SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2003

HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes.
- Working time – 2 hours.
- Write using black or blue pen.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may **NOT** be awarded for messy or badly arranged work.
- Hand in your answer booklets in 3 sections. Section A (Questions 1,2 and 3); Section B (Questions 4 and 5) and Section C (Questions 6 and 7).
- Start each **NEW** section in a separate answer booklet.

Total Marks - 84 Marks

- Attempt Sections A - C
- All questions are of equal value.

Examiner: *B. Opferkuch*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

Total marks-84.
Attempt Questions 1-7.
All questions are of equal value.

Answer each Section in a SEPARATE writing booklet. Extra writing booklets are available

Section A Use a SEPARATE writing booklet

Question 1 (12 marks) Marks

(a) Differentiate

(i) $x\sin 3x$ 1

(ii) e^{1-x^2} 1

(b) Find the acute angle between the lines $3y = 2x + 8$ and $5x - y - 9 = 0$. 2

(c) Evaluate

(i) $\displaystyle\int_0^2 \frac{dx}{4+x^2}$ 2

(ii) $\displaystyle\int_0^1 \frac{x^2}{2+x^3}\,dx$ 2

(d) The letters of the word INTEGRAL are arranged in a row. 2

If one of these arrangements is selected at random, what is the probability that the vowels are in the same position?

(e) Solve the inequality $\dfrac{\theta-4}{\theta} > 0$. 2

Section A continued.

Question 2. (12 marks)

Marks

(a) If α, β and γ are the roots of the equation $2x^3-5x^2-3x+1=0$, evaluate

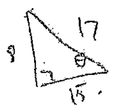
(i) $\alpha+\beta+\gamma$ and $\alpha\beta\gamma$. 1

(ii) $\alpha^2+\beta^2+\gamma^2$. 2

(b) Use the substitution $u=x^2+4$ to find the exact value of $\displaystyle\int_0^{2\sqrt{3}} \frac{x}{\sqrt{x^2+4}}\,dx$. 3

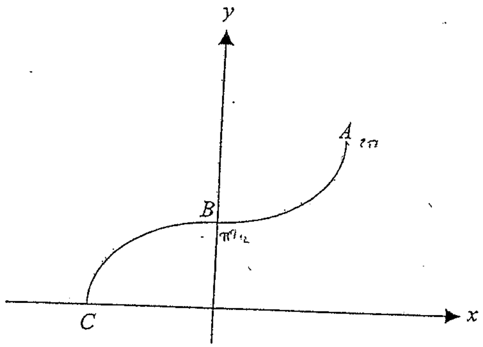
(c) Determine the exact value of $\cos\left(\tan^{-1}\left(\frac{8}{15}\right)\right)$. 2

$\tan\theta = \frac{8}{15}$



$\cos\theta =$

(d)



The diagram shows the graph of $y=\pi+2\sin^{-1}3x$.

(i) Find the coordinates of A and C. 2

(ii) Find the gradient of the tangent at B. 2

Section A continued.

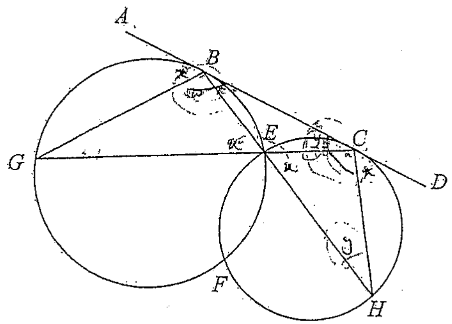
Question 3. (12 marks)

Marks

(a) A function is defined as $f(x)=1+e^{2x}$.
Find the inverse function $f^{-1}(x)$ and state the domain and range. 2

(b) Consider the quadratic expression $Q(x)=(5k-4)x^2-6x+(6k+3)$, where k is a constant.
Find the values of k for which $Q(x)=0$ has rational roots. 3

(c)



$ABCD$ is a common tangent to the two circles.

(i) Prove that $\angle ABG = \angle DCH$. 2

(ii) Prove that $\Delta BCG \,|||\, \Delta BCH$. 2

(d) Consider the series $2^N+2^{N-1}+2^{N-2}+\ldots..+2^{1-N}+2^{-N}$, where N is a positive integer.

(i) Find an expression in terms of N for the number of terms in the series. 2

(ii) Find an expression in terms of N for the sum of the series. 1

Section B Use a SEPARATE writing booklet.

