SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2004

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using black or blue pen.
- Board approved calculators may be used.
- All *necessary* working should be shown in every question if full marks are to be awarded.
- Marks may **NOT** be awarded for messy or badly arranged work.
- Hand in your answer booklets in 3 sections.
  Section A (Questions 1 - 3),
  Section B (Questions 4 - 5) and
  Section C (Questions 6 - 8).
- Start each section in a **NEW** answer booklet.

Total Marks - 120 Marks

- Attempt Sections A - C
- All questions are NOT of equal value.

Examiner: *E. Choy*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

Total marks – 120
Attempt Questions 1 – 8
All questions are of equal value

Answer each section in a SEPARATE writing booklet. Extra writing booklets are available.

SECTION A (Use a SEPARATE writing booklet)

Question 1 (15 marks) **Marks**

(a) Evaluate

(i) $\displaystyle\int_0^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}}$ **1**

(ii) $\displaystyle\int_0^{1} \sqrt{4-x^2}\,dx$ **1**

(iii) $\displaystyle\int_{-1}^{2} x\sqrt{2-x}\,dx$ **1**

(b) Evaluate

(i) $\displaystyle\int_1^{2} \frac{e^{2x}}{e^x-1}dx$ **2**

(ii) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{4+5\sin x}dx$ **4**

(c) (i) If $I_n = \displaystyle\int_0^{\frac{\pi}{4}} \tan^{2n}x\,dx$, $n \geq 0$, show that $I_n + I_{n-2} = \dfrac{1}{2n-1}$ **3**

(ii) Hence, evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \tan^6 x\,dx$ **1**

(d) Evaluate $\displaystyle\int_1^{e} x\ln\left(x^2\right)dx$ **2**

Question 2 (15 marks) **Marks**

(a) (i) Sketch on the same axes the graphs 2

$$y = x + 3 \text{ and } y = 2|x|.$$

(ii) Hence or otherwise:

(α) Solve for x, $2|x| < x + 3$. 2

(β) Sketch the curve $y = \dfrac{2|x|}{x+3}$. 3

(b) Let $f(x) = \dfrac{3}{x-1}$.

On separate diagrams sketch the graphs of the following:

(i) $y = f(|x|)$ 2

(ii) $y^2 = f(x)$ 3

(iii) $y = e^{f(x)}$ 3

SECTION A continued

Question 3 (15 marks) **Marks**

(a) If $z = -1 + i\sqrt{3}$ and $w = 2\text{cis}\dfrac{\pi}{6}$

(i) Find $|z|$. 1

(ii) $\arg z$. 1

(iii) Express z in the form $r\text{cis}\theta$. 1

(iv) Express $z^6 \div w^3$ in the form $r\text{cis}\theta$. 1

(b) (i) Express $\sqrt{5-12i}$ in the form $a + ib$. 2

(ii) Hence describe the locus of the point which represents z on the Argand diagram if 2

$$|z^2 - 5 + 12i| = |z - 3 + 2i|$$

(c) The origin and the points representing the complex numbers z, $\dfrac{1}{z}$ and $z + \dfrac{1}{z}$ are joined to form a quadrilateral.
Write down the conditions for z so that the quadrilateral will be

(i) a rhombus; 1

(ii) a square. 1

(d) (i) Find the equation and sketch the locus of z if 2

$$|z - i| = \text{Im}(z)$$

(ii) Find the least value of $\arg z$ in (i) above. 3

END OF SECTION A

SECTION B (Use a SEPARATE writing booklet)

Question 4 (15 marks) — Marks

(a) $3-i$ is a zero of $P(z)=z^3-4z^2-2z+m$, where m is a real number. **3**

Find m.

(b) If α, β and γ are the roots of $x^3+px+q=0$, find a cubic equation whose roots are α^2, β^2 and γ^2. **3**

(c) Given a real polynomial $Q(x)$, show that if α is a root of $Q(x)-x=0$, then α is also a root of $Q(Q(x))-x=0$. **3**

(d) Use the following identity to answer the following questions.

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$

(i) Solve $16x^5-20x^3+5x=0$ **3**

(ii) Hence show that **3**

$$\cos\frac{\pi}{10}\cos\frac{3\pi}{10}\cos\frac{7\pi}{10}\cos\frac{9\pi}{10}=\frac{5}{16}$$

SECTION B continued

Question 5 (15 marks) — Marks

(a) Let $z=\cos\theta+i\sin\theta$, show that

(i) $z^n+z^{-n}=2\cos n\theta$ **1**

(ii) $z^n-z^{-n}=2i\sin n\theta$ **1**

(b) (i) Show that for any integer k that **2**

$$\left[z-\left(\cos\frac{k\pi}{4}+i\sin\frac{k\pi}{4}\right)\right]\left[z-\left(\cos\frac{(8-k)\pi}{4}+i\sin\frac{(8-k)\pi}{4}\right)\right]=z^2-2z\cos\frac{k\pi}{4}+1$$

(ii) Hence simplify the following products

(α) $\left[z-\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]\left[z-\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)\right]$ **1**

(β) $\left[z-\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)\right]\left[z-\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)\right]$ **1**

(c) Using the results of (b) above, factorise z^4+1 into 2 real quadratic factors. **2**

(d) Using (a) and (c) above, prove the identity **2**

$$\cos 2\theta = 2\cos^2\theta - 1$$

(e) The complex numbers $z=x+iy$, $z_1=-x+iy$ and $z_2=-\frac{2}{z}$ are represented by the points P, P_1 and P_2 in the Argand diagram respectively.

(i) Show that O, P_1 and P_2 are collinear where O is the origin. **3**

(ii) Show that $OP_1 \times OP_2 = 2$ **2**

END OF SECTION B

SECTION C (Use a SEPARATE writing booklet)

Question 6 (15 marks) **Marks**

(a) A particle of mass m is projected vertically upwards with a velocity of u ms^{-1}, with air resistance proportional to its velocity.

