SYDNEY BOYS HIGH SCHOOL

MOORE PARK, SURRY HILLS

2005

YEAR 12

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Working time – 2 Hours.
- Reading Time – 5 minutes.
- Write using black or blue pen.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may not be awarded for messy or badly arranged work
- Hand in your answer booklets in 4 sections. Section A (Questions 1 and 2), Section B (Questions 3 and 4), Section C (Questions 5 and 6) and Section D (Question 7)

Total Marks - 84

- Attempt questions 1 – 7
- All QUESTIONS are of equal value.

Examiner: *A. Fuller*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right) x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ x > 0$

Total marks - 84
Attempt Questions 1 - 7
All questions are of equal value

Answer each SECTION in a SEPARATE writing booklet.

Section A

Marks

Question 1 (12 marks)

(a) Simplify $\dfrac{3^n}{3^{n+1}-3^n}$ **1**

(b) Evaluate $\lim\limits_{x\to 0}\dfrac{\sin 5x}{4x}$ **1**

(c) The remainder when $x^3-3x^2+px-14$ is divided by $x-3$ is 1. Find the value of p. **2**

(d) Given that $\log_a 2 = x$, find $\log_a(2a)$ in terms of x. **2**

(e) Find the coordinates of the point P that divides the interval from A (-1,5) to B (6,-4) externally in the ratio 3 : 2 . **2**

(f) Find, to the nearest minute, the acute angle between The lines $3x+2y-5=0$ and $x-5y+7=0$. gradients. **2**

(g) Solve the inequality $\dfrac{2}{x}\le 1$ **2**

Question 2 (12 marks)

(a) Differentiate with respect to x

(i) $y=\tan^3(5x+4)$ **2**

(ii) $y=\ln\left(\dfrac{2x+3}{3x+4}\right)$ **2**

(iii) $y=\cos\left(e^{1-5x}\right)$ **2**

(b) 30 girls, including Miss Australia, enter a Miss World Competition. The first six places are announced.

(i) How many different announcements are possible? **1**

(ii) How many different announcements are possible if Miss Australia is assured a place in the first six? **2**

(c) If $f(x)=\tan^{-1}(2x)$ evaluate:

(i) $f(\tfrac{1}{2})$ **1**

(ii) $f'(\tfrac{1}{2})$ **2**

End of Section

Section B (Use a SEPARATE writing booklet)

Marks

Question 3 (12 marks)

(a) (i) State the natural domain and the corresponding range of $y = 3\cos^{-1}(x-2)$ — 2

(ii) Hence, or otherwise sketch $y = 3\cos^{-1}(x-2)$ — 1

(b) Find $\int x\sqrt{16+x^2}\,dx$ using the substitution $u = 16 + x^2$ — 2

(c) Find the general solution of $\sin 2\theta = \sqrt{3}\cos 2\theta$ — 2

(d) The roots of the equation $4x^3 + 6x^2 + c = 0$, where c is a non-zero constant, are α, β, and $\alpha\beta$. — 5

(i) Show that $\alpha\beta \neq 0$.

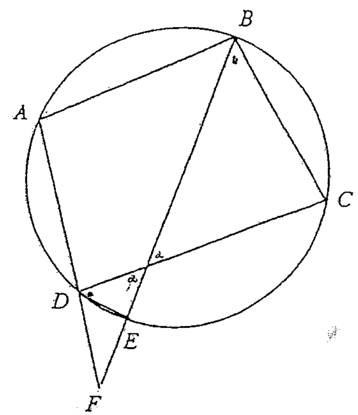
(ii) Show that $\alpha\beta + \alpha^2\beta + \alpha\beta^2 = 0$ and deduce the value of $\alpha + \beta$.

(iii) Show that $\alpha\beta = -\dfrac{1}{2}$.

Question 4 (12 marks)

(a) If $\tan\theta = 2$ and $0 < \theta < \dfrac{\pi}{2}$ evaluate $\sin\left(\theta + \dfrac{\pi}{4}\right)$. — 3

(b) In the diagram ABCD is a cyclic quadrilateral. The bisector of $\angle$ABC cuts the circle at E, and meets AD produced at F.

