



SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2006
TRIAL HIGHER SCHOOL
CERTIFICATE

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using black or blue pen.
- Board approved calculators may be used.
- All *necessary* working should be shown in every question if full marks are to be awarded.
- Marks may **NOT** be awarded for messy or badly arranged work.
- Hand in your answer booklets in 3 sections.
 - Section A (Questions 1 - 2),
 - Section B (Questions 3 - 4)
 - Section C (Questions 5 - 6)
 - Section D (Questions 7 - 8).
- Start each **NEW** section in a separate answer booklet.

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

Total Marks - 120 Marks

- Attempt Sections A - D
- All questions are of equal value.

Examiner: E. Choy

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln(x + \sqrt{x^2 - a^2}), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln(x + \sqrt{x^2 + a^2})$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Total marks – 120
 Attempt Questions 1 – 8
 All questions are of equal value

Answer each section in a SEPARATE writing booklet. Extra writing booklets are available.

SECTION A (Use a SEPARATE writing booklet)

Question 1 (15 marks)

Marks

(a) By first completing the square, evaluate the following integrals

(i) $\int_{-1}^0 \frac{dx}{\sqrt{3-2x-x^2}}$ 2

(ii) $\int_0^1 \sqrt{x(1-x)} dx$ ✓ 2

(b) Integrate the expressions below

(i) $\int \frac{1}{x \ln x} dx$ 1

(ii) $\int x \ln x dx$ 2

tc

(iii) $\int \frac{x+1}{x^2+x+1} dx$ 2

(c) Use the technique of integration by parts to evaluate

$$\int_0^{\frac{1}{2}} \cos^{-1} x dx$$

(d) (i) Find real numbers A , B , and C so that

$$\frac{10}{(3+x)(1+x^2)} = \frac{A}{3+x} + \frac{Bx+C}{1+x^2}$$

for all $x \neq -3$

(ii) Use part (i) above and the substitution $t = \tan \theta$ to find

$$\int \frac{10d\theta}{3 + \tan \theta}$$

SECTION A continued

Question 2 (15 marks)

Marks

(a) (i) Write the complex number $-\sqrt{3} + i$ in modulus-argument form. 1
 (ii) Hence, use de Moivre's Theorem to find $(-\sqrt{3} + i)^{10}$ in the form $a + ib$, for real values a and b . 2

(b) Sketch each of the following regions on separate Argand diagrams

(i) $-1 < \operatorname{Re}(z) < 2$ and $0 < \operatorname{Im}(z) < 3$ 2

(ii) $z\bar{z} - (1-i)z - (1+i)\bar{z} < 2$ 2

(iii) $0 < \arg[(1-i)z] < \frac{\pi}{6}$ 2

(c) (i) Find the square roots of the complex number $-3 + 4i$ 2

(ii) Find the roots of the quadratic equation $x^2 - (4 - 2i)x + (6 - 8i) = 0$ 2

(d) The locus of a point P , which moves in the complex plane, is represented by the equation $|z - (3 + 4i)| = 5$

(i) Sketch the locus of the point P . 2

(ii) Find the modulus of z when $\arg z = \tan^{-1}(\frac{1}{2})$. 2

SECTION B (Use a SEPARATE writing booklet)

Question 3 (15 marks)

(a) Find a cubic equation with roots α , β and γ such that

$$\left. \begin{aligned} \alpha\beta\gamma &= 5 \\ \alpha + \beta + \gamma &= 7 \\ \alpha^2 + \beta^2 + \gamma^2 &= 29 \end{aligned} \right\}$$

3

(b) The polynomial $P(x)$ is defined by $P(x) = x^4 + Ax^2 + B$, where A and B are real positive numbers.

(i) Explain why $P(x)$ has no real zeroes. 2

(ii) If two of the zeroes of $P(x)$ are ib and $-id$ where b and d are real show that: 4

$$b^4 + d^4 = A^2 - 2B$$

(c) Given that $f(x) = x^3 - 3ax + b$, where a and b are real numbers then:

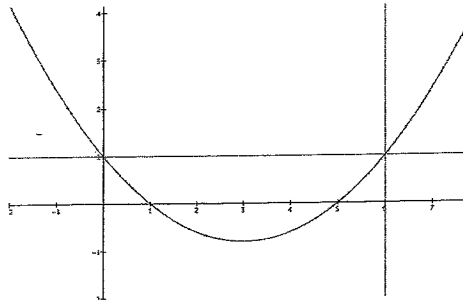
✓ (i) Show that $y = f(x)$ has turning points if $a > 0$, and find their coordinates. 3

✓ (ii) Show that $f(x)$ has three distinct real zeroes if $b^2 < 4a^3$. 3

SECTION B continued

Question 4 (15 marks)

(a)



The sketch above shows the parabola $y = f(x)$, where

$$f(x) = \frac{1}{5}(x-1)(x-5).$$

Without any use of calculus, draw careful sketches of the following curves, showing all intercepts, asymptotes and turning points.

