SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2005
HIGHER SCHOOL CERTIFICATE
TRIAL PAPER

Mathematics Extension 2

General Instructions

- Reading Time – 5 Minutes
- Working time – 3 Hours
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- Each Section is to be returned in a separate bundle.
- All necessary working should be shown in every question.

Total Marks – 120

- Attempt questions 1 – 8

Examiner: *C.Kourtesis*

NOTE: This is a trial paper only and does not necessarily reflect the content or format of the final Higher School Certificate examination paper for this subject.

Section A
(Start a new answer sheet.)

Question 1. (15 marks)

	Marks
(a) Evaluate $\int_0^2 \frac{3}{4+x^2}dx$.	2
(b) Find $\int \cos x \sin^4 x\, dx$.	1
(c) Use integration by parts to find $\int te^{-t}dt$.	2
(d) (i) Find real numbers a and b such that $\frac{1}{x(\pi-2x)} = \frac{a}{x} + \frac{b}{\pi-2x}$.	2
(ii) Hence find $\int \frac{dx}{x(\pi-2x)}$.	2
(e) Evaluate $\int_{-3}^{3}(2-\lvert x\rvert)dx$.	2
(f) (i) Use the substitution $x = a - t$ to prove that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$.	2
(ii) Hence evaluate $\int_0^{\frac{\pi}{2}} \log_e(\tan x)dx$	2

Question 2. (15 marks)

Marks

(a) If $z=2+i$ and $w=-1+2i$ find 2

$$\operatorname{Im}(z-w).$$

(b) On an Argand diagram shade the region that is satisfied by both the conditions 2

$$\operatorname{Re}(z)\geq 2 \text{ and } |z-1|\leq 2.$$

(c) If $|z|=2$ and $\arg z=\theta$ determine 3

(i) $\left|\frac{i}{z^2}\right|$ (ii) $\arg\left(\frac{i}{z^2}\right)$

(d) If for a complex number z it is given that $\overline{z}=z$ where $z\neq 0$, determine the locus of z. 2

(e) A complex number z is such that $\arg(z+2)=\frac{\pi}{6}$ and $\arg(z-2)=\frac{2\pi}{3}$. 3

Find z, expressing your answer in the form $a+ib$ where a and b are real.

(f) The complex numbers z_1, z_2 and z_3 are represented in the complex plane by the points P, Q and R respectively. If the line segments PQ and PR have the same length and are perpendicular to one another, prove that: 3

$$2z_1^2+z_2^2+z_3^2=2z_1(z_2+z_3)$$

Section B
(Start a new answer sheet.)

Question 3. (15 marks)

Marks

(a) If $2-3i$ is a zero of the polynomial z^3+pz+q where p and q are real, find the values of p and q. 3

(b) If α, β and γ are roots of the equation $x^3+6x+1=0$ find the polynomial equation whose roots are $\alpha\beta$, $\beta\gamma$ and $\alpha\gamma$. 2

(c) Consider the function $f(x)=3\left(\frac{x+4}{x}\right)^2$.

(i) Show that the curve $y=f(x)$ has a minimum turning point at $x=-4$ and a point of inflexion at $x=-6$. 5

(ii) Sketch the graph of $y=f(x)$ showing clearly the equations of any asymptotes. 2

(d) Use mathematical induction to prove that 3

$$n!>2^n \text{ for } n>3 \text{ where } n \text{ is an integer.}$$

Question 4 (15 marks)

(a) If $f(x)=\sin x$ for $-\pi \le x \le \pi$ draw neat sketches, on separate diagrams, of:

(i) $y=[f(x)]^2$ **2**

(ii) $y=\dfrac{1}{f\left(x+\frac{\pi}{2}\right)}$ **2**

(iii) $y^2=f(x)$ **2**

(iv) $y=f\left(\sqrt{|x|}\right)$ **2**

(b) Show that the equation of the tangent to the curve $x^{\frac{1}{2}}+y^{\frac{1}{2}}=a^{\frac{1}{2}}$ at the point $P(x_0,y_0)$ on the curve is $xx_o^{-\frac{1}{2}}+yy_0^{-\frac{1}{2}}=a^{\frac{1}{2}}$. **3**

(c) Consider the polynomial $P(x)=x^5-ax+1$. By considering turning points on the curve $y=P(x)$, prove that $P(x)=0$ has three distinct roots if **4**

$a>5\left(\frac{1}{2}\right)^{\frac{8}{5}}$.

Section C
(Start a new answer booklet)

Question 5 (15 marks)

Marks

(a) A particle of mass m is thrown vertically upward from the origin with initial speed V_0. The particle is subject to a resistance equal to mkv, where v is its speed and k is a positive constant.

(i) Show that until the particle reaches its highest point the equation of motion is **1**

$$\ddot{y}=-(kv+g)$$

where y is its height and g is the acceleration due to gravity.