Question 4. (12 marks)

Marks

(a) Consider the function $f(\theta)=\dfrac{\sin\theta+\sin\frac{\theta}{2}}{1+\cos\theta+\cos\frac{\theta}{2}}$

(i) Show that $f(\theta)=t$ where $t=\tan\frac{\theta}{2}$. **3**

(ii) Write down the general solution of $f(\theta)=1$. **1**

(a) A certain particle moves along the straight line in accordance with the law: $t=2x^2-5x+3$, where x is measured in centimetres and t in seconds.

Initially, the particle is $1\cdot5$ centimetres to the right of the origin O, and moving away from O.

(i) Show that the velocity, v cms^{-1}, is given by **1**

$$v=\frac{1}{4x-5}$$

(ii) Find an expression for the acceleration, a cms^{-2}, of the particle, in terms of x. **2**

(iii) Find the velocity and acceleration of the particle when: **3**

(α) $x=2$ cm

(β) $t=6$ seconds

(iv) Describe carefully in words the motion of the particle. **2**

Section B continued.

Question 5.

Marks

a) (i) Prove the identity $\dfrac{\cos y-\cos(y+2\alpha)}{2\sin\alpha}=\sin(y+\alpha)$ **2**

(ii) Hence prove by mathematical induction that for positive integers n, **4**

$$\sin\alpha+\sin3\alpha+\sin5\alpha+\ldots+\sin(2n-1)\alpha=\frac{1-\cos2n\alpha}{2\sin\alpha}.$$

(b) (i) Show that the curve $y=\dfrac{x^3+4}{x^2}$ has one stationary point and no points of inflexion. **2**

(ii) Write down the equation(s) of any asymptotes. **1**

(iii) Sketch the curve. **1**

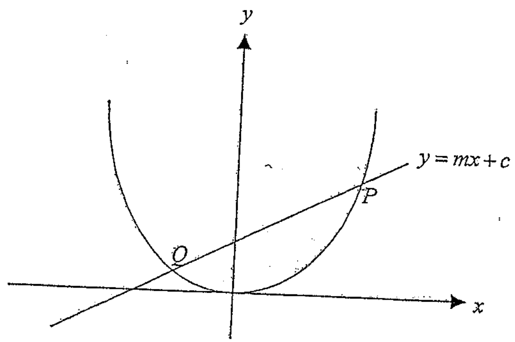
(iv) Hence, use the graph to find the values of k for which the equation $x^3-kx^2+4=0$ has 3 real roots. **2**

Section C Use a SEPARATE writing booklet.

Question 6. (12 marks)

Marks

The straight line $y = mx + c$ meets the parabola $x = 2t,\ y = t^2$ in real distinct points P and Q which correspond respectively to the values $t = p$ and $t = q$.



(i) Prove that $pq = -c$. 2

(ii) Prove that $p^2 + q^2 = 4m^2 + 2c$. 2

(iii) Show that the equation of the normal to the parabola at P is $x + py = 2p + p^3$. 2

(iv) The point N is the point of intersection of the normals to the parabola at P and Q. 2
Show that the coordinates at N are $\left(-pq(p+q),\ \left(2 + p^2 + pq + q^2\right)\right)$

(v) If the chord PQ is free to move while maintaining a fixed gradient.

(α) Show that the locus of N is a straight line. 2

(β) Hence, or otherwise, show that this straight line is a normal to the parabola. 2

Section C continued.

Question 7. (12 marks)

Marks

(a) When the polynomial $P(x)$ is divided by $(x+4)$ the remainder is 5 and when $P(x)$ is divided by $(x-1)$ the remainder is 9. Find the remainder when $P(x)$ is divided by $(x-1)(x+4)$. 3

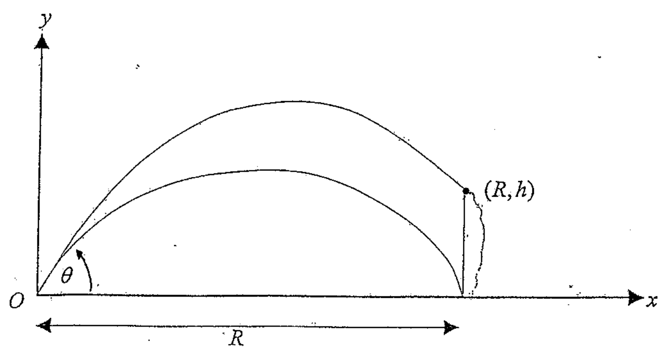
(b) A projectile is fired from a point on horizontal ground with initial speed V ms^{-1} and angle of projection θ.
The cartesian equation of the path is given by

$$y = x\tan\theta - \frac{gx^2}{2V^2\cos^2\theta}$$

where x and y are the horizontal and vertical displacements of the particle from O, the point of projection.
The acceleration due to gravity is g and air resistance has been neglected.

