SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2006

YEAR 12
TRIAL HIGHER SCHOOL
CERTIFICATE

Mathematics Extension 1

General Instructions

- Working time – 2 Hours
- Reading time – 5 Minutes
- Write using black or blue pen
- Board approved calculators may be used
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may not be awarded for messy or badly arranged work.
- Hand in your answer booklets in 4 sections. Section A (Questions 1 and 2), Section B (Questions 3 and 4), Section C (Questions 5 and 6) and Section D (Question 7)

Total Marks – 84

- Attempt Questions 1 – 7.
- All QUESTIONS are of equal value.

Examiner: *R. Boros*

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Section A – Start a new booklet

Marks

Question 1. (12 marks)

a)

(i) Evaluate $\int_0^1 \frac{x}{x^2+1}\,dx$ leaving your answer in exact form. 2

(ii) Evaluate $\int_{-2}^{2\sqrt{3}} \frac{1}{x^2+4}\,dx$ leaving your answer in exact form. 2

b) Find the gradient of the tangent to the curve $y=\tan^{-1}(\sin x)$ at $x=0$. 2

c) Solve for x, $\frac{1}{x+1}<3$. 2

d) Give the general solution of the equation, $\cos\left(\theta+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$. 2

e) If $f(x)=8x^3$, then find the inverse function $f^{-1}(x)$. 2

Question 2. (12 marks)

a) Find the co-ordinates of the point P that divides the interval $A(-4,-6)$ and $B(6,-1)$ externally in the ratio 3:1. 2

b) (i) Sketch the graph of $y=|2x-4|$. 2

(ii) Using your graph, or otherwise, solve the inequation $|2x-4|>x$. 2

c) Use the substitution $u=1+x$ to evaluate, $\int_{-1}^{3} x\sqrt{1+x}\,.dx$. 2

d) Solve for n, $2\times {}^nC_4 = 5\times {}^nC_2$. 2

e) What is the least distance between the circle $x^2+y^2+2x+4y=1$ and the line $3x+\cancel{4y}=6$? (Leave your answer in exact form.) 2

4y

End of Section A

Section B – Start a new booklet

Marks

Question 3. (12 marks)

a) If the roots of the equation, $x^4-2x^3-5x+1=0$, are t_1, t_2, t_3, t_4,

find $\sum_1^4 (t_i t_j t_k)^{-1}$, such that $i\neq j\neq k$. 2

b) State the domain and range of the function $y=2\sin^{-1}\left(\frac{x}{3}\right)$.

Hence sketch the curve. 3

c) A bowl of water heated to $100^\circ C$ is placed in a coolroom where the temperature is maintained at $-5^\circ C$. After t minutes, the temperature $T^\circ C$ of the water is changing so that $\frac{dT}{dt}=-k(T+5)$.

(i) Prove that $T=Ae^{-kt}-5$ satisfies this equation and find the value of A. 1

(ii) After 20 minutes, the temperature of the water has fallen to $40^\circ C$. How long, to the nearest minute, will the water need to be in the coolroom before ice begins to form, (i.e. the temperature falls to $0^\circ C$). 2

d) (i) Show that the equation $\ln x+x^2-4x=0$ has a root lying between $x=3$ and $x=4$. 2

(ii) By taking $x=4$ as a first approximation, use one application of Newton's Method to obtain another approximation for the root, to 2 decimal places. Is this newer approximation a better one? Explain. 2

Question 4. (12 marks) **Marks**

a) The points $P\left(2ap, ap^2\right)$ and $Q\left(2aq, aq^2\right)$ lie on the parabola $x^2 = 4ay$. It is given that the chord PQ has equation $y = \left(\dfrac{p+q}{2}\right)x - apq$.

(i) Show that the gradient of the tangent at P is p. **1**

(ii) Prove that if PQ passes through the focus, then the tangent at P is parallel to the normal at Q. **2**

b) A committee of five is to be formed from 4 Liberal senators, 3 Labor senators and 2 Democrat senators.