(i) Show that after a time t seconds, the height above the ground is **4**

$$x_1 = \frac{g+ku}{k^2}\left(1-e^{-kt}\right)-\frac{gt}{k},$$

where k is a constant and g is the acceleration due to gravity.

(ii) At the same time another particle of mass m is released from rest, from a height h metres vertically above the first particle. You may assume that at time t seconds, its distance from the ground is given by: **4**

$$x_2 = h+\frac{g}{k^2}\left(1-e^{-kt}\right)-\frac{gt}{k}$$

Show that the two particles will meet at time T where

$$T=\frac{1}{k}\ln\left(\frac{u}{u-kh}\right)$$

(b) A vehicle of mass m moves in a straight line subject to a resistance $P+Qv^2$, where v is the speed and P and Q are constants with $Q>0$.

(i) Form an equation of motion for the acceleration of the vehicle. **1**

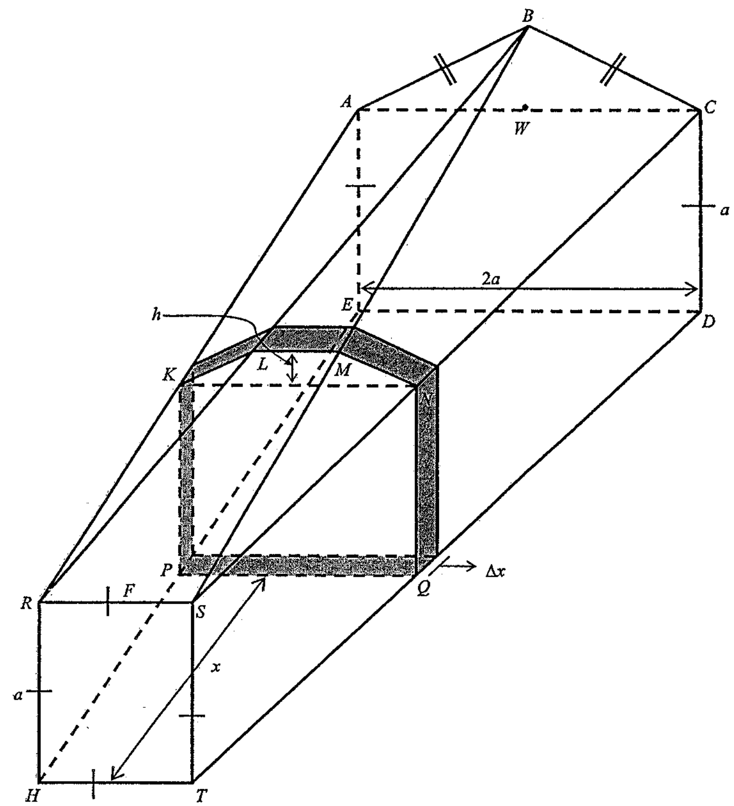
(ii) Hence show that if $P=0$, the distance required to slow down from speed $\frac{3U}{2}$ to speed U is $\frac{m}{Q}\ln\left(\frac{3}{2}\right)$. **3**

(iii) Also show that if $P>0$, the distance required to stop from speed U is given by **3**

$$D=\lambda\ln\left(1+kU^2\right)$$

where k and λ are constants

Question 7 (15 marks) **Marks**



The diagram above shows a solid with a trapezoidal base $EDTH$ of length b metres.
The front end $HTSR$ is a square with side length a metres.
The back is the pentagon $ABCDE$ which consists of the rectangle $ACDE$ with length $2a$ metres and width a metres, surmounted by the equilateral triangle ABC.

Consider a slice of the solid, parallel to the front and the back, with face formed by both the trapezium $KLMN$ and the rectangle $KNQP$, which has thickness Δx and is at a distance x metres from HT.

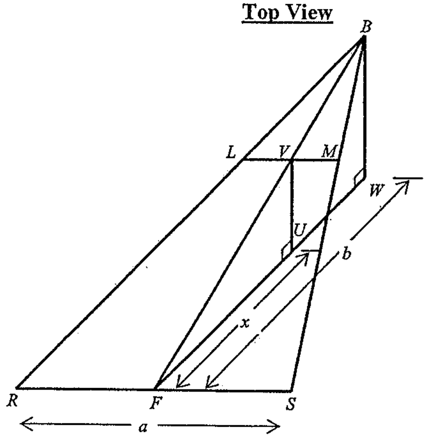
Question 7 continued on page 9

Question 7 continued

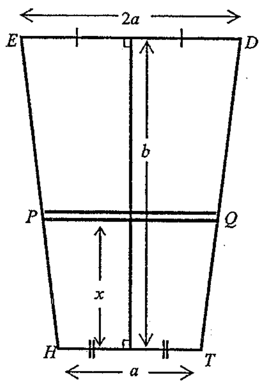
Marks

(i) Show that the height, BW, of the equilateral triangle ABC is $\sqrt{3}a$ metres. **2**

Top View



Base of Solid



(ii) Given that the perpendicular height of the trapezium $KLMN$ is h metres ie $VU = h$, use the similar triangles BWF and VUF, in the Top View, to find h in terms of a, b and x. **3**

(iii) Given that the triangles BLM and BRS are similar, show that **3**

$$LM = \frac{a(b-x)}{b}$$

(iv) Using the cross section of the base, find the length of PQ in terms of a, b and x. **3**

(v) Find the volume of the solid. **4**

Question 8 starts on page 10

SECTION C continued

Question 8 (15 marks)

Marks

(a) If $a > 0$, $b > 0$ and $a + b = t$ show that **3**

$$\frac{1}{a} + \frac{1}{b} \geq \frac{4}{t}$$

(b) There are n $(n > 1)$ different boxes each of which can hold up to $n+2$ books. Find the probability that:

(i) No box is empty when n different books are put into the boxes at random. **1**

(ii) Exactly one box is empty when n different books are put into the boxes at random. **2**

(iii) No box is empty when $n+1$ different books are put into the boxes at random. **2**

(iv) No box is empty when $n+2$ different books are put into the boxes at random. **2**

(c) $PQRS$ is a cyclic quadrilateral such that the sides PQ, QR, RS and SP touch a circle at A, B, C and D respectively.