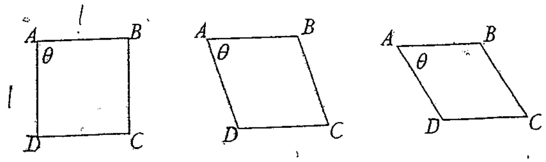


(i) Copy the diagram showing the above information

(ii) Give a reason why $\angle$CDE=$\angle$CBE — 1

(iii) Show that DE bisects $\angle$CDF — 3

(c)



A square ABCD of side 1 unit is gradually 'pushed over' to become a rhombus. The angle at A (θ) decreases at a constant rate of $0 \cdot 1$ radians per second.

(i) At what rate is the area of the rhombus ABCD decreasing when $\theta = \frac{\pi}{6}$? **2**

(ii) At what rate is the shorter diagonal of the rhombus ABCD decreasing when $\theta = \frac{\pi}{3}$? **3**

End of Section

Section C (Use a SEPARATE writing booklet)

Marks

Question 5 (12 marks)

(a) Two boys decide to settle an argument by taking turns to toss a die. The first person to throw a six wins.

(i) What is the probability that the first person wins on his second throw? **1**

(ii) What is the probability that the first person will win the argument? **2**

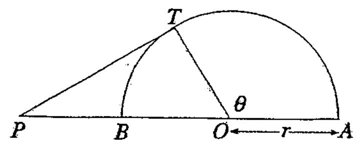
(b) $P(2at, at^2)$, $t > 0$ is a point on the parabola $x^2 = 4ay$. The normal to the parabola at P cuts the x axis at X and the y axis at Y.

(i) Show that the normal at P has equation $x + ty - 2at - at^3 = 0$ **2**

(ii) Find the co-ordinates of X and Y **1**

(iii) Find the value of t such that P is the midpoint of XY **2**

(c)



The point T lies on the circumference of a semicircle, radius r and diameter AB, as shown. The point P lies on AB produced and PT is the tangent at T.
The arc AT subtends an angle of θ at the centre, O, and the area of ΔOPT is equal to that of the sector AOT. **4**

(i) Show that $\theta + \tan\theta = 0$.

(ii) Taking 2 as an approximation to θ, use Newton's method once to find a better approximation to two decimal places.

Question 6 (12 marks)

(a) A particle is oscillating in simple harmonic motion such that its displacement x metres from a given origin O satisfies the equation $\frac{d^2x}{dt^2} = -4x$ where t is the time in seconds

(i) Show that $x = \alpha\cos(2t + \beta)$ is a possible equation of motion for this particle, where α and β are constants **2**

(ii) The particle is observed initially to have a velocity of 2 metres per second and a displacement from the origin of 4 metres. Find the amplitude of the oscillation. **2**

(iii) Determine the maximum velocity of the particle **2**

(b) Prove by Mathematical Induction that **3**

$$\sum_{r=1}^{n} r^3 = 1^3 + 2^3 + 3^3 + \ldots\ldots + n^3 = \frac{1}{4}n^2(n+1)^2$$

(c) Consider the function $f(x) = \frac{x}{\sqrt{1-x^2}}$

(i) Find the domain of $f(x)$ **1**

(ii) Find $f^{-1}(x)$, the inverse function of $f(x)$ **2**

End of Section

Section D (Use a SEPARATE writing booklet)

Question 7 (12 marks) **Marks**

(a) A projectile fired with velocity V and at an angle of 45° to the horizontal, just clears the tops of two vertical posts of height $8a^2$, and the posts are $12a^2$ apart. There is no air resistance, and the acceleration due to gravity is g.