(i) How many different committees can be formed that have 3 Liberals, 1 Labor and 1 Democrat? **1**

(ii) If the committee is to be chosen at random, what is the probability that there will be a Liberal majority in the committee? **2**

c) (i) Express $7\cos\theta - \sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0^\circ \le \alpha \le 90^\circ$. **2**

(ii) Hence solve $7\cos\theta - \sin\theta = 5$ for $0^\circ \le \theta \le 360^\circ$, giving your answer to the nearest degree. **2**

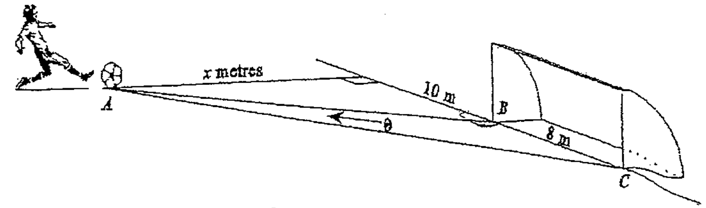
d) Find the values of the constants a and b if $x^2 - 2x - 3$ is a factor of the polynomial $P(x) = x^3 - 3x^2 + ax + b$. **2**

End of Section B

Section C – Start a new booklet **Marks**

Question 5. (12 marks)

a)



A soccer player A is x metres from a goal line of a soccer field. He takes a shot at the goal BC, with the ball not leaving the ground.

(i) Show that the angle θ within which he must shoot is given by $\theta = \tan^{-1}\left(\dfrac{8x}{180 + x^2}\right)$ when he is 10 metres to one side of the near goal post and 18 metres to the same side of the far post. **2**

(ii) Find the value of x which makes this angle a maximum. (Leave your answer in exact form). **2**

b) A particle moves in a straight line such that its velocity V m/s is given by $V = 2\sqrt{2x-1}$ when it is x metres from the origin. If $x = \dfrac{1}{2}$ when $t = 0$ find:

(i) the acceleration. **1**

(ii) an expression for x in terms of t. **2**

c) Find the volume of the solid obtained by rotating $y = \sin^{-1} x$ about the y-axis between $y = -\dfrac{\pi}{4}$ and $y = \dfrac{\pi}{4}$. Answer in exact form. **3**

d) The perimeter of a circle is increasing at 3 cm/s. Leaving your answer in terms of π, find the rate at which the area is increasing when the perimeter is 1m. **2**

Question 6. (12 marks) **Marks**

a)

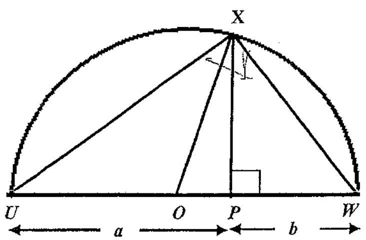
Consider the following three expressions involving n, where n is a positive integer: $5^n+3,\ 7^n+5,\ 5^n+7$

(i) By substituting values of n, show that 7^n+5 is the only one of these expressions which could be divisible by 6 for all positive integers n. 1

(ii) Use mathematical induction to show that the expression 7^n+5 is in fact divisible by 6 for all positive integers n. 2

b)

Not to scale



In the diagram UXW is a semi-circle with O as a midpoint of diameter UW. The point P lies on UW and XP is perpendicular to UW. The length of $UP=a$ units and $PW=b$ units are shown.

(i) Explain why $OX=\dfrac{a+b}{2}$. 1

(ii) Show that $\triangle UXP\ |||\ \triangle XWP$. 1

(iii) Deduce that $XP=\sqrt{ab}$. 1

(iv) By using the diagram show that $\dfrac{a+b}{2}\geq\sqrt{ab}$. 1

c) The displacement x metres of a particle from the origin is given by $x=5\cos\left(3t-\dfrac{\pi}{6}\right)$, where t is the time lapsed in seconds.

(i) Show that $\ddot{x}=-9x$. 1

(ii) Find the period of the motion 1

Marks

d) Suppose that $(5+2x)^{12}=\sum_{k=0}^{12}a_kx^k$.