(i) Use the given equation to show that the maximum range R on the horizontal plane is given by $R = \dfrac{V^2}{g}$. 2

(ii) Show that to hit a target h metres above the ground at the same horizontal distance R using the same angle of projection θ, the speed of projection must be increased to $\dfrac{V^2}{\sqrt{V^2 - gh}}$. 4

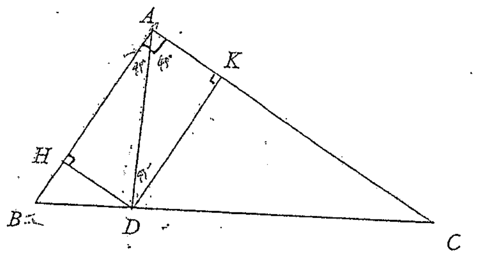


Section C continued.

Question 7.

Marks

(c)



In the triangle ABC, $\angle BAC = 90°$. AD bisects $\angle BAC$. $DH \perp AB$ and $DK \perp AC$.

Copy the diagram.

(i) Show that $\dfrac{AD}{DH} = \sqrt{2}$. 1

(ii) By considering the areas of the triangles or otherwise, show that $\dfrac{\sqrt{2}}{AD} = \dfrac{1}{AB} + \dfrac{1}{AC}$. 2

THIS IS THE END OF THE PAPER

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V. Good effort

QUESTION 1.

(a) (i) $\frac{d}{dx}(x\sin 3x) = x.3\cos 3x + \sin 3x.$ ✓

(ii) $\frac{d}{dx}\left[e^{1-x^2}\right] = -2xe^{1-x^2}$ ✓

(b). $y = \frac{2}{3}x + \frac{8}{3}$ $y - 5x - 9 = 0.$

$\therefore m_1 = \frac{2}{3}$ $\therefore m_2 = 5.$ ✓

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$

$= \left|\frac{\frac{2}{3} - 5}{1 + \frac{10}{3}}\right| = \left|\frac{-\frac{13}{3}}{\frac{13}{3}}\right| = 1$

$\therefore \theta = \tan^{-1}(1) = \frac{\pi}{4}.$ ✓

(c). (i). $\left[\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2 = \tan^{-1}1 = \frac{\pi}{4}.$ ✓

(ii). $\frac{1}{3}\int_0^1 3x^2(2+x^3)^{-1}\,dx. = \frac{1}{3}\left[\ln|2+x^3|\right]_0^1$ ✓

$= \frac{1}{3}\left[\ln 3 - \ln 2\right]$

$= \frac{1}{3}\ln\frac{3}{2}.$ ✓

(d). $\frac{5!}{8!} = \frac{1}{8\times6\times7} = \frac{1}{336}$ ✓

12

(e). $\frac{\theta - 4}{\theta} > 0.$ $*\ \theta \neq 0.$

Case 1: $\theta - 4 > 0 \ \cap\ \theta > 0.$
$\theta > 4 \ \cap\ \theta > 0$
$\therefore \underline{\theta > 4}$ ✓

Case 2: $\theta - 4 < 0 \ \cap\ \theta < 0$
$\theta < 4 \ \cap\ \theta < 0.$
$\underline{\theta < 0.}$ ✓

QUESTION 2.

(a). (i). $\alpha + \beta + \gamma = -\frac{b}{a} = \frac{5}{2}.$ ✓

$\alpha\beta\gamma = \frac{-d}{a} = -\frac{1}{2}.$ ✓

(ii). $(\alpha+\beta+\gamma)^2 = 2(\alpha\beta + \alpha\gamma + \beta\gamma) + \alpha^2 + \beta^2 + \gamma^2$

$\therefore \alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\Sigma\alpha\beta).$ ✓