(i) If the projectile is at a point $P(x, y)$ at time t, Derive expressions for x and y in terms of t. **2**

(ii) Hence, show that the equation of the path of the projectile is $y = x - \dfrac{gx^2}{V^2}$ **2**

(iii) Using the information in (ii) show that the range of the projectile is $\dfrac{V^2}{g}$ **2**

(iv) If the first post is b units from the origin, show that **2**

(α) $\dfrac{V^2}{g} = 2b + 12a^2$

(β) $8a^2 = b - \dfrac{gb^2}{V^2}$

(v) Hence or otherwise prove that $V = 6a\sqrt{g}$ **4**

End of paper

SYDNEY BOYS HIGH SCHOOL
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AUGUST 2005

Trial Higher School Certificate Examination

YEAR 12

Mathematics Extension 1

Sample Solutions

Section	Marker
A	RD
B	RB
C	FN
D	AMG

Section A

Q1 (a) $\dfrac{3^n}{3^{n+1}-3^n} = \dfrac{3^n}{3^n(3-1)}$

$= \frac{1}{2}$ (1)

(b) $\lim_{x\to 0} \dfrac{\sin 5x}{4x} = \lim_{5x\to 0} \dfrac{\sin 5x}{5x} \times \dfrac{5}{4}$

$= \frac{5}{4}$ (1)

(c) $P(3) = 27 - 27 + 3p - 14 = 1$

$\therefore 3p = 15$

$p = 5$ (2)

(d) $\log_a 2a = \log_a 2 + \log_a a$

$= x + 1$ (2)

(e) $P \equiv \left(\dfrac{-3\times 6 + 2\times -1}{-3+2}, \dfrac{-3\times -4 + 2\times 5}{-3+2} \right)$

$\equiv \left(\dfrac{-20}{-1}, \dfrac{22}{-1} \right)$

$\equiv (20, -22)$ (2)

(f) $\tan\theta = \left| \dfrac{m_1 - m_2}{1 + m_1 m_2} \right|$

$= \left| \dfrac{-\frac{3}{2} - \frac{1}{5}}{1 + -\frac{3}{2}\times\frac{1}{5}} \right|$

$= \left| \dfrac{-\frac{15}{10} - \frac{2}{10}}{1 - \frac{3}{10}} \right|$

$= \left| -\frac{17}{7} \right|$

$= \frac{17}{7}$

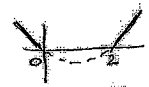
$\therefore \theta = 67^\circ 37'$ (2)

(g) $\dfrac{2}{x} \le 1$

$2x \le x^2$

$0 \le x^2 - 2x$

$x(x-2) \ge 0$



$x < 0$ or $x \ge 2$ (2)

[12]

Q2

(a) (i) $y = \tan^3(5x+4)$

$y' = 3\tan^2(5x+4) . \sec^2(5x+4) . 5$

$= 15\tan^2(5x+4) . \sec^2(5x+4)$

(ii) $y = \ln\left(\frac{2x+3}{3x+4}\right)$

$= \ln(2x+3) - \ln(3x+4)$

$y' = \frac{1}{2x+3} \times 2 - \frac{1}{3x+4} \times 3$

$= \frac{2}{2x+3} - \frac{3}{3x+4}$

(iii) $y = \cos(e^{1-5x})$

$y' = -\sin(e^{1-5x}) . e^{1-5x} . -5$

$= 5\sin(e^{1-5x}) . e^{1-5x}$

(b) (i) $30 \times 29 \times 28 \times 27 \times 26 \times 25$

$= 427\ 518\ 000$

(ii) Choose the six finalists and then consider where they may be placed:

${}^{1}C_1 \times {}^{29}C_5 \times 6!$

$= 118\,755 \times 270$

$= 85\ 503\ 600$

(c) (i) $f(\frac{1}{2}) = \tan^{-1} 1$

$= \frac{\pi}{4}$

(ii) $f'(x) = \frac{1}{1+(2x)^2} \times 2$

$= \frac{2}{1+4x^2}$

$f'(\frac{1}{2}) = \frac{2}{1+4\times\frac{1}{4}}$

$= 1$

Section B

3 (a) (i) $y = 3\cos^{-1}(x-2)$

Domain $-1 \leq x-2 \leq 1$

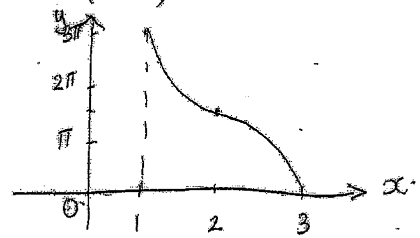
$+2 \quad +2 \quad +2$

$1 \leq x \leq 3.$

Range $0 \leq \cos^{-1}(x-2) \leq \pi$

$0 \leq 3\cos^{-1}(x-2) \leq 3\pi$

(ii) $y = 3\cos^{-1}(x-2)$



(b) $\int x\sqrt{16+x^2}\,dx$ using $u = 16+x^2$

$u = 16+x^2$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$\frac{du}{2} = x\,dx$