(ii) Prove that the particle reaches its greatest height in time T given by **4**

$$kT=\log_e\left[1+\frac{kV_0}{g}\right].$$

(iv) If the highest point reached is at a height H above the ground prove that **4**

$$V_0=Hk+gT.$$

(b) If α and β are roots of the equation $z^2-2z+2=0$

(i) find α and β in mod-arg form. **3**

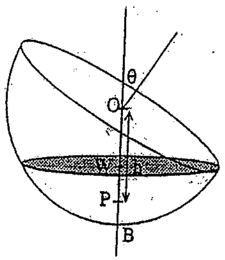
(ii) show that $\alpha^n+\beta^n=\sqrt{2^{n+2}}.\left[\cos\frac{n\pi}{4}\right]$. **3**

Question 6 (15 marks)

(a) A group of 20 people is to be seated at a long rectangular table, 10 on each side. There are 7 people who wish to sit on one side of the table and 6 people who wish to sit on the other side. How many seating arrangements are possible? **2**

(b) The area enclosed by the curves $y=\sqrt{x}$ and $y=x^2$ is rotated about the y axis through one complete revolution. Use the cylindrical shell method to find the volume of the solid that is generated. **3**

(c) The diagram shows a hemi-spherical bowl of radius r. The bowl has been tilted so that its axis is no longer vertical, but at an angle θ to the vertical. At this angle it can hold a volume V of water.



The vertical line from the centre O meets the surface of the water at W and meets the bottom of the bowl at B. Let P between W and B, and let h be the distance OP.

(i) Explain why $V=\int_{r\sin\theta}^{r}\pi\left(r^2-h^2\right)dh$. **3**

(ii) Hence show $V=\frac{r^3\pi}{3}\left(2-3\sin\theta+\sin^3\theta\right)$. **2**

(d) (i) Show that $x^4+y^4\ge 2x^2y^2$. **2**

(ii) If $P(x,y)$ is any point on the curve $x^4+y^4=1$ prove that $OP\le 2^{\frac{1}{4}}$, where O is the origin. **3**

Section D
(Start a new answer booklet)

Question 7 (15 marks)

(a) How many sets of 5 quartets (groups of four musicians) can be formed from 5 violinists, 5 viola players, 5 cellists, and 5 pianists if each quartet is to consist of one player of each instrument? **2**

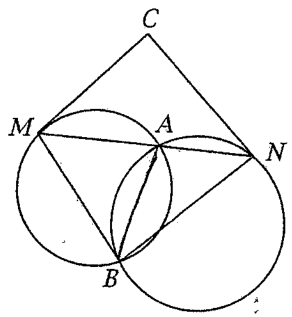
(b) (i) If $t=\tan\theta$, prove that **2**

$$\tan 4\theta=\frac{4t\left(1-t^2\right)}{1-6t^2+t^4}.$$

(ii) If $\tan\theta\tan 4\theta=1$ deduce that $5t^4-10t^2+1=0$. **2**

(iii) Given that $\theta=\frac{\pi}{10}$ and $\theta=\frac{3\pi}{10}$ are roots of the equation $\tan\theta\tan 4\theta=1$, find the exact value of $\tan\frac{\pi}{10}$. **4**

(c)



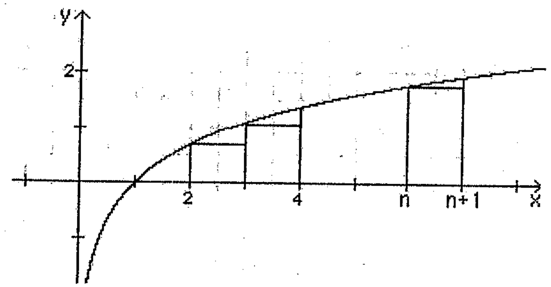
Two circles intersect at A and B. A line through A cuts the circles at M and N. The tangents at M and N intersect at C. **5**

(i) Prove that $\angle CMA+\angle CNA=\angle MBN$.

(ii) Prove M, C, N, B are concyclic.

Question 8 (15 marks)

(a) 6



The diagram above shows the graph of $y=\log_e x$ for $1 \le x \le n+1$.

(i) By considering the sum of the areas of inner and outer rectangles show that

$$\ln(n!) < \int_1^{n+1} \ln x\, dx < \ln((n+1))!$$

(ii) Find $\int_1^{n+1} \ln x\, dx$.

(iii) Hence prove that

$$e^n > \frac{(n+1)^n}{n!}$$

(b) If a root of the cubic equation $x^3+bx^2+cx+d=0$ is equal to the reciprocal of another root, prove that 3

$$1+bd=c+d^2.$$

This question continues on the next page.

(c) A stone is projected from a point O on a horizontal plane at an angle of elevation α and with initial velocity U metres per second. The stone reaches a point A in its trajectory, and at that instant it is moving in a direction perpendicular to the angle of projection with speed V metres per second. 6

Air resistance is neglected throughout the motion and g is the acceleration due to gravity.

If t is the time in seconds at any instant, show that when the stone is at A:

(i) $V=U\cot\alpha$

(ii) $t=\dfrac{U}{g\sin\alpha}$.

This is the end of the paper.