$= \left(\frac{5}{2}\right)^2 - 2\left(-\frac{3}{2}\right).$

$= \frac{25}{4} + 3 = \frac{37}{4}.$ ✓

(b). let $u = x^2 + 4.$ when $x = 0;\ u = 4.$
$x = 2\sqrt{3};\ u = 16.$

$\frac{du}{dx} = 2x.$

$dx = \frac{du}{2x}$

$\Rightarrow \int_4^{16} \frac{x}{\sqrt{u}} \times \frac{du}{2x} = \frac{1}{2}\int_4^{16} u^{-1/2}\,du.$ ✓

$= \frac{1}{2}\left[\frac{u^{1/2}}{1/2}\right]_4^{16} = 4 - 2 = \underline{2}.$ ✓

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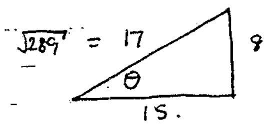
(c). $\cos\left[\tan^{-1}\left(\frac{8}{15}\right)\right] \equiv \cos\theta$

$\tan\theta = \frac{8}{15}.$ $\Rightarrow$



✓

$\therefore \cos\theta = \frac{15}{17}.$ ✓

(d). Domain: $-1 \leq 3x \leq 1.$
$-\frac{1}{3} \leq x \leq \frac{1}{3}.$

Range: $-\pi \leq 2\sin^{-1}3x \leq \pi.$
$0 \leq \pi + 2\sin^{-1}3x \leq 2\pi \Rightarrow 0 \leq y \leq 2\pi$

$\Rightarrow A\left(\frac{1}{3}, 2\pi\right)$ ✓ $C\left(-\frac{1}{3}, 0\right).$ ✓

(ii) $B(0,\pi)$. $y = \pi + 2\sin^{-1}\left(\frac{x}{\frac{1}{3}}\right)$.

$$\frac{dy}{dx} = \frac{2}{\sqrt{(\frac{1}{3})^2 - x^2}} = \frac{6}{\sqrt{1-9x^2}}$$ ✓

at B; $m = \frac{6}{\sqrt{1}} = 6$.

$\Rightarrow y - \pi = 6(x) \Rightarrow y = 6x + \pi$. ✓

QUESTION 3.

(a) $f(x) = 1 + e^{2x}$. Domain; all real x.
Range; $y > 1$.

$f[f^{-1}(x)] = 1 + e^{2f^{-1}(x)} = x$. ✓

$e^{2f^{-1}(x)} = x - 1$.

$f^{-1}(x) = \frac{\ln|x-1|}{2} = \ln\sqrt{x-1}$ ✓

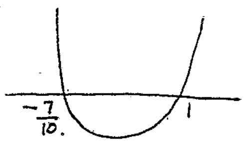
$\therefore$ D: $x > 1$
R: all real y. ✓

(b). $Q(x) = (5k-4)x^2 - 6x + (6k+3) = 0$.
$\Delta > 0$.

$$\begin{aligned}\Delta &= 36 - 4(5k-4)(6k+3).\\ &= 36 - 4(30k^2 - 9k - 12).\\ &= 36 - 120k^2 + 36k + 48\\ &= \cancel{36} - 120k^2 + 36k + 84 > 0.\end{aligned}$$

$10k^2 - 3k - 7 < 0$. ✓

$(10k+7)(k-1) < 0$.



$\Rightarrow -\frac{7}{10} < k < 1$. ✓

(c). In circle BEFC: let $\angle ABC = \alpha$.
$\angle ABC = \angle BEC = \alpha$ ($\angle$ between chord BC & tangent AO is equal to the $\angle$ in the alternate segment.).

$\angle BEC = CEH = \alpha$ (vert opp $\angle$'s are =). ✓

$\therefore \angle CEH = \angle DCH = \alpha$ (alt segment theorem for circle CEFH). ✓

$\Rightarrow \angle ABC = \angle DCH$.

(ii) In ΔBCG & ΔBCH:
let $\angle CBH = \beta$.
$\therefore \angle BGC = \beta$ (alternate segment theorem). ✓
From ΔCBE; $\angle DCG = \alpha - \beta$ (exterior $\angle$ of Δ = sum of opp interior)
$\therefore \angle CHB = \alpha - \beta$ (alternate segment theorem).
Since $\angle CBH = \angle BGC$
& $\angle BCG = \angle CHB$. ✓
$\Delta BCG \parallel\!\mid \Delta BCH$ (equiangular).