Prove that:

(i) AC is perpendicular to BD. **2**

(ii) Let the midpoints of AB, BC, CD and DA be E, F, G and H respectively.
Show that E, F, G and H lie on a circle. **3**

End of paper

1(a)(i) $I = \left[\sin^{-1} \frac{x}{2}\right]_0^{\sqrt3}$,

$= \pi/3$.

(ii) $I = \int_0^{\pi/6} 4\cos^2\theta\, d\theta$, put $x = 2\sin\theta$, $dx = 2\cos\theta\, d\theta$; $x=1, \theta=\pi/6$; $x=0, \theta=0$

$= 2\int_0^{\pi/6}(1+\cos 2\theta)\,d\theta$,

$= 2\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/6}$,

$= \frac{\pi}{3}+\frac{\sqrt3}{2}$.

(iii) $I = \int_0^3 (2-u)\,u^{1/2}\,du$, put $u=2-x$, $du=-dx$; $x=-1, u=3$; $x=2, u=0$

$= \int_0^3 (2u^{1/2}-u^{3/2})\,du$,

$= \left[\frac{4u^{3/2}}{3} - \frac{2u^{5/2}}{5}\right]_0^3$,

$= 4\sqrt3 - \frac{18\sqrt3}{5}$,

$= \frac{2\sqrt3}{5}$.

1(b)(i) $I = \int_{e-1}^{e^2-1} \frac{1+u}{u}\,du$, put $u = e^x-1$, $du = e^x dx$; $x=1, u=e-1$; $x=2, u=e^2-1$

$= \left[\ln u + u\right]_{e-1}^{e^2-1}$,

$= \ln(e^2-1)+e^2-1-\ln(e-1) - (e-1)$,

$= e^2-e+\ln(e+1)$.

(ii) $I = \int_0^1 \frac{1}{4+5\times\frac{2t}{1+t^2}} \times \frac{2\,dt}{1+t^2}$, put $t=\tan\frac{x}{2}$, $dt = \frac12\sec^2\frac{x}{2}\,dx$, i.e. $dx = \frac{2\,dt}{1+t^2}$

$= \int_0^1 \frac{dt}{2+2t^2+5t}$, $x=0, t=0$; $x=\frac{\pi}{2}, t=1$

$= \int_0^1 \frac{dt}{(2t+1)(t+2)}$

Now, $\frac{1}{(2t+1)(t+2)} = \frac{A}{2t+1} + \frac{B}{t+2}$

$1 = A(t+2) + B(2t+1)$

Let $t=-\frac12$, $3A/2 = 1$, $A = 2/3$

$t=-2$, $-3B=1$, $B=-1/3$

So, $I = \frac13\int_0^1\left(\frac{2}{2t+1} - \frac{1}{t+2}\right)dt$,

$= \frac13\left[\ln(2t+1) - \ln(t+2)\right]_0^1$,

$= \frac13\left\{\ln\frac31 - \ln\frac32\right\}$,

$= \frac{\ln 2}{3}$.

(c)(i) $I_n = \int_0^{\pi/4} (\tan^2 x)^n\,dx$,

$= \int_0^{\pi/4} (\tan^2 x)^{n-1}(\sec^2 x - 1)\,dx$,

$= \int_0^{\pi/4} \tan^{2n-2} x\; d\tan x - I_{n-1}$,

$I_n + I_{n-1} = \left[\frac{\tan^{2n-1} x}{2n-1}\right]_0^{\pi/4}$,

$= \frac{1}{2n-1}$

(ii) $I_3 = \frac15 - I_2$,

$I_2 = \frac13 - I_1$, $= \frac13 - \int_0^{\pi/4} \tan^2 x\,dx$

$= \frac13 - \int_0^{\pi/4} (\sec^2 x - 1)\,dx$

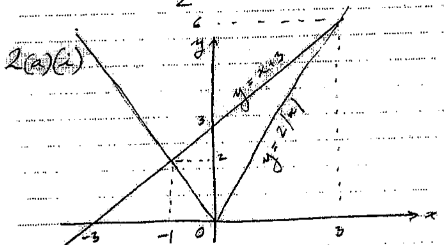
1(d) $I = \left[x^2 \ln x\right]_1^e - \int_1^e x\,dx$, $u = \ln(x^2)$, $u' = \frac{2}{x}$, $v' = x$, $v = \frac{x^2}{2}$

$= e^2 - 0 - \left[\frac{x^2}{2}\right]_1^e$,

$= e^2 - \frac{e^2}{2} + \frac12$,

$= \frac{e^2+1}{2}$

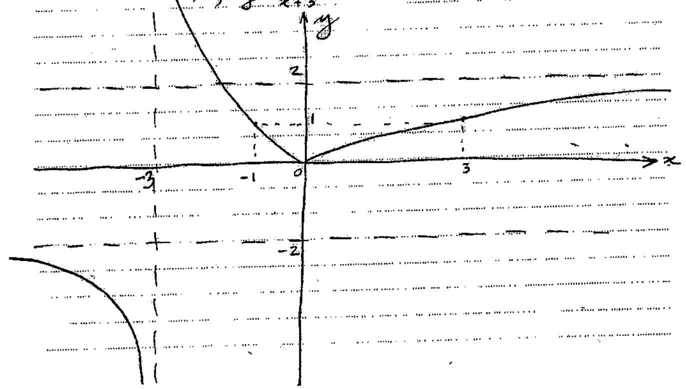
2(a)(i)



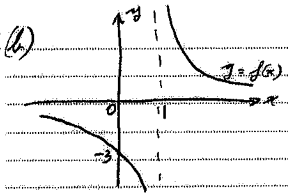
(ii)(α) From the graph $-1 < x < 3$.