Section A

(1) (i)
$$\int_0^2 \frac{3}{4+x^2}\,dx = \frac{3}{2}\int_0^2 \frac{3}{4+x^2}\,dx$$
$$= \frac{3}{2}\left[\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2$$
$$= \frac{3}{2}\left[\tan^{-1}(1) - 0\right]$$
$$= \frac{3\pi}{8}$$

(ii)
$$\int \cos x \sin^4 x\,dx = \int u^4\,du \quad [u = \sin x]$$
$$= \frac{u^5}{5} + c$$
$$= \frac{\sin^5 x}{5} + c$$

(iii)
$$\int \underset{g}{t}\,\underset{f'}{e^{-t}}\,dt = fg - \int fg'\,dt$$
$$= -te^{-t} - \int \left(-e^{-t} \times 1\right)dt$$
$$= -te^{-t} + \int e^{-t}\,dt$$
$$= -te^{-t} - e^{-t} + c$$

(d) (i)
$$1 \equiv a(\pi - 2x) + bx$$
$$x = 0 \Rightarrow a = \frac{1}{\pi}$$
$2a = b$ [coefficients of x]
$$\therefore b = \frac{2}{\pi}$$

$$a = \frac{1}{\pi}, b = \frac{2}{\pi}$$

(ii)
$$\int \frac{dx}{x(\pi - 2x)} = \frac{1}{\pi}\int \left(\frac{1}{x} - \frac{-2}{\pi - 2x}\right)dx$$
$$= \frac{1}{\pi}\ln x - \frac{1}{\pi}\ln(\pi - 2x) + c$$
$$= \frac{1}{\pi}\ln\left(\frac{x}{\pi - 2x}\right) + c$$

(e)
$$\int_{-3}^{3}(2 - |x|)\,dx = 2\int_0^3 (2 - |x|)\,dx \qquad [\text{Q } 2 - |x| \text{ is even}]$$
$$= 2\int_0^3 (2 - x)\,dx \qquad [\text{Q } 2 - |x| = 2 - x, x > 0]$$
$$= 2\left[2x - \frac{1}{2}x^2\right]_0^3$$
$$= 2\left[6 - \frac{9}{2}\right]$$
$$= 3$$

(f) (i)
$$x = a - t \Rightarrow dx = -dt$$
$$x = 0 \Rightarrow t = a$$
$$x = a \Rightarrow t = 0$$
$$\int_0^a f(x)\,dx = \int_a^0 f(a - t)(-dt)$$
$$= \int_0^a f(a - t)\,dt \qquad \left[\text{Q} \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right]$$
$$= \int_0^a f(a - x)\,dx \qquad [\text{Q choice of variable irrelevant}]$$

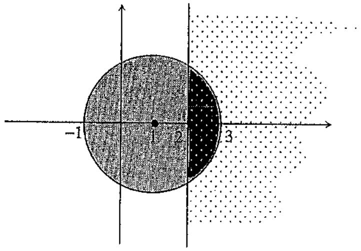
(ii)
$$I = \int_0^{\frac{\pi}{2}} \ln(\tan x)\,dx$$
$$= \int_0^{\frac{\pi}{2}} \ln\left(\tan\left(\frac{\pi}{2} - x\right)\right)dx$$
$$= \int_0^{\frac{\pi}{2}} \ln(\cot x)\,dx$$
$$2I = \int_0^{\frac{\pi}{2}} \ln(\tan x)\,dx + \int_0^{\frac{\pi}{2}} \ln(\cot x)\,dx$$
$$= \int_0^{\frac{\pi}{2}} \left[\ln(\tan x) + \ln(\cot x)\right]dx$$
$$= \int_0^{\frac{\pi}{2}} \ln 1\,dx$$
$$= 0$$
$$\therefore 2I = 0$$
$$\therefore I = 0$$
$$\therefore \int_0^{\frac{\pi}{2}} \ln(\tan x)\,dx = 0$$

(2) (a) $z=2+i, w=-1+2i$

$\therefore z-w=3-i$

$\therefore \operatorname{Im}(z-w)=-1$

(b)



(c) (i) $\left|\frac{i}{z^2}\right|=\frac{|i|}{|z|^2}=\frac{1}{4}$

(ii) $\arg\left(\frac{i}{z^2}\right)=\arg i-\arg\left(z^2\right)$

$=\frac{\pi}{2}-2\arg z$

$=\frac{\pi}{2}-2\theta$

(d) $z=\bar{z} \Rightarrow z$ is purely real

So the locus is $y=0$, except $x=0$.

Alternatively:

Let $z=x+iy, (z\neq 0)$

$\therefore \bar{z}=x-iy$

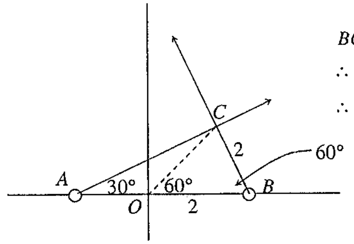
$\therefore z=\bar{z} \Rightarrow x+iy=x-iy$

$\therefore 2iy=0 \Rightarrow y=0$

$\therefore z$ is a purely real number excluding 0

(e) $\arg(z+2)=\frac{\pi}{6}, \arg(z-2)=\frac{2\pi}{3}$.

z is represented by the point C, the intersection of the two rays.

