16\times 0$,

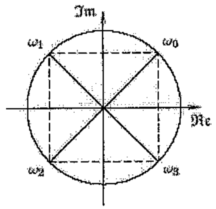
$= -16$.

$\therefore\ \omega_0 = \sqrt{2}+\sqrt{2}i$,

$\omega_1 = -\sqrt{2}+\sqrt{2}i$,

$\omega_2 = -\sqrt{2}-\sqrt{2}i$,

$\omega_3 = \sqrt{2}-\sqrt{2}i$.



[2] (ii) $z^4+16 = \left(z-\sqrt{2}-\sqrt{2}i\right)\left(z-\sqrt{2}+\sqrt{2}i\right)\left(z+\sqrt{2}+\sqrt{2}i\right)\left(z+\sqrt{2}-\sqrt{2}i\right)$,

$= \left(z^2-2\sqrt{2}z+4\right)\left(z^2+2\sqrt{2}z+4\right)$.

[2] (iii) Method 1:

$\omega = 2\text{cis}\,\frac{\pi}{4}$,

$\omega^3 = 2\text{cis}\,\frac{3\pi}{4}$,

$\omega^5 = 32\text{cis}\,\frac{5\pi}{4}$,

$\omega^7 = 128\text{cis}\,\frac{7\pi}{4}$.

$\omega+\frac{\omega^3}{4}+\frac{\omega^5}{16}+\frac{\omega^7}{64} = 2\text{cis}\,\frac{\pi}{4}+2\text{cis}\,\frac{3\pi}{4}+2\text{cis}\,\frac{5\pi}{4}+2\text{cis}\,\frac{7\pi}{4}$,

$= \omega_0+\omega_1+\omega_2+\omega_3$, the sum of the roots,

$= 0$.

Method 2:

$\frac{64\omega+16\omega^3+4\omega^5+\omega^7}{64} = \left(16\omega\left(4+\omega^2\right)+\omega^5\left(4+\omega^2\right)\right)\times\frac{1}{64}$,

$= \omega\left(16+\omega^4\right)\left(4+\omega^2\right)\times\frac{1}{64}$.

But $16+\omega^4=0$,

$\therefore\ \omega+\frac{\omega^3}{4}+\frac{\omega^5}{16}+\frac{\omega^7}{64}=0$

Method 3:

$\omega^4 = -16$,

$\omega+\frac{\omega^3}{4}+\frac{\omega^5}{16}+\frac{\omega^7}{64} = \omega+\frac{\omega^3}{4}+\frac{-16\omega}{16}+\frac{-16\omega^3}{64}$,

$= \omega-\omega+\frac{\omega^3}{4}-\frac{\omega^3}{4}$,

$= 0$.

Method 4:

In the geometric series given, $a=\omega$ and $r=\frac{\omega^2}{4}$.

$S_4 = \frac{\omega\left(1-\left(\frac{\omega^2}{4}\right)^4\right)}{1-\frac{\omega^2}{4}}$, note that $\frac{\omega^8}{256}=\frac{(-16)^2}{256}=1$,

$= \frac{\omega(1-1)}{1-\frac{\omega^2}{4}}$,

$= 0$.

(4)(a)(i) $\int_0^a x\sqrt{a-x}\,dx$

$$= \int_a^0 (a-u)\sqrt{u}\,(-du)$$
$$= \int_0^a (a-u)u^{1/2}\,du$$
$$= \int_0^a (au^{1/2} - u^{3/2})du$$
$$= \left[\frac{2a}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right]_0^a$$
$$= \left(\frac{2a^2}{3} - \frac{2a^2}{5}\right)\sqrt{a}$$
$$= \frac{4a^2}{15}\sqrt{a} = \frac{4a^{5/2}}{15}$$

$u = a - x \Rightarrow x = a - u; dx = -du$
$x = 0 \Rightarrow u = a$
$x = a \Rightarrow u = 0$

(ii) $\int_0^1 \frac{\sin^{-1} x}{\sqrt{1+x}}dx$

$$= \int_0^1 \left((1+x)^{-1/2} \times \sin^{-1} x\right)dx$$
$$= 2\sqrt{1+x}\sin^{-1} x\Big|_0^1 - \int_0^1 \frac{2\sqrt{1+x}}{\sqrt{1-x^2}}dx$$
$$= \sqrt{2}\pi - 2\int_0^1 \frac{1}{\sqrt{1-x}}dx$$
$$= \sqrt{2}\pi + 2\int_0^1 -(1-x)^{-1/2}\,dx$$
$$= \sqrt{2}\pi + 2\times 2\sqrt{1-x}\Big|_0^1$$
$$= \sqrt{2}\pi - 4$$

(b)(i) $\int_0^1 \frac{(2dt/1+t^2)}{1+(1-t^2/1+t^2)+(2t/1+t^2)}$

$$= \int_0^1 \frac{2dt}{1+t^2+1-t^2+2t}$$
$$= \int_0^1 \frac{2dt}{2+2t}$$
$$= \int_0^1 \frac{dt}{1+t}$$
$$= \left[\ln|1+t|\right]_0^1$$
$$= \ln 2$$