(β) If $x<0$, $y = \frac{-2x}{x+3} = -2 + \frac{6}{x+3}$,

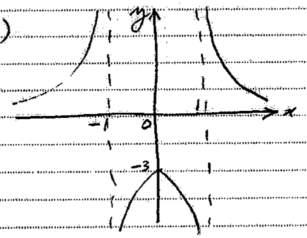
$x \geq 0$, $y = \frac{2x}{x+3} = 2 - \frac{6}{x+3}$



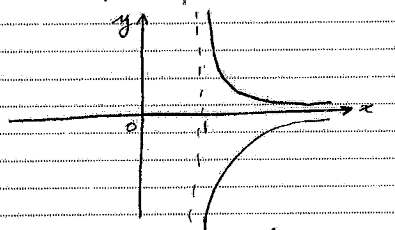
2(b)



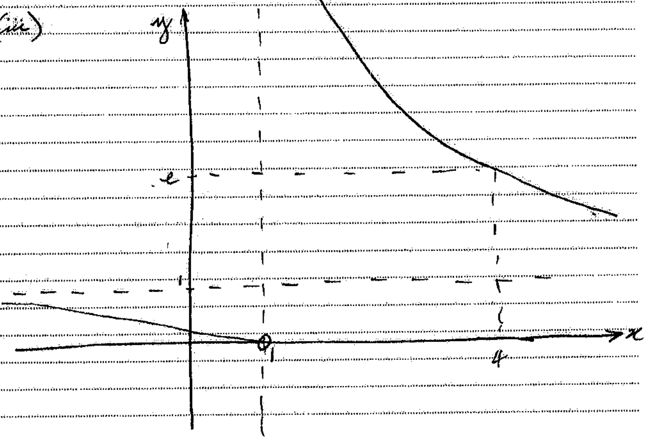
(i)



(ii)



(iii)



3(a)(i) $|z| = \sqrt{1+3}$,

$= 2.$

(ii) $\arg z = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right)$,

$= \frac{2\pi}{3}$

(iii) $z = 2 \text{ cis } {}^{2\pi}/_{3}$.

(iv) $z^6 \div w^3 = \frac{2^6}{2^3} \text{cis}\left(6 \times \frac{2\pi}{3} - 3 \times \frac{\pi}{6}\right)$

$= 8 \text{ cis}(7\pi/2)$ or $8 \text{ cis}(-\pi/2)$

$= -8i.$

(b)(i) $5 - 12i = a^2 + 2abi - b^2$

$a^2 + b^2 = 13$

$a^2 - b^2 = 5$

$ab = -6$

$2a^2 = 18$

$a = \pm 3$

$b = \mp 2$

$\therefore \sqrt{5-12i} = \pm(3 - 2i)$

3(b)(ii) $z^2 - (5 - 12i) = z^2 - (3 - 2i)^2$

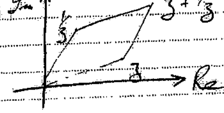
$= (z + 3 - 2i)(z - 3 + 2i)$

$|z + 3 - 2i||z - 3 + 2i| = |z - 3 + 2i|$

$|z + 3 - 2i| = 1$

So locus of z is a circle centre $(-3, 2)$, radius 1.

(c)(i)



$|z| = \left|\frac{1}{z}\right|$ for a rhombus

so if $z = r \text{ cis } \theta$

$r = \frac{1}{r}$

$r^2 = 1$

$r = 1$ (taking +ve root)

$\therefore |z| = 1$ is the condition.

(ii) $\pm iz = {}^{1}/_{z}$

$\theta \pm {}^{\pi}/_{2} = -\theta + n\pi$

$2\theta = \pm {}^{\pi}/_{2} + n\pi$

$\theta = \pm {}^{\pi}/_{4}, \pm {}^{3\pi}/_{4}$

$\therefore$ Conditions are $|z| = 1$.

$\arg z = \pm {}^{\pi}/_{4}, \pm {}^{3\pi}/_{4}$.

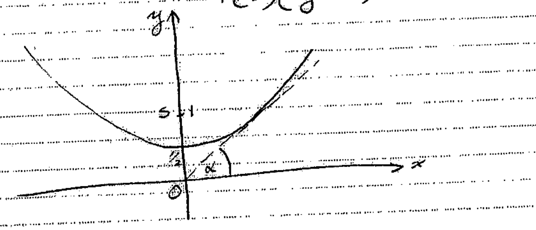
(d)(i) Let $z = x + iy$

$\sqrt{x^2 + (y-1)^2} = y$

$x^2 + y^2 - 2y + 1 = y^2$

$x^2 = 2y - 1$

$= 4(\tfrac{1}{2})(y - \tfrac{1}{2})$



(ii) $y = \frac{x^2+1}{2}$; parabola

$y = mx$, tangent

$mx = \frac{x^2+1}{2}$

$x^2 - 2mx + 1 = 0$

$\Delta = 0$ at tangent

$4m^2 - 4 = 0$

$m = \pm 1$

$\alpha = \pi/4, \ 3\pi/4$

$\therefore$ Minimum argument $\pi/4$.