(e) $2^N + 2^N.2^{-1} + 2^N.2^{-2} + \ldots + 2.2^{-N} + 2^{-N}$

(i) $r = \frac{1}{2}$

$\therefore T_n = ar^{n-1}$

$2^{-N} = 2^N(\frac{1}{2})^{n-1}$

$\frac{2^{-N}}{2^N} = 2^{-n+1} \Rightarrow 2^{-2N} = 2^{-n+1}$

$\therefore 2N = n - 1$

$\therefore 2N + 1 = n$

(ii) $S_n = \frac{a(1-r^n)}{1-r} = \frac{2^N(1-(\frac{1}{2})^{2N+1})}{1-\frac{1}{2}}$

$= \frac{2^N}{\frac{1}{2}}(1 - 2^{-2N-1})$

$= 2^{N+1}(1 - 2^{-2N-1})$

$= 2^{N+1} - 2^{-N}$

$= 2^{-N}(2^{2N+1} - 1)$

$= \frac{1}{2^N}(2^{2N+1} - 1)$

(a). $f(\theta) = \dfrac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}{\cancel{1+}\ 2\cos^2\frac{\theta}{2} \cancel{-1} + \cos\frac{\theta}{2}}$

$= \dfrac{\sin\frac{\theta}{2}\left[2\cos\frac{\theta}{2} + 1\right]}{\cos\frac{\theta}{2}\left[2\cos\frac{\theta}{2} + 1\right]} = \tan\frac{\theta}{2} \doteq t.$ ✓

(ii). $\tan\frac{\theta}{2} = 1.$ ✓

$\frac{\theta}{2} = \pi n + \frac{\pi}{4} \Rightarrow \theta = 2\pi n + \frac{\pi}{2}$ ✓ for any integer n.

(b). $t = 2x^2 - 5x + 3.$

$t=0, \quad x = 1.5.$

(i) $V = \dfrac{dx}{dt} \Rightarrow \dfrac{dt}{dx} = 4x - 5.$

$\dfrac{dx}{dt} = \dfrac{1}{4x-5}.$ ✓

(ii) $a = \dfrac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{1}{2}\dfrac{d}{dx}\left(\dfrac{1}{(4x-5)^2}\right) = \frac{1}{2}\dfrac{d}{dx}(4x-5)^{-2}$

$= \frac{1}{2} \times -2(4x-5)^{-3} \times 4.$ ✓

$= \dfrac{-4}{(4x-5)^3}$ ✓

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(iii). (α) at $x=2$; $v = \frac{1}{8-5} = \frac{1}{3}$ cm/s. $a = \dfrac{-4}{(8-5)^3} = \dfrac{-4}{27}$ cm/s^2. ✓

(β) $a = \dfrac{dv}{dt}$ OR; $6 = 2x^2 - 5x + 3.$

$2x^2 - 5x - 3 = 0.$

$(2x+1)(x-3) = 0$

$x = 3$ ✓ only.

$\Rightarrow a = \dfrac{-4}{(12-5)^3} = \dfrac{-4}{343}$ cm/s^2 ✓ $\quad v = \frac{1}{12-5} = \frac{1}{7}$ cm/s.

?? (iv). As $x \to \infty$, $v \to 0$, the particle is slowing down to a stop.

QUESTION 5.

(a) (i). LHS $= \dfrac{\cos y - (\cos y\cos 2\alpha - \sin y \sin 2\alpha)}{2\sin\alpha}.$

$= \dfrac{\cos y - \cos y \cos 2\alpha + \sin y \sin 2\alpha}{2\sin\alpha}$ ✓

$= \dfrac{\cos y - \cos y(1 - 2\sin^2\alpha) + 2\sin\alpha\cos\alpha\sin y}{2\sin\alpha}.$

$= \dfrac{\cancel{\cos y} - \cancel{\cos y} + \cancel{2}\sin^{\cancel{2}}\alpha\cos y + \cancel{2\sin\alpha}\cos\alpha\sin y}{\cancel{2\sin\alpha}}$ ✓

$= \sin\alpha\cos y + \cos\alpha\sin y = \sin(\alpha+y) = \sin(y+\alpha)$

$=$ RHS

(ii). STEP 1.; Test for $n=1$.

~~LHS: = $\sin(\alpha)$~~ $\quad$ RHS $= \dfrac{1-\cos 2\alpha}{2\sin\alpha} = \dfrac{1 - 1 - 2\sin^2\alpha}{2\sin\alpha}$ ✓

$= \sin\alpha = $ LHS.