becomes $\int \frac{du}{2} . u^{\frac{1}{2}}$

$\frac{1}{2}\int u^{\frac{1}{2}}\,du$

$= \frac{1}{2} . u^{\frac{3}{2}} . \frac{2}{3} + C$

$= \frac{1}{3} u^{1\frac{1}{2}} + C$

~~$\frac{1}{3} (16+x^2)\sqrt{16+x^2} + C$~~

or $\frac{1}{3} [\sqrt{16+x^2}]^3 + C.$

(c) general soln to $\sin 2\theta = \sqrt{3}\cos 2\theta$

$\frac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta = \frac{\sqrt{3}}{1}$

So $2\theta = \frac{\pi}{3} + k\pi$

$\theta = \frac{\pi}{6} + \frac{k}{2}\pi$ where $k = 0, \pm1, \pm2, \pm3, \ldots$

(d) $4x^3 + 6x^2 + c = 0$

$c \neq 0$, roots are $\alpha, \beta, \alpha\beta$. $a=4$, $b=6$, $c=0$, $d=c$.

(i) sum of roots $\alpha+\beta+\alpha\beta = -\frac{b}{a} = -\frac{6}{4}$

product $\alpha\beta\alpha\beta = (\alpha\beta)^2 = -\frac{d}{a} = -\frac{c}{4}$

product in twos $\alpha\beta + \alpha^2\beta + \alpha\beta^2 = \frac{c}{a} = \frac{0}{4} = 0$

now since $(\alpha\beta)^2 = -\frac{c}{4}$ and $c \neq 0$

then $\alpha\beta \neq 0$ /

(ii) From above, since $\alpha\beta + \alpha^2\beta + \alpha\beta^2 = \frac{0}{4} = 0$

then $\alpha\beta(1+\alpha+\beta) = 0$

So $\alpha\beta = 0$, but it cannot from (i)

So $1+\alpha+\beta = 0$

$\alpha+\beta = -1$ //

(iii) since $\alpha+\beta+\alpha\beta = -\frac{6}{4} = -\frac{3}{2}$ from above

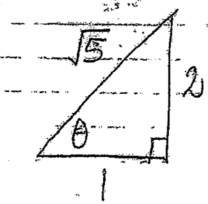
and $\alpha+\beta = -1$

$-1 + \alpha\beta = -1\frac{1}{2}$

$\alpha\beta = -1\frac{1}{2} + 1 = -\frac{1}{2}$ //

4 (a) $\tan\theta = 2$, $0 < \theta < \frac{\pi}{2}$

evaluate $\sin(\theta + \frac{\pi}{4})$



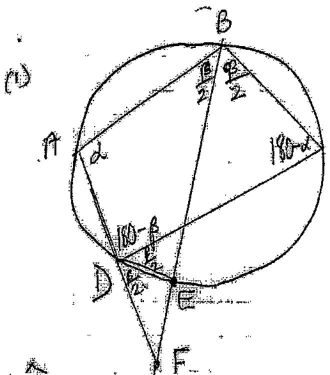
$\sin(\theta+\frac{\pi}{4}) = \sin\theta\cos\frac{\pi}{4} + \cos\theta\sin\frac{\pi}{4}$

$= \frac{2}{\sqrt{5}} \times \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{5}} \times \frac{1}{\sqrt{2}}$

$= \frac{2}{\sqrt{10}} + \frac{1}{\sqrt{10}} = \frac{3}{\sqrt{10}} \times \frac{\sqrt{10}}{\sqrt{10}} = \frac{3\sqrt{10}}{10}$

$(0.94868\ldots)$

(b) (i)



cyclic quad: $\alpha + (180-\alpha) = 180°$

$\frac{\beta}{2} + \frac{\beta}{2} + (180-\beta) = 180°$

(ii) $C\hat{B}E = \frac{\beta}{2}$ stands on minor arc CE.