(i) Use the binomial theorem to write the expression for a_k. 1

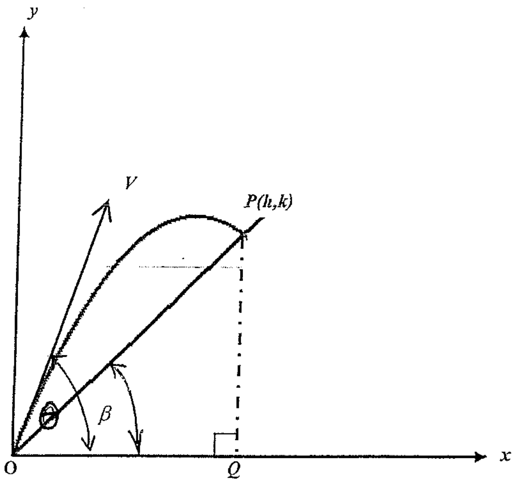
(ii) Show that $\dfrac{a_{k+1}}{a_k}=\dfrac{24-2k}{5k+5}$ 2

End of Section C

Section D – Start a new booklet

Marks

Question 7. (12 marks)



A projectile is fired from the origin with a velocity V and an angle of elevation θ, where $\theta \neq 90^\circ$. You may assume that $x = Vt\cos\theta$ and $y = -\frac{1}{2}gt^2 + Vt\sin\theta$, where x and y are the horizontal and vertical displacements of the projectile in metres from O at time t seconds after firing, and g is the acceleration due to gravity.

(i) Show that the Cartesian equation of the flight of the projectile is: **1**

$$y = x\tan\theta - \frac{g}{2V^2\cos^2\theta}x^2$$

(ii) Suppose the projectile is fired up a plane inclined at β to the horizontal so that $0^\circ \leq \beta \leq \theta$. If the projectile strikes the plane at $P(h,k)$, show that: **2**

$$h = \frac{(\tan\theta - \tan\beta)2V^2\cos^2\theta}{g}$$

(iii) Hence, show that the range OP of the projectile can be given by **4**

$$OP = \frac{2V^2\sin(\theta-\beta)\cos\theta}{g\cos^2\beta}$$

Marks

(iv) Given the fact that $2\sin(x-\beta)\cos x = \sin(2x-\beta) - \sin\beta$. Show that the maximum value of the range of OP is given by: **4**

$$\frac{V^2}{g(1+\sin\beta)}$$

(v) If the angle of inclination of the plane is 14°, at what angle to the horizontal should the projectile be fired in order to attain maximum range? **1**

End of Examination

Question 1:

a) i) $\int_0^1 \frac{x}{x^2+1}\,dx = \frac{1}{2}\int_0^1 \frac{2x}{x^2+1}\,dx$ ½

$= \frac{1}{2}\left[\ln(x^2+1)\right]_0^1$ ½

no ½ −1

$= \frac{1}{2}\left[\ln 2 - \ln 1\right]$

$= \frac{1}{2}\ln 2$ 1

ii) $\int_{-2}^{2\sqrt{3}} \frac{1}{x^2+4}\,dx = \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_{-2}^{2\sqrt{3}}$ ½

$= \left(\frac{1}{2}\tan^{-1}\frac{2\sqrt{3}}{2}\right) - \left(\frac{1}{2}\tan^{-1}\frac{-2}{2}\right)$

$= \left(\frac{1}{2}\tan^{-1}\sqrt{3}\right) - \left(\frac{1}{2}\tan^{-1}(-1)\right)$

$= \left(\frac{1}{2}\cdot\frac{\pi}{3}\right) - \left(\frac{1}{2}\cdot\frac{-\pi}{4}\right)$ ½

$= \frac{\pi}{6} - -\frac{\pi}{8}$

$= \frac{7\pi}{24}$ 1

b) Find the gradient of the tangent to the curve $y = \tan^{-1}(\sin x)$ at $x = 0$

$\frac{dy}{dx} = \frac{1}{1+(\sin x)^2} \times (\cos x)$ −1 for no × $\cos x$.