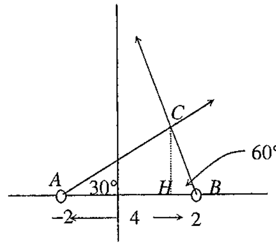


$BC=4\cos 60°=2$

$\therefore \Delta BOC$ is equilateral

$\therefore z=2\operatorname{cis}60°=1+i\sqrt{3}$

Alternatively



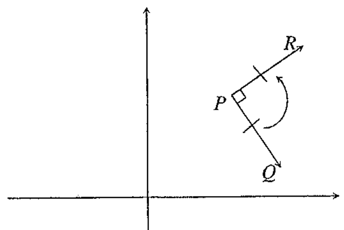
$BC=4\cos 60°=2$

$CH=2\sin 60°=\sqrt{3}$

$BH=2\cos 60°=1$

$\therefore z=1+i\sqrt{3}$

(f)



$\overrightarrow{PR} \perp \overrightarrow{PQ}$

$\left|\overrightarrow{PR}\right|=\left|\overrightarrow{PQ}\right|$

$\overrightarrow{PR}$ represents z_3-z_1

$\overrightarrow{PQ}$ represents z_2-z_1

$i(z_2-z_1)=z_3-z_1$

$\therefore i^2(z_2-z_1)^2=(z_3-z_1)^2$

$\therefore -(z_2^2-2z_2z_1+z_1^2)=z_3^2-2z_1z_3+z_1^2$

$\therefore 2z_1^2+z_2^2+z_3^2=2z_1z_3+2z_2z_1=2z_1(z_2+z_3)$

Section B

Question (3)

(a) $z^3 + pz + q = 0$

If $(2-3i)$ is a zero then

$(2-3i)^3 + (2-3i)p + q = 0$

$(2-3i)^3 = (2-3i)^2(2-3i)$
$= (-5-12i)(2-3i)$
$= -46 - 9i$ 1

$\therefore -46 - 9i + (2-3i)p + q = 0$.

Equating real parts.

$-46 + 2p + q = 0$ 1

$2p + q = 46$ —— ①

Equating imaginary parts.

$-9 - 3p = 0 \Rightarrow 3p = -9$

$\therefore \boxed{p = -3}$ —— ②

Subst ② into ①

We have 1

$-6 + q = 46$

$\boxed{q = 52}$ —— ③

[3]

(b) $x^3 + 6x + 1 = 0$

If α, β, γ are the roots then $\alpha\beta\gamma = -1$

Now, $\alpha\beta = \frac{\alpha\beta\gamma}{\gamma} = -\frac{1}{\gamma}$

Similarly, $\beta\gamma = -\frac{1}{\alpha}$ 1

and $\alpha\gamma = -\frac{1}{\beta}$

Let $y = -\frac{1}{x} \Rightarrow x = -\frac{1}{y}$

i.e. the polynomial equation is $\left(-\frac{1}{y}\right)^3 + 6\left(-\frac{1}{y}\right) + 1 = 0$. 1

$-1 - 6y^2 + y^3 = 0$

$\boxed{y^3 - 6y^2 - 1 = 0}$ [2]

(c) $y = 3\left(\frac{x+4}{x}\right)^2$

$\frac{dy}{dx} = 6\left(\frac{x+4}{x}\right)\left(\frac{-4}{x^2}\right)$

$= \frac{-24(x+4)}{x^3}$

$\frac{dy}{dx} = 0 \Rightarrow x = -4$ 2

When $x = -4$, $y = 0$.

$\therefore (-4, 0)$ is a stationary pt.

$\frac{d^2y}{dx^2} = -24\left[\frac{x^3 - (x+4)3x^2}{x^6}\right]$

$= 24\left[\frac{2x^3 + 12x^2}{x^6}\right]$

$= 48x^2\left(\frac{x+6}{x^4}\right)$

$f''(x) = 0, \Rightarrow x = -6$.

When $x = -6$, $y = 3\left(\frac{4}{36}\right) = \frac{1}{3}$ 2

~~$\therefore (-6, \frac{1}{3})$ could be a pt of inflexion~~

Test:

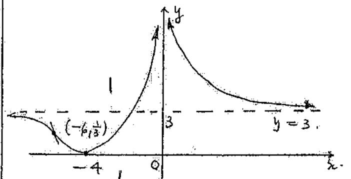
x	-7	-6	-5
$f''(x)$	$-$	0	$+$

$\therefore (-6, \frac{1}{3})$ is a pt. of inflexion

Also, $f''(-4) = \frac{48}{4^4}(2) > 0$

$\therefore (-4, 0)$ is a min turning pt

$x \neq 0 \quad \therefore$ y-axis is a vertical asymptote. [5]



[2]

Note: $\lim_{x \to \pm\infty} 3\left(\frac{x^2 + 8x + 16}{x^2}\right) = 3$

i.e. horizontal asymptote at $y = 3$.

(d) Let $S(n)$ be the proposition that $n! > 2^n$, $n > 3$, $n \in Z^+$.

For $n = 4$

$4! = 24 > 2^4 = 16$

$\therefore S(4)$ is true.

Assume $S(k)$ is true.

Prove true for $n = k+1$.

$(k+1)! = (k+1)k!$

$> (k+1) \cdot 2^k$

$\because k > 3$, then $k+1 > 4$

$> 4 \cdot 2^k$

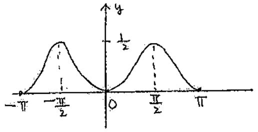
$2 \cdot 2^{k+1}$

$> 2^{k+1}$

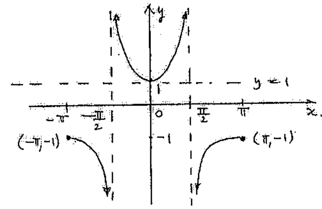
Question (4).