NB The vertex of the parabola is at $(3, -\frac{4}{5})$.

(i) $y = \frac{1}{f(x)}$

(ii) $y = [f(x)]^2$

(iii) $y = \tan^{-1}[f(x)]$

(iv) $y = f(\ln x)$

(b) Suppose the function $f(x) = O(x) + E(x)$, where $O(x)$ is odd and $E(x)$ is even.

(i) By considering $f(-x)$, find an expression for $O(x)$ in terms of f .

(ii) Hence write down $O(x)$ when $f(x) = e^x$.

SECTION C starts on page 5

Marks

2

3

3

3

2

2

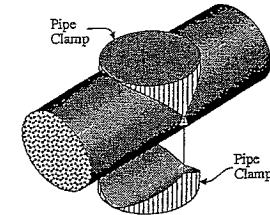
SECTION C (Use a SEPARATE writing booklet)

Question 5 (15 marks)

Marks

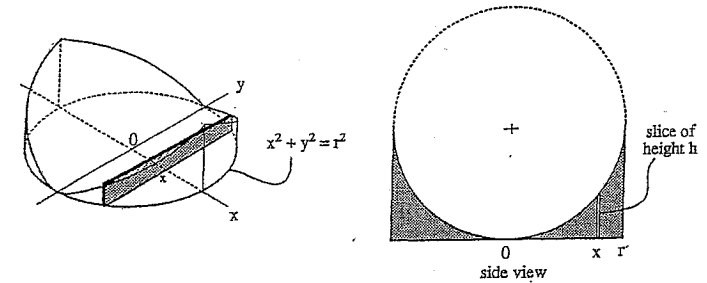
(a)

A pipe-clamp is made of two identical pieces. Each piece has a circular base of radius r units and the other face is curved so as to fit flush against the pipe held between the two pieces.



The pipe also has a radius of r units.

A vertical slice, of thickness Δx , taken x units from the centre of the base is in the shape of a rectangle with one side in the circular base and of height necessary to reach the cylindrical pipe as shown in the diagram below:



(i) Show that the height of the slice taken x units from O is given by

$$h = r - \sqrt{r^2 - x^2}$$

(ii) Show that the volume, ΔV , of such a slice is given by

$$\Delta V = [2r\sqrt{r^2 - x^2} - 2(r^2 - x^2)] \Delta x$$

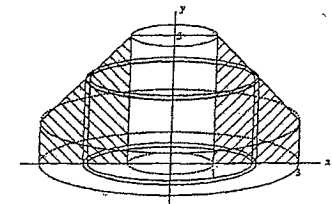
(iii) Hence, find by integration, the volume of ONE piece of the pipe-clamp.

(b) (i) Show that the volume, ΔV , of a right cylindrical shell of height H , with inner radius r and thickness Δr is given by the formula

$$\Delta V = 2\pi r H \Delta r$$

where Δr is sufficiently small so that $(\Delta r)^2$ may be neglected.

(b) (ii) A metal umbrella base is formed by rotating the area enclosed between $x=1$, $x=3$, $y=0$ and $y=4-x$ about the y -axis as shown.



Using the method of cylindrical shells, find the volume of the umbrella base.

SECTION C continued

Question 6 (15 marks)

- (a) A point T moves so that the sum of its distances from the point $(-2, 0)$ and $(2, 0)$, on a Cartesian plane, is 6 units.
- (i) Show that the locus of T is an ellipse \mathcal{E} with the equation $\frac{x^2}{9} + \frac{y^2}{5} = 1$ 2
- (ii) Find the equation of the auxiliary circle, \mathcal{A} , of \mathcal{E} 1
- (iii) Find the eccentricity, coordinates of the foci and the equations of the directrices of the ellipse, \mathcal{E} 2
- (iv) Draw a neat sketch, showing the ellipse and its auxiliary circle. 1
- (v) A line parallel to the y -axis meets the positive x -axis at N and the curves \mathcal{E} and \mathcal{A} at P and Q respectively. Given the coordinates $N(3\cos\theta, 0)$, find the coordinates of P and Q (where P and Q are in the first quadrant). 1
- (vi) Find the equations of the tangents at P and Q . 2
- (vii) If R is the point of intersection of the tangents at P and Q :
- (α) Show that R lies on the major axis of \mathcal{E} . 2
- (β) Prove that the product of the lengths ON and OR is independent of the positions of P and Q on the curves. 2