$= \frac{\cos x}{1+\sin^2 x}$ 1

when $x = 0$

$\frac{dy}{dx} = \frac{\cos 0}{1+\sin^2 0}$

$= \frac{1}{1}$

$= 1$ 1

c) solve for x, $\frac{1}{x+1} < 3$

$x + 1 \neq 0$

$\therefore x \neq -1$ ½

× by $(x+1)^2$ $\quad \frac{(x+1)^{\cancel{2}}}{\cancel{x+1}} < 3(x+1)^2$ ½

$x+1 < 3x^2+6x+3$

$0 < 3x^2+6x+3-(x+1)$

$0 < 3x^2+5x+2$

$\frac{(3x+3)(3x+2)}{3}$ $\quad 0 < (x+1)(3x+2)$ ½

$= \frac{\cancel{3}(x+1)(3x+2)}{\cancel{3}}$ $\quad x < -1,\ x > -2/3$ ½

d) General solution for

$\cos(\theta + \pi/4) = 1/\sqrt{2}$

$\cos(\theta + \pi/4) = \cos \pi/4$ ½

$\theta + \pi/4 = 2n\pi \pm \pi/4$ 1

$\theta = 2n\pi \pm \pi/4 - \pi/4$ ½

e) $f(x) = 8x^3$ find inverse function $f^{-1}(x)$

$f(f^{-1}(x)) = x = f^{-1}(f(x))$ ½

$f(f^{-1}(x)) = x$

$\therefore 8(f^{-1}(x))^3 = x$ $\qquad f^{-1}(f(x)) = \frac{\sqrt[3]{8x^3}}{2}$

$f^{-1}(x)^3 = x/8$ $\qquad = \frac{2x}{2}$

$\therefore f^{-1}(x) = \sqrt[3]{\frac{x}{8}}$ $\qquad = x$ ½

$= \frac{\sqrt[3]{x}}{2}$ ½

$\therefore f^{-1}(x) = \frac{\sqrt[3]{x}}{2}$ ½

−1 for $\sqrt[3]{8} = 2\sqrt{2}$

−1 for $\frac{\sqrt[3]{x}}{8}$

Question 2.

a) $A(-4,-6)$ $\quad B(6,-1)$ divide externally in ratio $3:1$

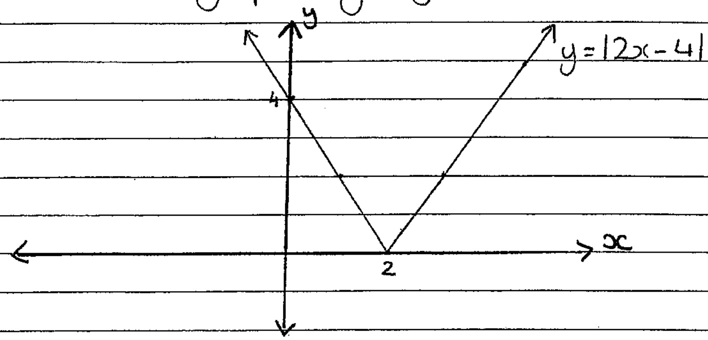
x-co-ordinate $= \dfrac{(3\times 6)-(1\times -4)}{3-1} = 11$ ½

y-co-ordinate $= \dfrac{(3\times -1)-(1\times -6)}{3-1} = \dfrac{3}{2}$ ½

$\therefore$ The co-ordinates of P are $(11, \tfrac{3}{2})$ 1

1½ if x_2, x_1, $y_2\, y_1$ are switched.

b) i) sketch the graph of $y=|2x-4|$



2

ii) solve $|2x-4|>x$

$\sqrt{(2x-4)^2} > x$

$(2x-4)^2 > x^2$

$4x^2-16x+16 > x^2$

$3x^2-16x+16 > 0$

$\dfrac{(3x-12)(3x-4)}{3} > 0$

$\dfrac{\cancel{3}(x-4)(3x-4)}{\cancel{3}} > 0$

$x>4, \; x<\tfrac{4}{3}$ $\quad$ 2

c) Use $u = 1+x$ to evaluate $\int_{-1}^{3} x\sqrt{1+x}\,dx$

$u = 1+x$ $\qquad$ limits:

$\therefore \dfrac{du}{dx} = 1 \quad \therefore x = u-1$ $\qquad$ $x=3 \therefore u=1+3 \Rightarrow u=4$ ½

$du = dx$ $\qquad$ $x=-1 \therefore u=1-1 \Rightarrow u=0$

$\int_0^4 (u-1)\cdot\sqrt{u}\,du$ ½

$= \int_0^4 (u-1)\cdot u^{1/2}\,du$

$= \int_0^4 \left(u^{3/2} - u^{1/2}\right) du$

$= \left[\dfrac{u^{5/2}}{5/2} - \dfrac{u^{3/2}}{3/2}\right]_0^4$ ½

$= \left[\dfrac{2}{5}.4^{5/2} - \dfrac{2}{3}.4^{3/2}\right] - \left[\dfrac{2}{5}.0^{5/2} - \dfrac{2}{3}.0^{3/2}\right]$