Section B

Question 4

(a) $P(3-i) = (3-i)^3 - 4(3-i)^2 - 2(3-i) + m = 0$

$= 18 - 26i - 32 + 24i - 6 + 2i + m = 0$

$\Rightarrow (m-20) + i(0) = 0$

$\therefore m = 20$

(b) Required equation is $P(\sqrt{x}) = (\sqrt{x})^3 + p\sqrt{x} + q = 0$

ie $x\sqrt{x} + p\sqrt{x} = -q$

$\Rightarrow [x\sqrt{x} + p\sqrt{x}]^2 = [-q]^2$

$\therefore x^3 + 2px^2 + p^2x - q^2 = 0$

(c) If α is a root $\Rightarrow Q(\alpha) - \alpha = 0$

ie $Q(\alpha) = \alpha$

Now $Q[Q(\alpha)] - \alpha = Q[\alpha] - \alpha$
$= \alpha - \alpha$
$= 0$
} using $Q(\alpha) = \alpha$

(d) (i) let $x = \cos\theta \Rightarrow \cos 5\theta = 16x^5 - 20x^3 + 5x = 0$

Now $\cos 5\theta = 0$

$\Rightarrow 5\theta = 2k\pi \pm \frac{\pi}{2}$

$\theta = \frac{2k\pi}{5} \pm \frac{\pi}{10}$

using $\theta = \frac{2k\pi}{5} + \frac{\pi}{10}$ when $k = 0, \ \theta = \frac{\pi}{10}$

$k = 1, \ \theta = \frac{\pi}{2}$

$k = -1, \ \theta = -\frac{3\pi}{10}$

$k = 2, \ \theta = \frac{9\pi}{10}$

$k = -2, \ \theta = -\frac{7\pi}{10}$

Now roots of $16x^5 - 20x^3 + 5x = 0$ are of the form $x = \cos\theta$ ie $x = \cos\frac{\pi}{10}, \cos\frac{\pi}{2}, \cos\left(-\frac{3\pi}{10}\right)$, $\cos\frac{9\pi}{10}, \cos\left(-\frac{7\pi}{10}\right)$

$\Rightarrow$ Roots are $\cos\frac{\pi}{10}, \cos\frac{3\pi}{10}, \cos\frac{9\pi}{10}, \cos\frac{7\pi}{10}, \cos\frac{\pi}{2}$

[since $\cos\theta = \cos(-\theta)$]

(ii) $16x^4 - 20x^2 + 5 = 0$ has roots $x = \cos\frac{\pi}{10}, \cos\frac{3\pi}{10}, \cos\frac{7\pi}{10}, \cos\frac{9\pi}{10}$ $\therefore \cos\frac{\pi}{10} \cdot \cos\frac{3\pi}{10} \cdot \cos\frac{7\pi}{10} \cdot \cos\frac{9\pi}{10} = \frac{5}{16}$ Prod. of Roots

Question 5

(a) $z^n = \cos(n\theta) + i\sin(n\theta)$ — (A)

$z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$

$\therefore z^{-n} = \cos(n\theta) - i\sin(n\theta)$ — (B)

(i) $z^n + z^{-n} = 2\cos n\theta$ (A)+(B)

(ii) $z^n - z^{-n} = 2i\sin n\theta$ (A)−(B)

(b) (i)

$\left[z - \text{cis}\frac{k\pi}{4}\right]\left[z - \text{cis}\frac{(8-k)\pi}{4}\right]$

$= z^2 - \left[\text{cis}\frac{k\pi}{4} + \text{cis}\left(-\frac{k\pi}{4}\right)\right]z + \left[\text{cis}\frac{k\pi}{4} \cdot \text{cis}\left(-\frac{k\pi}{4}\right)\right]$

$= z^2 - 2\cos\frac{k\pi}{4} z + \text{cis}\, 0$ (from (a))

$= z^2 - 2\cos\frac{k\pi}{4} z + 1$

(ii) (α) let $k = 1$ in (i)

$\Rightarrow$ LHS $= z^2 - 2z\cos\frac{\pi}{4} + 1$

$= z^2 - \frac{2z}{\sqrt{2}} + 1$

(β) let $k = 3$ in (i)

$\Rightarrow$ LHS $= z^2 - 2z\cos\frac{3\pi}{4} + 1$

$= z^2 + \frac{2}{\sqrt{2}}z + 1$

(c) Since non-real zeros occur in conjugate pairs $\Rightarrow$ quadratic factors

$z^4 + 1 = \left[z - \text{cis}\frac{\pi}{4}\right]\left[z - \text{cis}\frac{7\pi}{4}\right]\left[z - \text{cis}\frac{3\pi}{4}\right]\left[z - \text{cis}\frac{5\pi}{4}\right]$

$= \left[z^2 - \frac{2}{\sqrt{2}}z + 1\right]\left[z^2 + \frac{2}{\sqrt{2}}z + 1\right]$

using b(ii)

$= \left[z^2 - \sqrt{2}z + 1\right]\left[z^2 + \sqrt{2}z + 1\right]$

(d) $z^2 + z^{-2} = 2\cos 2\theta$

$\frac{z^4+1}{z^2} = 2\cos 2\theta$

$z^4 + 1 = 2z^2\cos 2\theta$

$[z + 2\sqrt{z} + 1][z - 2\sqrt{z} + 1] = 2z^2\cos 2\theta$

$\left[\left(z + \frac{1}{z}\right) + \sqrt{2}\right]\left[\left(z + \frac{1}{z}\right) - \sqrt{2}\right] = 2\cos 2\theta$

$[2\cos\theta + \sqrt{2}][2\cos\theta - \sqrt{2}] = 2\cos 2\theta$

$4\cos^2\theta - 2 = 2\cos 2\theta$

$2\cos^2\theta - 1 = \cos 2\theta$

(e)