$y = \sin x$, $-\pi \leq x \leq \pi$

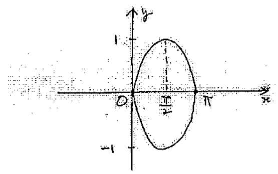
(i) $y = \sin^2 x$
$= \frac{1}{2}(1 - \cos 2x)$



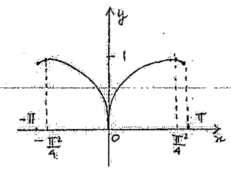
(ii) $y = \frac{1}{\sin(x + \frac{\pi}{2})} = \frac{1}{\cos x}$.



(iii) $y^2 = \sin x$.
$y = \pm\sqrt{\sin x}$



(iv)



(b) $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ /

$\therefore \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}y^{-\frac{1}{2}} \cdot \frac{dy}{dx} = 0$.

$\therefore \frac{dy}{dx} = -\frac{x^{-\frac{1}{2}}}{y^{-\frac{1}{2}}} = -\sqrt{\frac{y}{x}}$.

$\left.\frac{dy}{dx}\right|_{x=x_0, y=y_0} = -\sqrt{\frac{y_0}{x_0}}$.

Equation of tgt.

$y - y_0 = \frac{-y_0^{\frac{1}{2}}}{x_0^{\frac{1}{2}}}(x - x_0)$

$\therefore x_0^{\frac{1}{2}} y - x_0^{\frac{1}{2}} y_0 = x_0 y_0^{\frac{1}{2}} - x y_0^{\frac{1}{2}}$.

$\Rightarrow y_0^{\frac{1}{2}} x + x_0^{\frac{1}{2}} y = x_0^{\frac{1}{2}} y_0^{\frac{1}{2}}(y_0^{\frac{1}{2}} + x_0^{\frac{1}{2}})$

$\because (x_0, y_0)$ is on the curve ①

$\Rightarrow x_0^{\frac{1}{2}} + y_0^{\frac{1}{2}} = a^{\frac{1}{2}}$

and divide both sides of ① by $x_0^{\frac{1}{2}} y_0^{\frac{1}{2}}$ we have,

$x_0^{-\frac{1}{2}} x + y_0^{-\frac{1}{2}} y = a^{\frac{1}{2}}$

(c)

$p(x) = x^5 - ax + 1$

$p'(x) = 5x^4 - a$

$p''(x) = 20x^3$

Stationary pts when

$5x^4 - a = 0$

$x^4 = \frac{a}{5}$

$\therefore \quad x = \pm\left(\frac{a}{5}\right)^{\frac{1}{4}}$ 1

For $x = \left(\frac{a}{5}\right)^{\frac{1}{4}}$, $p''\left[\left(\frac{a}{5}\right)^{\frac{1}{4}}\right] > 0$

For $x = -\left(\frac{a}{5}\right)^{\frac{1}{4}}$, $p''\left[-\left(\frac{a}{5}\right)^{\frac{1}{4}}\right] < 0$ 1

When $x = \left(\frac{a}{5}\right)^{\frac{1}{4}}$, $y = \left(\frac{a}{5}\right)^{5/4} - a\left(\frac{a}{5}\right)^{\frac{1}{4}} + 1$

$\therefore \quad y = 1 - \left(\frac{4a}{5}\right)\left(\frac{a}{5}\right)^{\frac{1}{4}}$

For $x = -\left(\frac{a}{5}\right)^{\frac{1}{4}}$

$y = 1 + \left(\frac{4a}{5}\right)\left(\frac{a}{5}\right)^{\frac{1}{4}}$

For three distinct real roots, the curve cuts the x-axis in 3 points.

$x \to \infty \quad p(x) \to \infty \ (>0)$

and the turning points are on the opposite sides of the x-axis

$\Rightarrow$ The product of the y's < 0

$\therefore \left[1 + \left(\frac{4a}{5}\right)\left(\frac{a}{5}\right)^{\frac{1}{4}}\right]\left[1 - \left(\frac{4a}{5}\right)\left(\frac{a}{5}\right)^{\frac{1}{4}}\right] < 0$

$1 - \frac{16a^2}{25}\left(\frac{a}{5}\right)^{\frac{1}{2}} < 0$ 1

$\frac{16a^{5/2}}{5^{5/2}} > 1 \Rightarrow a^{5/2} > \left(\frac{5^{\frac{5}{2}}}{2^4}\right)$

$\therefore \quad a > \left(\frac{5^{5/2}}{2^4}\right)^{2/5} = \left[\frac{5^1}{2^{8/5}}\right]$ 1

$\Rightarrow \quad a > 5\left(\frac{1}{2}\right)^{\frac{8}{5}}$ [4]