$t = \tan\frac{x}{2}$
$\cos x = \frac{1-t^2}{1+t^2}, \sin x = \frac{2t}{1+t^2}$
$dx = \frac{2dt}{1+t^2}$

4 (b) (ii)

$$I = \int_0^{\frac{\pi}{2}} \frac{xdx}{1+\sin x+\cos x}$$
$$= \int_{\frac{\pi}{2}}^0 \frac{-\left(\frac{\pi}{2}-u\right)du}{1+\sin\left(\frac{\pi}{2}-u\right)+\cos\left(\frac{\pi}{2}-u\right)}$$
$$= \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2}-u\right)du}{1+\cos u+\sin u}$$

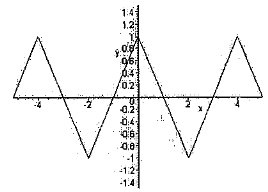
$u = \frac{\pi}{2} - x \Rightarrow x = \frac{\pi}{2} - u;\ dx = -du$
$x = 0 \Rightarrow u = \frac{\pi}{2}; x = \frac{\pi}{2} \Rightarrow u = 0$

$$\therefore 2I = \int_0^{\frac{\pi}{2}} \frac{\frac{\pi}{2}du}{1+\cos u+\sin u} = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{du}{1+\cos u+\sin u} = \frac{\pi}{2}\times\ln 2$$
$$\therefore I = \frac{\pi\ln 2}{4}$$

5 (a)

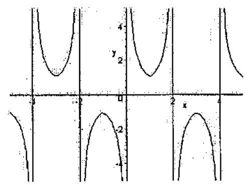
(i) $y = h(x+1)$

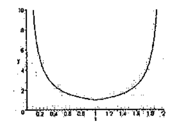
Move the curve 1 unit to the left



(ii) $y = \frac{1}{h(x)}$

Where $h(x) = 0$ there are vertical asymptotes.
Where $h \to 0^+, y \to \infty$
Where $h \to 0^-, y \to -\infty$
Where $h = 1$, the reciprocal is *pointed* ie not smooth.
(See the bottom diagram on the right.)

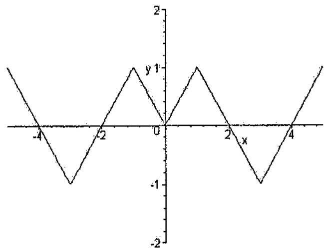




5 (a)

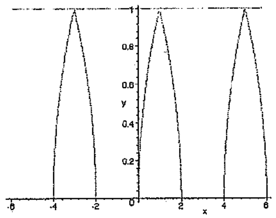
(iii) $y = h(|x|)$.

Erase the LHS of h and then reflect the RHS, so that the result is an even function.

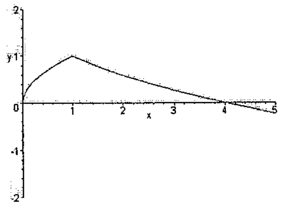


(iv) $y = \sqrt{h(x)}$

First erase the graph where $h < 0$.
Where $0 < h < 1 \Rightarrow \sqrt{h} > h$
Where $y = 1$, the graph is *pointed*, ie not smooth.
Where $y = 0$, vertical tangents.



(v) $y = h(\sqrt{x})$
Domain: $x \geq 0$
Note that $0 \leq x \leq 4 \Rightarrow 0 \leq \sqrt{x} \leq 2$
So $h\left(\sqrt{4}\right) = h(2) = 0$
The graph for $0 \leq x \leq 4$ will be the same y values for h over $0 \leq x \leq 2$.



5 (b)
First draw $9y = x(x-3)^2$
Clearly x intercepts are at $x = 0$ and $x = 3$ **with** $x = 3$ is a double root.

$y = x(x-3)^2/9 \Rightarrow z' = \frac{1}{9}\left(2x(x-3) + (x-3)^2\right) = \frac{(x-3)}{9}(2x + x - 3)$
$\therefore y' = \frac{1}{9}(x-3)(3x-3) = \frac{1}{3}(x-1)(x-3) = \frac{1}{3}(x^2 - 4x + 3)$
$\therefore y'' = \frac{1}{3}(2x-4)$
Stationary points when $y' = 0 \Rightarrow x = 1,3$ ie $\left(1, \frac{4}{9}\right)$ & $(3,0)$
At $x = 1$, $y'' < 0 \Rightarrow \left(1, \frac{4}{9}\right)$ is a maximum.

The graph in Fig I is the graph of z. The horizontal line is the line $y = 1$.

So with $y = \frac{1}{3}\sqrt{x(x-3)^2}$, the maximum turning point remains the same except it is now $\left(1, \frac{2}{3}\right)$.

Any part of the graph in Fig I below the $x-$ axis is not defined for the square root. Where $0 < y < 1$ we get $\sqrt{y} > y$ and where $y > 1$ we get $\sqrt{y} < y$.

The $x = 0$ intercept will have a vertical tangent, the $x = 3$ intercept is not smooth. This is shown in Fig 2.