$= \left[\dfrac{2\times 32}{5} - \dfrac{2\times 8}{3}\right]$

$= \dfrac{64}{5} - \dfrac{16}{3}$

$= \dfrac{112}{15}$ OR $7\tfrac{7}{15}$ ½

d) solve for n, $2\times {}^nC_4 = 5\times {}^nC_2$ $\qquad$ $-$½ for $n=-3$

$2\times\dfrac{n!}{(n-4)!\,4!} = 5\times\dfrac{n!}{(n-2)!\times 2!}$ 1

$\div$ by $n!$ $\quad \dfrac{2}{24}\cdot\dfrac{1}{(n-4)!} = \dfrac{5}{2}\cdot\dfrac{1}{(n-2)!}$

$\times$ by $(n-4)!$ $\quad \dfrac{2}{24} = \dfrac{5}{2}\cdot\dfrac{1}{(n-2)(n-3)}$

$\times$ by $\tfrac{2}{5}$ $\quad \dfrac{1}{30} = \dfrac{1}{(n-2)(n-3)}$

$\therefore n = 8.$ 1

Check:

$2\times {}^8C_4 = 140 = 5\times {}^8C_2$ ✓

2e) circle $x^2+y^2+2x+4y=1$

$\therefore (x+1)^2+(y+2)^2=6$ ½

$\therefore$ centre $(-1,-2)$ radius $\sqrt{6}$

line $3x+4y=6 \Rightarrow 3x+4y-6=0$

least distance between circle & line is the distance between the line & centre of the circle less the radius. ½

$d=\frac{|3\times -1)+(4\times -2)-6|}{\sqrt{3^2+4^2}}$

$=\frac{|-3-8-6|}{5}$

$=\frac{|-17|}{5}$ ½

$\therefore$ minimum distance $=\frac{17}{5}-\sqrt{6}$ ½

17/5 1 mark only.

SECTION B QUESTION 3

a) $\sum_{\substack{1 \\ i\neq j\neq k}}^{4}(t_it_jt_k)^{-1}=\frac{1}{t_1t_2t_3}+\frac{1}{t_2t_3t_4}+\frac{1}{t_3t_4t_1}+\frac{1}{t_4t_1t_2}$

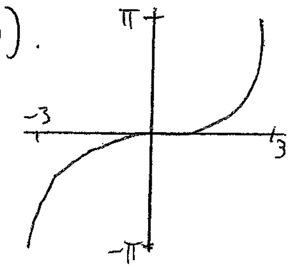
$=\frac{t_1+t_2+t_3+t_4}{t_1t_2t_3t_4}$

$t_1+t_2+t_3+t_4=-\frac{b}{a}$
$=2$

$t_1t_2t_3t_4=\frac{e}{a}$
$=1.$

So $\sum_{\substack{1 \\ i\neq j\neq k}}^{4}(t_it_jt_k)^{-1}=2.$

b).



Domain: $-1\le\frac{x}{3}\le 1$
$-3\le x\le 3.$

Range: $-\frac{\pi}{2}\le\sin^{-1}\left(\frac{x}{3}\right)\le\frac{\pi}{2}.$

$-\pi\le 2\sin^{-1}\left(\frac{x}{3}\right)\le\pi$

$-\pi\le y\le\pi.$

c i) LHS $= \frac{dT}{dt}$ RHS $= -k(T+5)$

$= -kAe^{-kt}$

$= -k(Ae^{-kt}-5+5)$
$= -kAe^{-kt}$
$=$ LHS.

(ii) Initial conditions

$100 = Ae^{-k0} - 5$
$A = 105.$

ii) After 20 minutes

$40 = 105e^{-20k} - 5.$

$\frac{45}{105} = e^{-20k}$

$-20k = \ln\frac{3}{7}$

$k = \frac{\ln\frac{3}{7}}{-20}.$

At 0°C.