$\Rightarrow$ Result is true for $n=1$.

STEP 2; Assume the result is true for $n=k$ where $1 < k < n$.

i.e

$\sin\alpha + \sin 3\alpha + \sin 5\alpha \ldots \sin(2k-1)\alpha = \dfrac{1-\cos 2k\alpha}{2\sin\alpha}$

STEP 3; Show the result is true for $n = k+1$.

rtp: $\sin\alpha + \sin 3\alpha \ldots \sin(2k+1)\alpha = \dfrac{1 - \cos(2k\alpha + 2\alpha)}{2\sin\alpha}.$

LHS $=$

$\dfrac{1-\cos 2k\alpha}{2\sin\alpha} + \sin(2k\alpha + \alpha)$ from step 2.

$= \dfrac{1 - \cos 2k\alpha + 2\sin\alpha\sin(2k\alpha+\alpha)}{2\sin\alpha}$ ✓ let $y = 2k\alpha$.

$= \dfrac{1 - \cancel{\cos y} + \cancel{2\sin\alpha} \times \dfrac{\cancel{\cos y} - \cos(y + 2\alpha)}{\cancel{2\sin\alpha}}}{\quad}$

$$= \frac{1-\cos(y+2\alpha)}{2\sin\alpha} = \frac{1-\cos(2k\alpha+2\alpha)}{2\sin\alpha} = RHS.$$

∴ The result is true for $n=k+1$

STEP 4;

Hence, if the result is true for $n=k$ & $n=k+1$, then it is true for all +ve integers n by PMI.

(b) (i). $y = \frac{x^3+4}{x^2}$ Quicker + easier to simplify $y = x + \frac{4}{x^2}$

$y' = 1 - \frac{8}{x^3}$

$$\frac{dy}{dx} = \cancel{\frac{x^2(3x^2+4) - 2x(x^3+4)}{x^4}} = \frac{x^2(3x^2) - 2x(x^3+4)}{x^4}$$

$$\cancel{= \frac{3x^4 + 4x^2 - 2x^4 - 8x}{x^4}} \qquad = \frac{3x^4 - 2x^4 - 8x}{x^4}$$

$$\cancel{= \frac{x^4 + 4x^2 - 8x}{x^4} = 0} \qquad = \frac{x^4 - 8x}{x^4} = 0.$$

$$\cancel{\Rightarrow x(x^3 + 4x - 8) = 0} \qquad x(x^3-8) = 0$$

$x \neq 0$ (from denominator) ; $x^3 = 8$. $\underline{x = 2}$ 1 stat pt.

$$\frac{d^2y}{dx^2} = \frac{x^4(4x^3-8) - 4x^3(x^4-8x)}{x^8}$$

(2,3) is a min t.p.

$$= \frac{4\cancel{x^7} - 8\cancel{x^4} - 4\cancel{x^7} + 32\cancel{x^4}}{x^{\cancel{8}4}}$$

$$= \frac{24}{x^4} > 0 \text{ for all } x.$$

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∴ NO PTS OF INFLEXION

concave up.

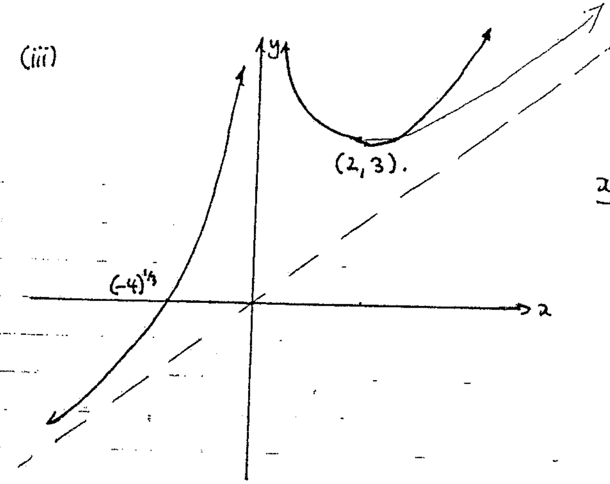
(ii) $\underline{x = 0}$ (vertical).

$y = \frac{1 + \frac{4}{x^3}}{\frac{1}{x^3}}$ as $x \to \infty$; $y \to \infty$.

$y \to x$

$y = x + \frac{4}{x^2}$

(iii)



at $x = 2$; $y = \frac{12}{4} = 3$.

$\frac{x^3+4}{x^4} = 0$.