$C\hat{D}E = \frac{\beta}{2}$ also since it stands on minor arc CE

(iii) now $A\hat{D}C + C\hat{D}E + E\hat{D}F = 180°$ straight line angle

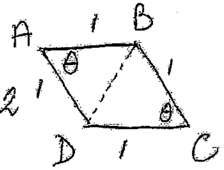
$(180-\beta) + \frac{\beta}{2} + E\hat{D}F = 180°$

$-\frac{\beta}{2} + E\hat{D}F = 0 \Rightarrow E\hat{D}F = \frac{\beta}{2}$

$\Rightarrow DE$ bisects $C\hat{D}F$

4 (c) Square ABCD side 1 unit

$\frac{d\theta}{dt} = -0.1$ radians/sec.



(i) area rhombus $A = \frac{1}{2} \times 1 \times 1 \times \sin\theta \times 2$

$A = \sin\theta$

Now $\frac{dA}{dt} = \frac{dA}{d\theta} \times \frac{d\theta}{dt}$

$= \cos\theta \times -0.1$

When $\theta = \frac{\pi}{6}$, $\frac{dA}{dt} = -0.1 \times \frac{\sqrt{3}}{2} = -\frac{1}{10} \times \frac{\sqrt{3}}{2}$

$= -\frac{\sqrt{3}}{20}$ units²/sec

$(-0.0866\ldots)$.

area is decreasing at a rate of $\frac{\sqrt{3}}{20}$ u²/s.

(ii) shorter diagonal BD.

$(BD)^2 = 1^2 + 1^2 - 2 \times 1 \times 1 \times \cos\theta$

$= 2 - 2\cos\theta$

$BD = \sqrt{2(1-\cos\theta)} = \sqrt{2}.(1-\cos\theta)^{\frac{1}{2}}$

$\frac{dBD}{dt} = \frac{dBD}{d\theta} \times \frac{d\theta}{dt}$

$= \sqrt{2} \times \frac{1}{2}(1-\cos\theta)^{-\frac{1}{2}} \times \sin\theta \times -0.1$

$= \frac{\sqrt{2} \times \sin\theta \times -0.1}{2.\sqrt{1-\cos\theta}}$

At $\theta = \frac{\pi}{3}$.

$\frac{dBD}{dt} = \frac{\sqrt{2} \times \frac{\sqrt{3}}{2} \times -0.1}{2.\sqrt{1-\frac{1}{2}}}$

$= \frac{\frac{\sqrt{6}}{2} \times -\frac{1}{10}}{2 \times \sqrt{\frac{1}{2}}}$

$= \frac{\frac{\sqrt{6}}{2} \times -\frac{1}{10}}{\frac{2}{\sqrt{2}}} \times \frac{\sqrt{2}}{2}$

$= -\frac{\sqrt{12}}{40} = -\frac{2\sqrt{3}}{40} = -\frac{\sqrt{3}}{20}$ u/s

shorter diagonal decreasing at $\frac{\sqrt{3}}{20}$ u/s

Section C

QUESTION 5

(a)

(i) $\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}=\frac{25}{216}$

(ii) $\frac{1}{6}+\left(\frac{5}{6}\right)^2\times\frac{1}{6}+\left(\frac{5}{6}\right)^4\times\frac{1}{6}$.......... geometric series

$$S_\infty=\frac{\frac{1}{6}}{1-\left(\frac{5}{6}\right)^2}=\frac{6}{11}$$

(b)(i) $y=\frac{x^2}{4a}, y^1=\frac{x}{2a}=\frac{2at}{2a}=t$ = gradient of tangent

gradient of normal $=-\frac{1}{t}$

eqn. of normal is $y-at^2=-\frac{1}{t}(x-2at)$

$yt-at^3=-x+2at$

$x+ty-2at-at^3=0$ as required.