Section C

Questions

(a)(i) $m\ddot{y} = -mkv - mg$

↑ $m\ddot{y}$ ↓ mg ↓ mkv

$\therefore \boxed{\ddot{y} = -(kv+g)}$

(ii) $\ddot{y} = \frac{dv}{dt} = -(kv+g)$

$\therefore dt = -(kv+g)^{-1}\,dv$

$\int_0^T 1\,dt = -\int_{v_0}^{0}(kv+g)^{-1}\,dv$

$T = \int_0^{v_0}(kv+g)^{-1}\,dv$

$= \int_0^{v_0}\frac{1}{kv+g}\,dv$

$= \frac{1}{k}\left[\ln(kv+g)\right]_0^{v_0}$

$= \frac{1}{k}\left(\ln(kv_0+g) - \ln g\right)$

$= \frac{1}{k}\ln\left(\frac{kv_0+g}{g}\right)$

$\therefore \boxed{kT = \ln\left(1 + \frac{kv_0}{g}\right)}$

(iii) $v \cdot \frac{dv}{dy} = -(kv + g)$

$\therefore \frac{dv}{dy} = -\frac{(kv+g)}{v}$

$dy = \frac{-v \cdot dv}{kv+g}$

$= -\frac{1}{k}\left(\frac{kv+g-g}{kv+g}\right) dv$

$\int_0^H 1 \cdot dy = -\frac{1}{k}\int_{V_0}^0 \frac{kv+g-g}{kv+g}\, dv$

$\therefore H = \frac{1}{k}\int_0^{V_0}\left(1 - \frac{g}{kv+g}\right) dv$

$= \frac{1}{k}\left[\int_0^{V_0} 1 \cdot dv - \int_0^{V_0} \frac{g}{kv+g}\, dv\right]$

$= \frac{1}{k}\left[\left[v\right]_0^{V_0} - \left[\frac{g}{k}\ln(kv+g)\right]_0^{V_0}\right]$

$= \frac{1}{k}\left[V_0 - \frac{g}{k}\left(\ln(kV_0+g) - \ln g\right)\right]$

$k.H = V_0 - \frac{g}{k}\ln\frac{kV_0+g}{g}$

$= V_0 - \frac{g}{k}kT$ from (ii)

$= V_0 - gT$

$\therefore \boxed{V_0 = kH + gT}$

(b) (i) $z = \frac{2 \pm \sqrt{4-8}}{2}$

$= \frac{2 \pm 2i}{2}$

$= 1 \pm i$

$= \sqrt{2}\,\text{cis} \pm \frac{\pi}{4}$

$\therefore \alpha, \beta$ are $\sqrt{2}\,\text{cis} \pm \frac{\pi}{4}$

(ii) $\alpha^n + \beta^n = \left(\sqrt{2}\,\text{cis}\frac{\pi}{4}\right)^n + \left(\sqrt{2}\,\text{cis}\frac{-\pi}{4}\right)^n$

$= (\sqrt{2})^n\left[\text{cis}\frac{n\pi}{4} + \text{cis}\frac{-n\pi}{4}\right]$

$= (\sqrt{2})^n\left(2\cos\frac{n\pi}{4}\right)$ (NB $z + \bar{z} = 2\,\text{Re}\,z$)

$= 2^{\frac{1}{2}n+1}\cos\frac{n\pi}{4}$

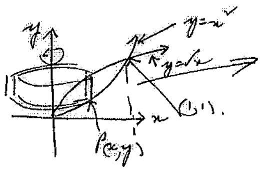
$= 2^{\frac{n+2}{2}}\cos\frac{n\pi}{4}$

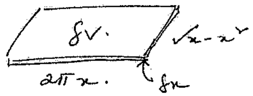
$\therefore \boxed{\alpha^n + \beta^n = \sqrt{2^{n+2}}\cos\frac{n\pi}{4}}$

QUESTION 6.

a $\boxed{{}^{10}P_7 \times {}^{10}P_6 \times 7!}$ (this is the same as $\binom{10}{7} \times \binom{10}{6} \times (7!)^2 \times 6!$)

b





$$V = \lim_{\delta x \to 0} \sum_{x=0}^{1} 2\pi x\,(\sqrt{x} - x^2)\,\delta x.$$

$$= 2\pi \int_0^1 (x^{3/2} - x^3)\,dx.$$

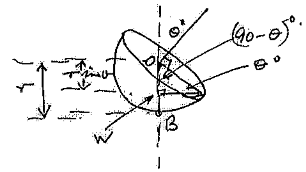
$$= 2\pi \left[\tfrac{2}{5} x^{5/2} - \tfrac{x^4}{4} \right]_0^1$$

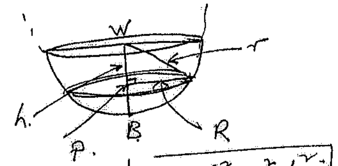
$$= 2\pi \left(\tfrac{2}{5} - \tfrac{1}{4} \right)$$

$$= 2\pi \times \tfrac{3}{20}$$

$$\boxed{= \tfrac{3\pi}{10} \ u^3}$$

(c) (i)





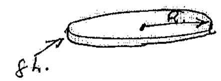
[now $R^2 = r^2 - h^2$.]