$x^3 = -4$.

$x = \sqrt[3]{-4}$

(iv). $x^3 + 4 = kx^2$.

$k = \frac{x^3+4}{x^2}$ $\Rightarrow$ $\underline{k > 3}$

QUESTION 6

$y = mx + c$. (A)

$x = 2t$, $y = t^2$ $\Rightarrow y = \left(\frac{x}{2}\right)^2 \Rightarrow \underline{x^2 = 4y}$ (B).

(i) $P(2p, p^2)$ $Q(2q, q^2)$.

Sub both in (A):

$p^2 = 2pm + c$ (1)

$q^2 = 2qm + c$ (2).

(1) + (2): $p^2 + q^2 = 2m(p+q) + 2c$.

$m_{PQ} = \frac{p^2-q^2}{2(p-q)} = \frac{p+q}{2} \Rightarrow p + q = 2m$

$\Rightarrow \underline{p^2 + q^2 = 2m(2m) + 2c = 4m^2 + 2c.}$

(ii) CHORD PQ:

$m_{PQ} = \frac{p+q}{2}$. $\Rightarrow y - p^2 = \frac{p+q}{2}(x - 2p)$.

$2y - \cancel{2p^2} = (p+q)x - \cancel{2p^2} - 2pq$

Now, $y = \frac{(p+q)}{2}x - pq \equiv mx + c$. $\Rightarrow pq = -c$

? $\Rightarrow \frac{1}{c} \times -\frac{c}{q} = -\frac{1}{q}$.

At Q, $m_T = q$ $\therefore$ $q \times -\frac{1}{q} = -1$

suggesting the locus is normal to the parabola.

QUESTION 7

(a) $P(x) = (x-1)(x+4)Q(x) + ax + b$.

- the degree of the remainder is 1 less the divisor.

$P(-4) = -4a + b = 5$ ①

$P(1) = a + b = 9$ ② ✓

② − ①: $5a = 4$

$a = \cancel{\frac{5}{4}}\ \frac{4}{5}$ ✗

Sub in ②: $-4 \times \frac{5}{4} + b = 5$.

$b = \cancel{10}\ 8\frac{1}{5}$ ✓

(b). $y = x\tan\theta - \frac{gx^2}{2V^2\cos^2\theta}$.

(i). at max range: $y = 0$ $\theta = \frac{\pi}{4}$. ✓

$0 = x - \frac{gx^2}{2V^2 \left(\frac{1}{\sqrt{2}}\right)^2}$

$x\left(1 - \frac{gx}{V^2}\right) = 0$.

$x = R = \frac{V^2}{g}$. ✓

(ii). Sub in (R, h). $\theta = \frac{\pi}{4}$.

$h = R - \frac{gR^2}{V^2}$. ✓

Solve for V;

$\frac{gR^2}{V^2} = R - h$.

$\frac{1}{\ \ } \qquad gR^2$

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(iii) m_N; $y' = \frac{x}{2}$ $\therefore m_T = \frac{2p}{2} = p$.

$\therefore m_N = -\frac{1}{p}$. ✓

$y - p^2 = -\frac{1}{p}(x - 2p)$

$py - p^3 = -x + 2p$. ✓

N_p: $x + py = 2p + p^3$ ✓ ③.

(iv). Similarly at Q;

N_q: $x + qy = 2q + q^3$ ④.

③ − ④; $y\cancel{(p-q)} = 2\cancel{(p-q)} + \cancel{(p-q)}(p^2+pq+q^2)$

$y = 2 + p^2 + pq + q^2$ ✓

Sub in ③: $x + \cancel{2p} + \cancel{p^3} + p^2q + pq^2 = \cancel{2p} + \cancel{p^3}$

$x + pq(p+q) = 0$.

$x = -pq(p+q)$ ✓

$\Rightarrow N\left(-pq(p+q),\ (2 + p^2 + pq + q^2)\right)$.

(v). (α) $p + q = \frac{x}{-pq} = +\frac{x}{c}$ from (i).

~~$(p+q)^2 = p^2 + q^2 + 2pq = +\frac{x^2}{c^2}$~~

~~$p^2 + q^2 = +\frac{x}{c} - 2pq$.~~ ✓

~~$= \frac{x^2}{c^2} + 2c$.~~

$\Rightarrow y = 2 + \frac{x^2}{c^2} + 2c - c$.