(i) let $\arg z = \theta$

$\therefore \arg z_1 = \pi - \theta$

$\arg z_2 = \arg\left(-\frac{2}{z}\right) = \arg(-2) - \arg(z)$

$\Rightarrow \arg z_2 = \pi - \theta$

$\arg z_1 = \arg z_2 = \pi - \theta$

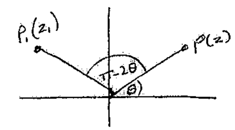
Since $|z_1| \neq |z_2|$ then O, P_1, P_2 collinear

$[z_1 = |z_1| \arg(\pi - \theta) \ ; \ z_2 = |z_2| \arg(\pi - \theta)]$

(ii) $OP_1 = \sqrt{(-x)^2 + (y)^2} = \sqrt{x^2+y^2}$

$OP_2 = \sqrt{\left(\frac{-2x}{x^2+y^2}\right)^2 + \left(\frac{2y}{x^2+y^2}\right)^2} = \frac{2}{\sqrt{x^2+y^2}}$

$\Rightarrow OP_1 \times OP_2 = \sqrt{x^2+y^2} \times \frac{2}{\sqrt{x^2+y^2}} = 2$



$\arg z = \theta$

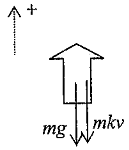
$\arg(z_1 - 0) = \pi - \theta$

$\arg(z_2 - 0) = \pi - \theta$

Section C

Q6

(a) With $R \propto v$, to make the algebra easier take $R = mkv$



(i)
$$m\frac{dv}{dt} = -(mg + mkv)$$
$$\frac{dv}{dt} = -(g + kv) \Rightarrow \frac{dt}{dv} = -\frac{1}{g+kv} = -\frac{1}{k}\left(\frac{k}{g+kv}\right)$$
$$\therefore t = -\frac{1}{k}\ln|g+kv| + c_1$$
$$(t = 0, v = u)$$
$$c_1 = \frac{1}{k}\ln|g+ku| \Rightarrow t = -\frac{1}{k}\ln\left|\frac{g+kv}{g+ku}\right|$$
$$\therefore \frac{g+kv}{g+ku} = e^{-kt}$$
$$\therefore g + kv = (g+ku)e^{-kt} \Rightarrow v = \frac{1}{k}\left[(g+ku)e^{-kt} - g\right]$$
$$x = \int \frac{1}{k}\left[(g+ku)e^{-kt} - g\right]dt$$
$$= \frac{1}{k}\left[\frac{g+ku}{-k}e^{-kt} - gt\right] + c_2$$
$$(t = 0, x = 0)$$
$$\therefore c_2 = \frac{g+ku}{k^2}$$
$$x = \frac{1}{k}\left[-\frac{g+ku}{k}e^{-kt} - gt\right] + \frac{g+ku}{k^2}$$
$$= \frac{g+ku}{k^2}\left(1 - e^{-kt}\right) - \frac{gt}{k}$$

QED

(ii) The two particles meet when $x_1 = x_2$

[**NB** You are allowed to assume the formula for x_2 !]

ie
$$\frac{g+ku}{k^2}\left(1-e^{-kt}\right) - \frac{gt}{k} = h + \frac{g}{k^2}\left(1-e^{-kt}\right) - \frac{gt}{k}$$
$$\therefore \frac{g}{k^2}\left(1-e^{-kt}\right) + \frac{u}{k}\left(1-e^{-kt}\right) - \frac{gt}{k} = h + \frac{g}{k^2}\left(1-e^{-kt}\right) - \frac{gt}{k}$$
$$\therefore \frac{u}{k}\left(1-e^{-kt}\right) = h$$
$$\therefore 1 - e^{-kt} = \frac{hk}{u} \Rightarrow e^{-kt} = 1 - \frac{hk}{u} = \frac{u-hk}{u}$$
$$\therefore -kt = \ln\left(\frac{u-hk}{u}\right) \Rightarrow kt = \ln\left(\frac{u}{u-hk}\right)$$
$$\therefore t = \frac{1}{k}\ln\left(\frac{u}{u-hk}\right)$$

(b) (i) $ma = mv\frac{dv}{dx} = -\left(P+Qv^2\right) \Rightarrow a = v\frac{dv}{dx} = -\frac{1}{m}\left(P+Qv^2\right)$

(ii) If $P = 0$ then $\frac{dv}{dx} = -\frac{Q}{m}v \Rightarrow \frac{dx}{dv} = -\frac{m}{Qv}$

If we transform the problem so that we take the *distance travelled* being from $x = 0$ (when $v = 3U/2$) to $x = D$ (when $v = U$) then

$$\int_0^D \frac{dx}{dv}dv = \int_0^D dx = -\frac{m}{Q}\int_{\frac{3U}{2}}^{U} \frac{dv}{v}$$

$$\therefore D = \left[-\frac{m}{Q}\ln|v|\right]_{\frac{3U}{2}}^{U} = -\frac{m}{Q}\ln\left(\frac{U}{\frac{3U}{2}}\right) = -\frac{m}{Q}\ln\left(\frac{2}{3}\right) = \frac{m}{Q}\ln\left(\frac{3}{2}\right)$$

QED

(iii) If $P > 0$ then $\frac{dv}{dx} = -\left(\frac{P+Qv^2}{mv}\right) \Rightarrow \frac{dx}{dv} = -\frac{mv}{P+Qv^2}$