(ii) $when\ y=0, x=2at+at^3$ X $(2at+at^3, 0)$

$when\ x=0, y=\frac{2at+at^3}{t}=2a+at^2$ Y $(0, 2a+at^2)$

(iii) Midpoint, P is $\left(at+\frac{at^3}{2}, a+\frac{at^2}{2}\right)$

$2at=at+\frac{at^3}{2}$ $\qquad at^2=a+\frac{at^2}{2}$

$4at=2at+at^3$ $\qquad 2at^2=2a+at^2$

$4=2+t^2$ $\qquad 2t^2=2+t^2$

$t=\pm\sqrt{2}$ $\quad t=\sqrt{2},\ t>0$

(c)(i) $\angle TOP=\pi-\phi$

$\tan\angle TOP=\frac{PT}{r}=-\tan\phi,\ PT=-r\tan\phi$

area Δ TOP = area sector TOA (given)

$\frac{1}{2}r\times PT=\frac{1}{2}r^2\phi$

$-r\tan\phi=r\phi$

$-\tan\phi=\phi$

$\phi+\tan\phi=0$ as required.

(ii) $a_1=a-\frac{f(a)}{f^1(a)}=a_1=2-\frac{f(2)}{f^1(2)}$

$2-\frac{2+\tan 2}{1+\sec^2 2}=2-\frac{-0\cdot185}{6\cdot774}$

$2\cdot03\ (2d.p.)$

QUESTION 6

(a)(i) if $x=\alpha\cos(2t+\beta)$

$\frac{dx}{dt}=-2\alpha\sin(2t+\beta)$

$\frac{d^2x}{dt^2}=-4\alpha\cos(2t+\beta)=-4x$ (a possible equation)

(ii) $v^2=n^2(\alpha^2-x^2)$, $n=2$ and $x=4$ when $v=2$

$4=4(\alpha^2-16)$

$\alpha=\sqrt{17}\ m$

(ii) Max velocity when displacement = 0

$v^2=4(17-0)$

$v=2\sqrt{17}m/s$

(b) When $n=1$, $1^3=\frac{1}{4}\times1^2\times2^2$. P(1) is true

Assume P(k) is true $1^3+2^3.....+k^3=\frac{1}{4}k^2(k+1)^2$

If $n=k+1$,

$1^3+2^3.....+k^3+(k+1)^3=\frac{1}{4}(k+1)^2(k+2)^2$

LHS $=\frac{1}{4}k^2(k+1)^2+(k+1)^3$ (using assumption)

$=(k+1)^2\left(\frac{1}{4}k^2+k+1\right)$

$=(k+1)^2\frac{1}{4}(k^2+4k+4)$

$=\frac{1}{4}(k+1)^2(k+2)^2$

= RHS

P$(k+1)$ is true if P(k) is true. P(1) is true.

$\therefore$, by Mathematical Induction, P(n) is true for any integer $n\geq1$

(c)(i) $1-x^2>0 \quad -1<x<1$

(ii) If $y=f(x)$, the inverse function is

$x=\frac{y}{\sqrt{1-y^2}}$

$x^2=\frac{y^2}{1-y^2}$

$x^2-x^2y^2=y^2$

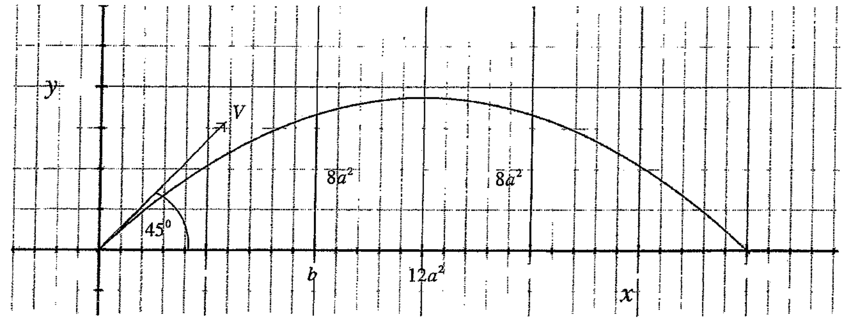
$y^2(1+x^2)=x^2$

$y^2=\frac{x^2}{1+x^2}$

$f^{-1}(x)=\frac{x}{\sqrt{1+x^2}}$ (odd function)