$0 = 105e^{-kt} - 5.$

$e^{-kt} = \frac{5}{105}$

$t = \frac{\ln\frac{1}{21}}{-k}.$

$t = 72$ minutes.

d i) Since $f(x) = \ln x + x^2 - 4x$ is a continuous function and

$f(3) = \ln 3 + 3^2 - 4\times 3$
$\approx -1.9 < 0$

and
$f(4) = \ln 4 + 4^2 - 4\times 4.$
$\approx 1.4 > 0$

Therefore $f(x) = \ln x - x^2 - 4x$ must have a root between $x=3, x=4.$

ii) $f'(x) = \frac{1}{x} + 2x - 4.$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$

$x_0 = 4$

$x_1 = 4 - \frac{\ln 4 + 16 - 16}{\frac{1}{4} + 8 - 4}$

$\approx 3.67.$

Yes, since we know $f(x)$ has a root between 3 and 4 and this approximation is close to 3 than than the first approximation of 4.

QUESTION 4.

i). $y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{x}{2a}$.

At $P(2ap, ap^2)$

$m_P = \frac{2ap}{2a}$

$= p$.

ii) Similarly to part (i) the tangent at Q is $m_Q = q$.

Thus the gradient of the normal will be $m = -\frac{1}{q}$.

Given that the chord goes through the focus $(0, a)$. then

$a = \left(\frac{p+q}{2}\right)0 - apq$.

$pq = -1$

$\therefore q = -\frac{1}{p}$.

Thus the gradient of the normal at Q will be $-\frac{1}{q} = -\frac{1}{\left(-\frac{1}{p}\right)}$

$= p$.

$\therefore$ Tangent at P is parallel to the normal at Q.

b) i) $\binom{4}{3}\binom{3}{1}\binom{2}{1} = 24$.

ii) $n(\text{Sample space}) = \binom{9}{5} = 126$.

$n(E) = n(4\ lib) + n(3\ lib)$

$= \binom{4}{4}\binom{5}{1} + \binom{4}{3}\binom{5}{2}$.

$= 5 + 40$

$= 45$.

$P(E) = \frac{n(E)}{n(S)}$.

$= \frac{45}{126}$

$= \frac{5}{14}$.

c) i) $R = \sqrt{49+1}$

$= 5\sqrt{2}$

$\tan\alpha = \frac{1}{7}$

$\alpha = 8°8'$

So $7\cos\theta - \sin\theta = 5\sqrt{2}\cos(\theta + 8°8')$.

(ii) $5\sqrt{2}\cos(\theta+8°8') = 5$.

$$\cos(\theta+8°8') = \frac{1}{\sqrt{2}}$$

$$\theta+8°8' = 45°, 315°$$

$$\theta \approx 37°, 307°.$$

d) $P(-1) = (-1)^3 - 3(-1)^2 + a(-1) + b$

$$= -4 - a + b.$$

$$P(3) = (3)^3 - 3(3)^2 + (3)a + b.$$

$$= 3a + b.$$

So $-4 - a + b = 0$ (A)
and $3a + b = 0$ (B)

from (A) $a = b - 4$
sub into (B)

$$3b - 12 + b = 0$$

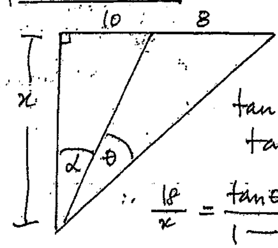
$$b = 3$$

so $a = -1$

$\therefore$ $a = -1$, $b = 3$.

Question (5)

(i)



$$\tan(\theta+\alpha) = \frac{18}{x}$$

$$\tan\alpha = \frac{10}{x}$$

$$\therefore \frac{18}{x} = \frac{\tan\theta + \frac{10}{x}}{1 - \tan\theta\cdot\frac{10}{x}}$$

$$(\tan\theta)x^2 - 8x + 180\tan\theta = 0$$

$$\tan\theta = \frac{8x}{180+x^2}$$ 2 [2]

$$\therefore \theta = \tan^{-1}\left(\frac{8x}{180+x^2}\right)$$

$$\frac{d\theta}{dx} = \frac{(180+x^2)8 - 8x(2x)}{(x^2+180)^2}$$ 1

i.e. $\frac{1440 - 8x^2}{(x^2+180)^2} = 0.$ [2]

$8(x^2 - 180) = 0$, $x^2 = 180$

$x = 6\sqrt{5}$.