If we transform the problem so that we take the *distance travelled* being from $x = 0$ (when $v = U$) to $x = D$ (when $v = 0$) then

$$\int_0^D \frac{dx}{dv}dv = \int_0^D dx = -\int_U^0 \frac{mvdv}{P+Qv^2} = -\frac{m}{2Q}\int_U^0 \frac{2vdv}{P+Qv^2}$$

$$D = -\frac{m}{2Q}\left[\ln\left|P+Qv^2\right|\right]_U^0 = -\frac{m}{2Q}\ln\left(\frac{P}{P+QU^2}\right)$$

$$= \frac{m}{2Q}\ln\left(\frac{P+QU^2}{P}\right)$$

$$= \frac{m}{2Q}\ln\left(1+\frac{Q}{P}U^2\right)$$

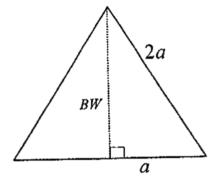
$$= \lambda\ln\left(1+kU^2\right)$$

where $\lambda = \frac{m}{2Q}$ and $k = \frac{Q}{P}$

QED

Q7

(i)



By Pythagoras' Theorem

$4a^2 = BW^2 + a^2$

$\therefore BW^2 = 3a^2$

$\therefore BW = \sqrt{3}a$

(ii) Since $\Delta BWF \,|||\, \Delta VUF$

$\therefore \frac{VU}{BW} = \frac{UF}{FW} \Rightarrow \frac{h}{\sqrt{3}a} = \frac{x}{b}$

$\therefore h = \frac{ax\sqrt{3}}{b}$

(iii) Since $\Delta BWF \,|||\, \Delta VUF$ then $\frac{VF}{BF} = \frac{VU}{BW} = \frac{h}{\sqrt{3}a}$

$BV = BF - VF$

$\Delta BLM \,|||\, \Delta RFS$ then $\frac{BV}{BF} = \frac{LM}{RS} \Rightarrow \frac{BF-VF}{BF} = \frac{LM}{a}$

$\therefore 1 - \frac{VF}{BF} = \frac{LM}{a} \Rightarrow 1 - \frac{h}{\sqrt{3}a} = \frac{LM}{a}$

$\therefore 1 - \frac{\frac{ax\sqrt{3}}{b}}{\sqrt{3}a} = \frac{LM}{a} \Rightarrow \frac{LM}{a} = 1 - \frac{x}{b} = \frac{b-x}{b}$

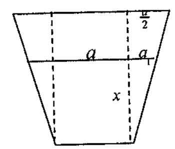
$\therefore LM = \frac{a(b-x)}{b}$

QED

(iv) **Clearly** when $x = 0$ then $PQ = a$ and when $x = b$ then $PQ = 2a$, so given the linear relationship of PQ in terms of x then

$$PQ - a = \frac{2a-a}{b}(x-0) \Rightarrow PQ = \frac{a}{b}x + a$$

Alternative solution



$PQ = a + 2a_1$

$\frac{a_1}{a/2} = \frac{x}{b} \Rightarrow 2a_1 = \frac{a}{b}x$

$\therefore PQ = \frac{a}{b}x + a$

(v) Area of slice is area of trapezium *KLMN* and rectangle *KNQP*

$$KLMN = \frac{1}{2}\times\frac{ax\sqrt{3}}{b}\times\left[\frac{a(b-x)}{b}+\frac{a}{b}x+a\right]$$

$$=\frac{a^2x\sqrt{3}}{b}$$

$$KNQP = a\times\left(\frac{a}{b}x+a\right)=a^2\left(\frac{x}{b}+1\right)$$

So cross sectional area is given by

$$\frac{a^2x\sqrt{3}}{b}+a^2\left(\frac{x}{b}+1\right)$$

$$=\frac{a^2x\sqrt{3}}{b}+a^2\left(\frac{x+b}{b}\right)$$

$$=\frac{a^2\left[x\left(1+\sqrt{3}\right)+b\right]}{b}$$

$$=\frac{a^2}{b}\left[x\left(1+\sqrt{3}\right)+b\right]$$

So the cross sectional volume is $\frac{a^2}{b}\left[x\left(1+\sqrt{3}\right)+b\right]\Delta x$

So the volume, V, is given by $\int_0^b \frac{a^2}{b}\left[x\left(1+\sqrt{3}\right)+b\right]dx$

$$V=\frac{a^2}{b}\int_0^b\left[x\left(1+\sqrt{3}\right)+b\right]dx$$

$$=\frac{a^2}{b}\left[\left(1+\sqrt{3}\right)\frac{x^2}{2}+bx\right]_0^b$$

$$=\frac{a^2}{b}\left[\left(1+\sqrt{3}\right)\frac{b^2}{2}+b^2\right]$$

$$=\frac{a^2b}{2}\left(3+\sqrt{3}\right)$$

[**NB** This is not a solid formed by rotation, so π *shouldn't* appear in the answer!]

Q8

(a)

Method 1	**Method 2**
$\frac{1}{a}+\frac{1}{b}-\frac{4}{t}=\frac{1}{a}+\frac{1}{b}-\frac{4}{a+b}$	$\left(\sqrt{a}-\sqrt{b}\right)^2\ge 0\Rightarrow a+b\ge 2\sqrt{ab}$
$=\frac{b(a+b)+a(a+b)-4ab}{ab(a+b)}$	$\therefore\frac{1}{a+b}\le\frac{1}{2\sqrt{ab}}\Rightarrow\frac{1}{\sqrt{ab}}\ge\frac{2}{a+b}$
$=\frac{a^2-2ab+b^2}{ab(a+b)}$	Also $\frac{1}{a}+\frac{1}{b}\ge\frac{2}{\sqrt{ab}}$
$=\frac{(a-b)^2}{ab(a+b)}$	So $\frac{1}{a}+\frac{1}{b}\ge\frac{2}{\sqrt{ab}}\ge\frac{4}{a+b}=\frac{4}{t}$
≥ 0	
$\therefore\frac{1}{a}+\frac{1}{b}\ge\frac{4}{t}$	

Method 3 (*reductio ad absurdum*)