NB $\frac{OW}{r} = \sin\theta$

$\boxed{OW = r\sin\theta}$

and $\boxed{OB = r}$

Consider the slice at P.



$$\delta V = \pi R^2.\delta h.$$

$$\therefore\ V = \lim_{\delta h \to 0} \sum_{h = r\sin\theta}^{h=r} .\pi R^2 \delta h.$$

$$= \pi \int_{r\sin\theta}^{r} R^2.\,dh.$$

$$= \pi \int_{r\sin\theta}^{r} (r^2 - h^2).\,dh.$$

(ii) $$V = \pi \left[r^2 h - \frac{h^3}{3} \right]_{r\sin\theta}^{r}$$

$$= \pi \left[r^3 - \frac{r^3}{3} - \left(r^3 \sin\theta - \frac{r^3 \sin^3\theta}{3} \right) \right]$$

$$= \pi \left[\frac{2r^3}{3} - r^3 \sin\theta + \frac{r^3 \sin^3\theta}{3} \right]$$

$$\boxed{= \frac{\pi r^3}{3} \left(2 - 3\sin\theta + \sin^3\theta \right)}$$

(d) (i) now $(x^2-y^2)^2 \geq 0$.

$x^4 - 2x^2y^2 + y^4 \geq 0$

$\therefore x^4 + y^4 \geq 2x^2y^2$

(ii) now $OP = \sqrt{x^2+y^2}$

$\therefore OP^4 = (x^2+y^2)^2$

$= x^4 + y^4 + 2x^2y^2$

$\leq x^4+y^4 + x^4+y^4$ (from (i))

$\leq 2.$ $(x^4+y^4=1)$

$\therefore OP \leq 2^{\frac{1}{4}}$

Section D

7. (a) How many sets of 5 quartets (groups of four musicians) can be formed from 5 violinists, 5 viola players, 5 cellists, and 5 pianists if each quartet is to consist of one player of each instrument?

> **Solution:** $\dfrac{5^4 \times 4^4 \times 3^4 \times 2^4 \times 1^4}{5!} = (5!)^3,$
> $= 1\,728\,000.$

(b) i. If $t = \tan\theta$, prove that

$$\tan 4\theta = \frac{4t(1-t^2)}{1-6t^2+t^4}.$$

> **Solution:**
> $$\begin{aligned} \text{L.H.S.} &= \frac{2\times\tan 2\theta}{1-(\tan 2\theta)^2}, \\ &= \frac{2\times\frac{2t}{1-t^2}}{1-\left(\frac{2t}{1-t^2}\right)^2}, \\ &= \frac{4t(1-t^2)}{(1-t^2)^2-4t^2}, \\ &= \frac{4t(1-t^2)}{1-6t^2+t^4}, \\ &= \text{R.H.S.} \end{aligned}$$

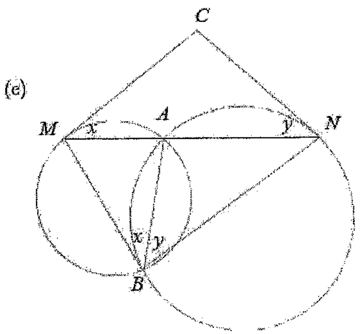
ii. If $\tan\theta\tan 4\theta = 1$, deduce that $5t^4 - 10t^2 + 1 = 0$.

> **Solution:**
> $$\begin{aligned} t\times\frac{4t(1-t^2)}{1-6t^2+t^4} &= 1, \\ 4t^2-4t^4 &= 1-6t^2+t^4, \\ 5t^4-10t^2+1 &= 0. \end{aligned}$$

iii. Given that $\theta = \dfrac{\pi}{10}$ and $\theta = \dfrac{3\pi}{10}$ are roots of the equation $\tan\theta\tan 4\theta = 1$, find the exact value of $\tan\dfrac{\pi}{10}$.

> **Solution:** Using the quadratic formula, $t^2 = \dfrac{10\pm\sqrt{100-20}}{10}$,
> $= \dfrac{5\pm2\sqrt{5}}{5}$.
>
> i.e., $t = \sqrt{\dfrac{5\pm\sqrt{5}}{5}}$ as $\tan\dfrac{\pi}{10}, \tan\dfrac{3\pi}{10} > 0$.
>
> Now, as $\tan\dfrac{\pi}{10} < \tan\dfrac{3\pi}{10}$, $\tan\dfrac{\pi}{10} = \sqrt{\dfrac{5-\sqrt{5}}{5}}$.

(e)



Two circles intersect at A and B. A line through A cuts the circles at M and N. The tangents at M and N intersect at C.

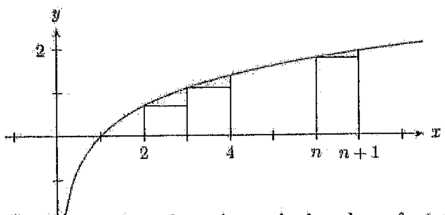
i. Prove that $\angle CMA + \angle CNA = \angle MBN$.

> **Solution:** Join AB.
> $$\begin{aligned} \angle CMA &= \angle MBA \text{ (angle in alternate segment)},\\ \angle CNA &= \angle ABN \text{ (angle in alternate segment)},\\ \therefore \angle CMA + \angle CNA &= \angle MBA + \angle ABN,\\ &= \angle MBN. \end{aligned}$$

ii. Prove M, C, N, B are concyclic.

> **Solution:** $\angle CMA + \angle CNA + \angle MCN = 180^\circ$ (angle sum of $\triangle CMN$),
> $\therefore \angle MBN + \angle MCN = 180^\circ$.
> So $MCNB$ is a cyclic quadrilateral (opposite angles supplementary).