We need to draw $y = \pm\frac{1}{3}\sqrt{x(x-3)^2}$: the $\pm$ means that the top part of the graph will be reflected.
The final graph is Fig 3. With turning points $\left(1, \pm\frac{2}{3}\right)$

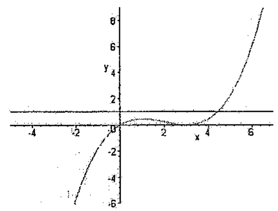


Fig I

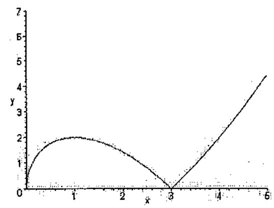


Fig II

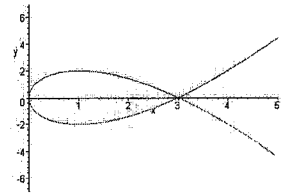
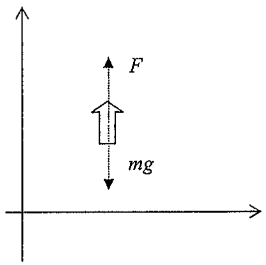
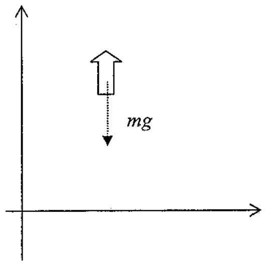


Fig III

Question 6

(i)

$0 \le t \le 2$	$t > 2$
	
$F = g(2-t)$ $\therefore ma = F - mg = g(2-t) - mg$ $\therefore a = \dfrac{g(2-t)}{0 \cdot 2} - g$ $\therefore a = 5g(2-t) - g = g(9-5t)$	After 2 seconds the only force acting is gravity $\therefore a = -g$

b(ii) (α) For $0 \le t \le 2$

$$\frac{dv}{dt} = g(9-5t)$$

$$v = 9gt - \frac{5}{2}gt^2 + C_1$$

$t=0$, $v=0$, $C_1=0$ $\}$ $\therefore v = gt\left[9 - \frac{5}{2}t\right]$ & $x = \frac{9gt^2}{2} - \frac{5}{6}gt^3 + C_3$ (*)

Max speed when $a = g(9-5t) = 0$

i.e. when $t = 9/5$

$\therefore$ Max speed $v = \frac{81g}{10}$ m/s

(β) When $t=2$, $v = 8g$

For $t \ge 2$, $a = -g$

$$\frac{dv}{dt} = -g$$

$$v = -gt + C_2$$

$C_2 = 10g$ i.e. $v = 10g - gt$ — (**)

For max height $v=0 \Rightarrow t=10$

From (*) when $t=2$, $x = \frac{34g}{3}$

From (**) $x = 10gt - \frac{gt^2}{2} + C_4$

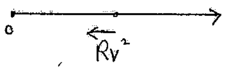
When $t=2$, $x = \frac{34g}{3} \Rightarrow C_4 = -\frac{20g}{3}$

$$\therefore x = -\frac{gt^2}{2} + 10gt - \frac{20g}{3}$$

When $t=10$, $x = \frac{130g}{3}$ m

Question 7

(i)



$m\ddot{x} = -Rv^2$ 1

since $m=1 \Rightarrow \ddot{x} = -Rv^2$

(ii) $\frac{dv}{dt} = -Rv^2$

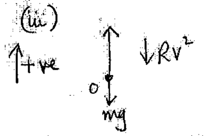
$\int_u^V \frac{dv}{v^2} = -R\int_0^t dt$

$\left[-\frac{1}{v}\right]_u^V = -Rt$ 3

$-\frac{1}{V} + \frac{1}{u} = -Rt$

$t = \frac{1}{R}\left(\frac{1}{V} - \frac{1}{u}\right)$

(iii)



1

$m\ddot{y} = -mg - Rv^2$

$m=1 \Rightarrow \ddot{y} = -(g + Rv^2)$

(iv) $\ddot{y} = -(g + Rv^2)$

Since $g = Ra^2$

$\Rightarrow \ddot{y} = -(Ra^2 + Rv^2)$

i.e. $\ddot{y} = -R(a^2 + v^2)$

$\frac{dv}{dt} = -R(a^2+v^2)$

$\int_u^0 \frac{dv}{a^2+v^2} = -R\int_0^t dt$

$\left[\frac{1}{a}\tan^{-1}\frac{v}{a}\right]_u^0 = -Rt$

$0 - \frac{1}{a}\tan^{-1}\frac{u}{a} = -Rt$

$\Rightarrow t = \frac{1}{Ra}\tan^{-1}\frac{u}{a}$ 3½

At this time 2nd particle is at rest.

Subst. into (ii) with $v=V$ 1½

$\Rightarrow \frac{1}{R}\left[\frac{1}{V} - \frac{1}{u}\right] = \frac{1}{Ra}\tan^{-1}\frac{u}{a}$

$\therefore \frac{1}{V} = \frac{1}{u} + \frac{1}{a}\tan^{-1}\frac{u}{a}$

where V = vel. of 1st particle when the second is momentarily at rest