Assume $\frac{1}{a}+\frac{1}{b}<\frac{4}{t}$

$\therefore\frac{a+b}{ab}<\frac{4}{t}$

$\therefore(a+b)^2<4ab \qquad (\because t=a+b)$

$\therefore(a+b)^2-4ab=(a-b)^2<0$

This last statement is clearly a contradiction as $k^2\ge 0, k\in\mathbb{R}$

So the original assumption was false

$$\therefore\frac{1}{a}+\frac{1}{b}<\frac{4}{t}$$

(b) (i) The total number of different outcomes:
The first book can go in any of n boxes, so there is a total of n^n different arrangements.
If there are to be no empty boxes, then the first book can go in any of n boxes, the next book only has $n-1$ boxes and so on. A total of $n!$

So the probability of no empty box is $\frac{n!}{n^n}$

(ii) For exactly one empty box, one box must have 2 books in it.
So we have to pick the empty box, this can be done in n ways.
Then we have to pick the box to have the two books, this can be in done in $n-1$ ways.

Then we have $\binom{n}{2}$ ways of picking the two books that will go in the one box, leaving $(n-2)!$ ways of arranging the other books.

A total of $n\times(n-1)\times\binom{n}{2}\times(n-2)!=\binom{n}{2}n!$

So the probability is $\frac{\binom{n}{2}n!}{n^n}$ or $\frac{n(n-1)n!}{2n^n}=\frac{(n-1)n!}{2n^{n-1}}$

(iii) With $n+1$ books to be distributed, this can be done in n^{n+1} ways because the first book has n boxes, the second book has n boxes and so on until the $(n+1)^{st}$ book.
With no box to be empty, 1 box must have 2 books in it.
We can choose this book in n ways. We can choose the 2 books in $\binom{n+1}{2}$ ways. The remaining books can be distributed in $(n-1)!$ ways.

A total of $n\times\binom{n+1}{2}\times(n-1)!=\binom{n+1}{2}n!$ ways.

So the probability is $\frac{n!\binom{n+1}{2}}{n^{n+1}}$ or $\frac{n(n+1)!}{2n^{n+1}}=\frac{(n+1)!}{2n^n}$

(iv) With $n+2$ books to be distributed over n boxes this can be done in n^{n+2} ways.
If no box is to be empty there are two cases:
Case 1: 1 box has 3 books in it;
Case 2: 2 boxes have 2 books in it.

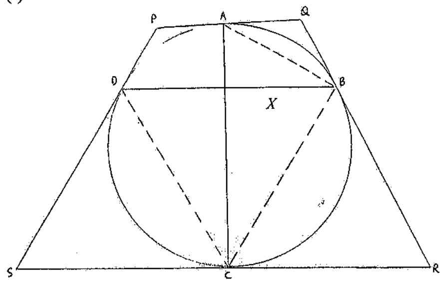
Case 1	Case 2
Pick the box to have 3 books, this can be done in n ways. Pick the 3 books, this can be done in $\binom{n+2}{3}$ ways. The remaining books can be distributed in $(n-1)!$ ways. A total of $\binom{n+2}{3}\times n\times(n-1)!$ ie $\frac{n(n+2)!}{6}$ ways	Pick the 2 boxes to have the 2 books this can be done in $\binom{n}{2}$ ways. Pick 2 books to go into the first of these boxes ie $\binom{n+2}{2}$ ways, then two books to go into the second box ie $\binom{n}{2}$ ways. Then the remaining books to be distributed in $(n-2)!$ ways. A total of $\binom{n}{2}^2\times\binom{n+2}{2}\times(n-2)!$ ie $\frac{n(n-1)(n+2)!}{8}$ ways

So a total number of $\frac{n(n+2)!}{6}+\frac{n(n-1)(n+2)!}{8}$ ways ie

$$\frac{4n(n+2)!+3n(n-1)(n+2)!}{24}=\frac{n(3n+1)(n+2)!}{24}\text{ ways}$$

So the probability is $\frac{n(3n+1)(n+2)!}{24n^{n+2}}=\frac{(3n+1)(n+2)!}{24n^{n+1}}$

(c) (i)

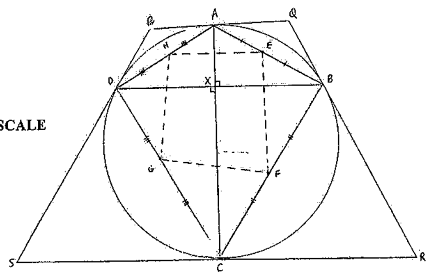


NOT TO SCALE

Let $\angle S = 2x$, then $\angle Q = 180 - 2x$ ($PQRS$ is a cyclic quadrilateral)
Also ΔSDC is isosceles, so $\angle SCD = 90 - x$.
$\angle DBC = \angle SCD = 90 - x$ (alternate segment theorem)
Similarly ΔSDC is isosceles, so $\angle QAB = x$.
Similarly $\angle BCA = \angle QAB = x$ (alternate segment theorem)
So $\angle CXB = 90°$ (angle sum of triangle)
$\therefore AC \perp BD$ **QED**

(c) (ii)

NOT TO SCALE



Lemma: The midpoints of a quadrilateral form a parallelogram

Proof: $AH : HD = AE : EB = 1:1$
$HE \parallel DB$ (Midpoint Theorem for Triangles)
Similarly $GF \parallel DB \Rightarrow HE \parallel FG$
Similarly $HG \parallel AC$ & $AC \parallel EF \Rightarrow HG \parallel EF$.
$\therefore EFGH$ is a parallelogram. **QED**

$\because AC \perp BD$, $HE \parallel DB$ & $GF \parallel DB$ and $HG \parallel AC$ & $AC \parallel EF$
$\therefore \angle HGF = \angle GFE = \angle FEH = \angle EHG = 90°$
$\therefore E, F, G$ and H are concyclic (All rectangles are concyclic)

QED