Section D

(7) (a)



(i)

$\ddot{x}=0$ $\qquad\qquad \ddot{y}=-g$

Integrate w.r.t. t $\qquad$ Integrate w.r.t. t

$\dot{x}=K$ $\qquad\qquad \dot{y}=-gt+L$

When $t=0$, $\dot{x}=\frac{V}{\sqrt{2}}$ $\qquad$ When $t=0$, $\dot{y}=\frac{V}{\sqrt{2}}$

$\therefore K=\frac{V}{\sqrt{2}}$ $\qquad \therefore L=\frac{V}{\sqrt{2}}$

$\therefore \dot{x}=\frac{V}{\sqrt{2}}$ $\qquad \therefore \dot{y}=\frac{V}{\sqrt{2}}-gt$

Integrate w.r.t. t $\qquad$ Integrate w.r.t. t

$x=\frac{Vt}{\sqrt{2}}+M$ $\qquad y=\frac{Vt}{\sqrt{2}}-\frac{1}{2}gt^2+N$

When $t=0$, $x=0$ $\qquad$ When $t=0$, $y=0$

$\therefore M=0$ $\qquad \therefore N=0$

$\therefore x=\frac{Vt}{\sqrt{2}}$ $\qquad \therefore y=\frac{Vt}{\sqrt{2}}-\frac{1}{2}gt^2$

(ii) From the equation for x:

$t=\frac{\sqrt{2}x}{V}$ $\qquad \therefore y=\frac{V}{\sqrt{2}}\frac{\sqrt{2}x}{V}-\frac{1}{2}g\left(\frac{\sqrt{2}x}{V}\right)^2$

$y=x-\frac{gx^2}{V^2}$

(iii) The range is achieved when $y = 0$

$$\therefore x - \frac{gx^2}{V^2} = 0$$

$$x\left(1 - \frac{gx}{V^2}\right) = 0$$

$$\therefore 1 - \frac{gx}{V^2} = 0$$

$$x = \frac{V^2}{g} \quad \text{(Range)}$$

(iv) (α) By symmetry the second post is b units from point of impact

$$\therefore (x_R =)\frac{V^2}{g} = 2b + 12a^2$$

(β) When $x = b,\ y = 8a^2$, in the equation from (ii):

$$8a^2 = b - \frac{gb^2}{V^2}$$

(v) From (α):

$$2b = \frac{V^2}{g} - 12a^2$$

$$\therefore b = \frac{V^2}{2g} - 6a^2$$

$$\therefore \frac{V^2}{2g} = b + 6a^2$$

$$\therefore V^2 = 2g\left(b + 6a^2\right)$$

$$= g\left(2b + 12a^2\right)$$

$$\therefore\ V = \sqrt{g}\sqrt{2b + 12a^2} \quad \text{———— (*)}$$

Hence it remains to prove that $2b = 24a^2$.

Now $\frac{g}{V^2} = \frac{1}{2b + 12a^2}$

So $8a^2 = b - \frac{gb^2}{V^2}$

$$= b - \frac{b^2}{2b + 12a^2}$$

$$= \frac{2b^2 + 12a^2b - b^2}{2b + 12a^2}$$

$$\therefore 16a^2b + 96a^4 = 2b^2 + 12a^2b - b^2$$

$$= b^2 + 12a^2b$$

$$\therefore b^2 - 4a^2b - 96a^4 = 0$$

$$\therefore b = \frac{4a^2 \pm \sqrt{16a^4 + 4 \times 96a^4}}{2}$$

$$= \frac{4a^2 \pm 4\sqrt{a^4 + 24a^4}}{2}$$

$$= \frac{4a^2 \pm 4 \times 5a^2}{2}$$

$$= 12a^2 \quad \text{(Neg result extraneous)}$$

$\therefore$ In equation (*)

$$V = \sqrt{g}\sqrt{36a^2}$$

$$= 6a\sqrt{g} \qquad \text{As required.}$$