8. (a)



The diagram above shows the graph of $y = \log_e x$ for $1 \le x \le n+1$.

i. By considering the sum of the areas of inner and outer rectangles, show that

$$\ln(n!) < \int_1^{n+1} \ln x\, dx < \ln((n+1)!)$$

> **Solution:**
> $$\begin{aligned} \text{Sum inner rectangles} &= \sum_{x=1}^{n} \ln x \times 1,\\ &= \ln 1 + \ln 2 + \ln 3 + \cdots + \ln n,\\ &= \ln n!\\ \text{Sum outer rectangles} &= \sum_{x=2}^{n+1} \ln x \times 1, \text{ or } \sum_{x=1}^{n} \ln(x+1) \times 1,\\ &= \ln 2 + \ln 3 + \ln 4 + \cdots + \ln(n+1),\\ &= \ln(n+1)! \end{aligned}$$
> $$\therefore \ln n! < \int_1^{n+1} \ln x\, dx < \ln(n+1)!$$

ii. Find $\int_1^{n+1} \ln x\, dx$.

> **Solution:**
> $$\begin{aligned} I &= \textstyle\int_1^{n+1} \ln x \times 1\, dx, & u &= \ln x & v' &= 1\\ &= [x \ln x]_1^{n+1} - \textstyle\int_1^{n+1} dx, & u' &= \frac{1}{x} & v &= x\\ &= (n+1)\ln(n+1) - 0 - [x]_1^{n+1},\\ &= (n+1)\ln(n+1) - (n+1-1),\\ &= (n+1)\ln(n+1) - n. \end{aligned}$$

iii. Hence prove that

$$e^n > \frac{(n+1)^n}{n!}$$

> **Solution:** From i., $\ln(n+1)! > \int_1^{n+1} \ln x\, dx$.
> $$\begin{aligned} \therefore \ln(n+1)! &> \ln(n+1)^{n+1} - n.\\ n &> \ln \frac{(n+1)^{n+1}}{(n+1)!},\\ &> \ln \frac{(n+1)^n}{n!}.\\ \therefore e^n &> \frac{(n+1)^n}{n!}. \end{aligned}$$

(b) If a root of the cubic equation $x^3 + bx^2 + cx + d = 0$ is equal to the reciprocal of another root, prove that

$$1 + bd = c + d^2.$$

> **Solution:** Let the roots be $\alpha, \frac{1}{\alpha}, \beta$.
> **Method 1:**
> $$\begin{aligned} \alpha \times \frac{1}{\alpha} \times \beta &= -d,\\ \beta &= -d. \end{aligned}$$
> Substitute in the equation for the root β:

$$-d^3+bd^2-cd+d=0,$$
$$cd+d^3=bd^2+d.$$
Divide by d $(d\neq 0)$,
$$c+d^2=bd+1.$$

Method 2:

$$\alpha+\frac{1}{\alpha}+\beta=-b,$$
$$1+\alpha\beta+\frac{\beta}{\alpha}=c,$$
$$\beta=-d.$$
$$\therefore\ \alpha+\frac{1}{\alpha}-d=-b\ \ldots\ \boxed{1}$$
$$1-\alpha d-\frac{d}{\alpha}=c\ \ldots\ \boxed{2}$$
$$1-c=d\left(\alpha+\frac{1}{\alpha}\right),$$
$$\therefore\ \alpha+\frac{1}{\alpha}=\frac{1-c}{d}.$$
Sub. in $\boxed{1}$, $\frac{1-c}{d}-d=-b$,
$$1-c-d^2=-bd,$$
i.e., $1+bd=c+d^2$.

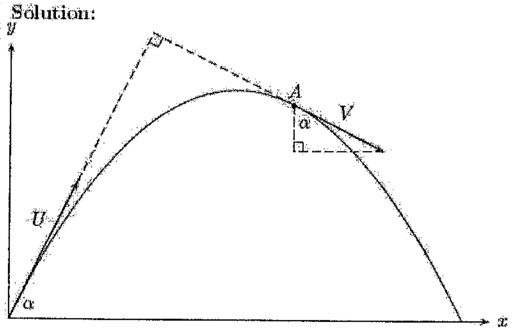
(c) A stone is projected from a point O on a horizontal plane at an angle of elevation α and with initial velocity U metres per second. The stone reaches a point A in its trajectory, and at that instant it is moving in a direction perpendicular to the angle of projection with speed V metres per second.

Air resistance is neglected throughout the motion and g is the acceleration due to gravity.

If t is the time in seconds at any instant, show that when the stone is at A:

i. $V=U\cot\alpha$

Solution:



$\ddot{x}=0$ $\qquad$ $\ddot{y}=-g$

$\dot{x}=U\cos\alpha$ $\qquad$ $\dot{y}=U\sin\alpha-gt$

At A, $U\cos\alpha=V\sin\alpha$,

i.e., $V=U\cot\alpha$

ii. $t=\dfrac{U}{g\sin\alpha}$

Solution: At A, $\dot{y}=-V\cos\alpha$ (now heading downwards),

i.e., $-U\cot\alpha\times\cos\alpha=U\sin\alpha-gt$,

$$gt=U\sin\alpha+U\frac{\cos\alpha}{\sin\alpha}.\cos\alpha,$$
$$=U\left(\frac{\sin^2\alpha+\cos^2\alpha}{\sin\alpha}\right).$$
$$\therefore\ t=\frac{U}{g\sin\alpha}.$$