Test: 1

θ	13	$6\sqrt{5}$	14
$\frac{d\theta}{dx}$	+	0	-1568

(b) $v = 2(2x-1)^{\frac{1}{2}}$.

(i) $v^2/2 = 4x - 2$

$\therefore \ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = 4.$ [1] 1

(ii) $\frac{dx}{dt} = 2(2x-1)^{\frac{1}{2}}$

$$\frac{dt}{dx} = \frac{1}{2\sqrt{2x-1}}$$

$$\therefore t = \frac{1}{2}(2x-1)^{\frac{1}{2}} + c$$

When $t = 0$, $x = \frac{1}{2}$. $\Rightarrow c = 0$.

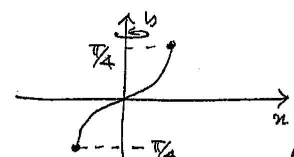
$\therefore 2t = \sqrt{2x-1}$ 2

$1 + 4t^2 = 2x$.

$\therefore x = \frac{1+4t^2}{2}$. [2]

$= 2t^2 + \frac{1}{2}$.

(c)



3

$\left(\sin^2 y = \frac{1-\cos 2y}{2}\right)$ 1

$$V = 2\pi\int_0^{\pi/4}\sin^2 y\,dy$$

$$= \pi\int_0^{\pi/4}(1-\cos 2y)\,dy$$

$$= \pi\left[y - \frac{\sin 2y}{2}\right]_0^{\pi/4}$$

$$= \pi\left(\frac{\pi}{4} - \frac{1}{2}\right)$$ [3]

(d) $\frac{dp}{dt} = 3$, $p = 2\pi r$

$\therefore r = \frac{p}{2\pi}$.

$$A = \pi\left(\frac{p^2}{4\pi^2}\right) = \frac{p^2}{4\pi}$$

12

$$\frac{dA}{dt} = \frac{dA}{dp}\cdot\frac{dp}{dt} = \frac{p}{2\pi}\times 3.$$ 2

$p = 100$

$\therefore \frac{dA}{dt} = \frac{150}{\pi}$ cm²/s

$= \frac{0.015}{\pi}$ m²/s. [2]

Question (6)

6(a) $n = 1$, $6 \nmid 8$.

(i) [1]

(ii) For $n = 1$, $7 + 5 = 12$ and $6 \mid 12$. 1

Assume $S(k)$ is true

i.e. $7^5 + 5 = 6M$, $M \in \mathbb{N}$

Consider $n = k+1$

$7^{k+1} + 5$ 2 [2]

$= 7\cdot 7^k + 5$

$= 7(7^5 + 5) - 30$

$= 42M - 30 = 6(7M - 5)$

$= 6N$ where $N = 7M - 5 \in \mathbb{N}$.

$\because$ $S(k+1)$ is true when $S(k)$ is true and $S(1)$ is true

$\therefore$ by mathematical induction

$6 \mid (7^n + 5)$ $\forall n \in \mathbb{N}$

QUESTION 7

(i) $t = \dfrac{x}{v\cos\vartheta}$

$$y = -\frac{gx^2}{2v^2\cos^2\vartheta} + \frac{vx\sin\vartheta}{v\cos\vartheta}$$

$$y = x\tan\vartheta - \frac{gx^2}{2v^2\cos^2\vartheta}$$

(ii)
At P, $y = k = h\tan\vartheta, x = h$

$$h\tan\beta = h\tan\vartheta - \frac{gh^2}{2v^2\cos^2\vartheta} \text{ from (i)}$$

$$\frac{gh^2}{2v^2\cos^2\vartheta} = h(\tan\vartheta - \tan\beta)$$

$$h = \frac{(\tan\vartheta - \tan\beta)2v^2\cos^2\vartheta}{g}$$

(iii)

$$OP = \frac{h}{\cos\beta}$$

$$= \frac{(\tan\vartheta - \tan\beta)2v^2\cos^2\vartheta}{g\cos\beta} \text{[from (ii)]}$$

$$= \frac{\left(\frac{\sin\vartheta}{\cos\vartheta} - \frac{\sin\beta}{\cos\beta}\right)2v^2\cos^2\vartheta}{g\cos\beta}$$