$y = \frac{x}{\cancel{c}} + (c+2)$. $\equiv mx + b$.

Since m is fixed gradient i.e. a constant, then $p^2 + q^2 = 4m^2 + 2c = k$

$\therefore y = 2 + p^2 + q^2 + pq$

$= 2 + k + \frac{k}{c}$

$= \frac{x}{c} + k + 2$

$= \boxed{mx + b}$ is a st. li

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(β). $y = \frac{x^2}{4}$ $\Rightarrow$ $m = \frac{x}{2}$.

T: $y - p^2 = p(x - 2p)$. ✓

$y - p^2 = px - 2p^2$.

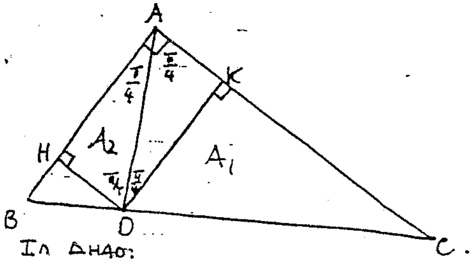
$y = px - p^2$. $m = p$ $-c$

Sub in $R = \frac{V^2}{g}$.

$$V^2 = \frac{g \times \frac{V^4}{g^2}}{\frac{V^2}{g} - h} = \frac{\frac{V^4}{\cancel{g}}}{\frac{V^2 - gh}{\cancel{g}}}$$

$$\Rightarrow V = \frac{V^2}{\sqrt{V^2 - gh}} \qquad V > 0.$$

(c).



In ΔHAD:

(i). $\angle HAD = \frac{\pi}{4}$ (given; AD bisects $\angle BAC$).

$\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{DH}{AD} \Rightarrow \frac{AD}{DH} = \sqrt{2}$.

(ii). In ΔADC;

$A_1 = AD.BC \sin \frac{\pi}{4} = \frac{AD.AC}{\sqrt{2}}$.

In ΔABD;

$A_2 = \frac{AD.AB}{\sqrt{2}}$

In ΔABC.

$\Sigma A = \frac{AB.AC}{2} = A_1 + A_2$.

$$\frac{AB.AC}{2} = \frac{AD.AC}{\sqrt{2}} + \frac{AB.AD}{\sqrt{2}}$$

$$\frac{1}{2}(AB.AC) = \frac{AD}{\sqrt{2}}(AD + AB)$$

$$\Rightarrow \frac{\sqrt{2}}{AD} = \frac{1}{AB} + \frac{1}{AC}. \qquad \text{(cross multiplication).}$$

i) . $\angle AHD = 90°$ (given)

$\angle HAD = 90/2 = 45°$ (AD bisects $\angle A$)

(i) $\therefore \angle ADH = 45°$ ($\angle$ sum Δ)

$\cos \angle ADH = \frac{HD}{AD} = \frac{1}{\sqrt{2}}$

$\therefore \frac{AD}{DH} = \sqrt{2}$

(ii) Area ~~ADB = $\frac{ADBD}{2}$~~

~~ABC = $\frac{ACAB}{2}$~~

~~ADC = $\frac{ADCD}{2}$~~

~~$\Delta ABC = \Delta ABD + \Delta ACD$~~

~~$\therefore \frac{ACAB}{2} = \frac{ADBD}{2} + \frac{ADCD}{2}$~~

(ii) In ΔADC; $A_{\Delta ADC} = \frac{1}{2} AD \times AC \sin \frac{\pi}{4}$

$= \frac{1}{2} \frac{AD.AC}{\sqrt{2}}$

In ΔABD; $A_{\Delta ABD} = \frac{1}{2} \frac{AD.AB}{\sqrt{2}}$

$\therefore$ in ΔABC

$A_{\Delta ABC} = (A_{\Delta ADC} + A_{\Delta ABC})$

$\frac{1}{2} \times AB \times AC = \frac{1}{2}\left(\frac{AD.AC}{\sqrt{2}} + \frac{AD.AB}{\sqrt{2}}\right)$

$= \frac{1}{\sqrt{2}} AD (DC + AB)$

Cross-multiplication gives.

$\frac{\sqrt{2}}{AD} = \frac{(AC + AB)}{AB \times AC}$

$\frac{\sqrt{2}}{AD} = \frac{AC}{AB.AC} + \frac{AB}{AB.AC}$

$= \frac{1}{AB} + \frac{1}{AC}$ as req^{d}.