$$= \frac{(\sin\vartheta\cos\beta - \sin\beta\cos\vartheta)2v^2\cos\vartheta}{g\cos^2\beta}$$

$$= \frac{2v^2\sin(\vartheta-\beta)\cos\vartheta}{g\cos^2\beta}$$

(iv)

$$OP = \frac{[\sin(2\vartheta-\beta) - \sin\beta]v^2}{g\cos^2\beta} \text{ (given)}$$

$$\frac{d(OP)}{(d\vartheta)} = \frac{2v^2}{g\cos^2\beta}[2\cos(2\vartheta-\beta)]$$

OP max/min $\cos(2\vartheta-\beta) = 0$

$2\vartheta - \beta = 90^0$

$$\vartheta = \frac{90^0+\beta}{2}$$

$$OP'' = \frac{4v^2}{g\cos^2\beta} \times -2\sin(2\vartheta-\beta)$$

always < 0 as $(2\vartheta-\beta) < 180^0$

$\therefore$ max val OP when $\vartheta = \dfrac{90^0+\beta}{2}$

$$\text{Max val. OP} = \frac{v^2(\sin 90^0 - \sin\beta)}{g(1-\sin^2\beta)}$$

$$= \frac{v^2(1-\sin\beta)}{g(1-\sin^2\beta)}$$

$$= \frac{v^2}{g(1+\sin\beta)}$$

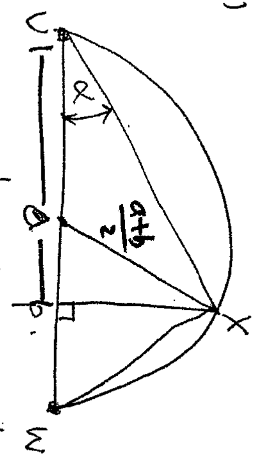
(v)

max val OP when $\vartheta = \dfrac{90^0+\beta}{2}$ [from (iv)]

$$\vartheta = \frac{90^0+14^0}{2}$$

$$\vartheta = 52^0$$

(b)



$OX = \frac{1}{2}UW = a+b$.
(radius = ½ diameter)
$\therefore OX = \dfrac{a+b}{2}$ —— (1) [1]

(ii) Let $\angle XUP = \alpha$.
$\angle UXW = 90^\circ$ (Angle in a semi circle)
$\therefore \angle XWP = 90^\circ - \alpha$.
In $\triangle UXP$, $\triangle XWP$.
$\angle XWP = 90 - \alpha \Rightarrow \angle XWP = \alpha$
(Angle sum of $\triangle XPW$.
$\therefore \triangle UXP \parallel\parallel \triangle XWP$ [1]
(equiangular
In $\triangle XPU$
$XP^2 + a^2 = XU^2$ $\quad \frac{XP}{PW} = \frac{UP}{XP}$
$(XP)^2 = a.b.$

036231139 [12]

In $\triangle XWP$
$XP^2 + b^2 = XW^2$.
In $\triangle UXW$
$(a+b)^2 = XU^2 + XW^2$
$a^2+b^2+2ab = (a^2+b^2) + 2XP^2$
$\therefore XP^2 = ab$ [1].
$\Longrightarrow XP = \sqrt{ab}$.
In $\triangle OXP$ $|OX| \perp XP$
(OX is the hypotenuse) [1]
$\therefore \dfrac{a+b}{2} > \sqrt{ab}$.

(c) $\dot{x} = -15\sin(3t - \frac{\pi}{6})$
(i) $\ddot{x} = -45\cos(3t - \frac{\pi}{6})$
$= -9[5\cos(3t - \frac{\pi}{6})]$
$= -9x$ [1].

(ii) $T = \dfrac{2\pi}{n} = \dfrac{2\pi}{3}$. [1].

(d) (i) $a_k = \binom{12}{k} 5^{12-k} 2^k$. [1]

(ii) $\dfrac{a_{k+1}}{a_k} = \dfrac{\binom{12}{k+1} 5^{11-k} 2^{k+1}}{\binom{12}{k} 5^{12-k} 2^k} = \dfrac{2}{5}\dfrac{(12-k)}{k+1}$ [2]