- (b) Given p red balls and m yellow balls, where $p - m + 1 > 0$, arranged in a row. Show that the number of ways of arranging them so that no two yellow balls appear together is given by: 2

$${}^{p+1}C_m$$

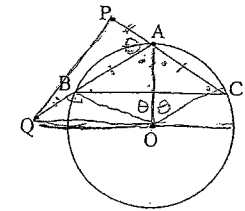
SECTION D starts on page 7

SECTION D (Use a SEPARATE writing booklet)

Question 7 (15 marks)

- (a) (i) Show that $z^5 + 1 = (z + 1)(z^4 - z^3 + z^2 - z + 1)$ 1
- (ii) If z is a solution to $z^5 + 1 = 0$ where $z \neq -1$, prove that $1 + z^2 + z^4 = z + z^3$. 1
- (iii) Hence show that $\cos \frac{\pi}{5} + \cos \frac{3\pi}{5} = \frac{1}{2}$ 3
- (b) For integer values of k where $k = 0, 1, 2, \dots$ define I_k as follows:
- $$I_k = \int_0^{\frac{\pi}{2}} \cos^k x \, dx$$
- (i) Express I_{k+2} in terms of k and I_k . 2
- (ii) Hence find an expression for I_{2n} , where $n = 0, 1, 2, \dots$ 2

- (c) In $\triangle ABC$, in the diagram on the right, $AB = AC$. Produce CA to P and AB to Q so that $AP = BQ$.



- (i) Show that $\angle OAP = \angle OBQ$. 3
- (ii) Prove that A, P, Q and O , the centre of circle ABC , are concyclic. 3

Question 8 (15 marks)

- (a) A particle is projected vertically upwards in a resisting medium where the resistance varies as the square of the velocity and k is the constant of variation. If the velocity of projection is $v_0 \tan \alpha$,
- (i) Show that the maximum height, H , reached is given by: 3
- $$H = \frac{1}{2k} \ln \left(\frac{g + kv_0^2 \tan^2 \alpha}{g} \right)$$
- (ii) Show that the particle returns to the point of projection with velocity $v_0 \sin \alpha$ given that v_0 is the terminal velocity. 4
- (iii) Show that the time of ascent is $\frac{v_0 \alpha}{g}$ 3
- (iv) Show that the time of descent is $\frac{v_0}{g} \ln(\sec \alpha + \tan \alpha)$ 2
- (b) Prove by induction that, for all integers n where $n > 1$, that 3
- $$\frac{4^n}{n+1} < \frac{(2n)!}{(n!)^2}$$

End of paper

$$1) a) i) \int_{-1}^0 \frac{dx}{\sqrt{3-2x-x^2}}$$

$$= \int_{-1}^0 \frac{dx}{\sqrt{-(x^2+2x+1)+4}}$$

$$= \int_{-1}^0 \frac{dx}{\sqrt{4-(x+1)^2}}$$

$$= \left[\sin^{-1}\left(\frac{x+1}{2}\right) \right]_{-1}^0$$

$$= \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0)$$

$$= \frac{\pi}{6}$$

$$ii) \int_0^1 \sqrt{x-x^2} dx$$

$$= \int_0^1 \sqrt{-(x^2-x+\frac{1}{4})+\frac{1}{4}} dx$$

$$= \int_0^1 \sqrt{\frac{1}{4} - (x-\frac{1}{2})^2} dx$$

$$\text{let } x-\frac{1}{2} = \frac{1}{2} \sin \theta$$

$$\frac{dx}{d\theta} = \frac{1}{2} \cos \theta$$

$$dx = \frac{1}{2} \cos \theta d\theta$$

$$\text{when } x=1, \theta = \frac{\pi}{2}$$

$$x=0, \theta = -\frac{\pi}{2}$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\frac{1}{4} - \frac{1}{4} \sin^2 \theta} \cdot \frac{1}{2} \cos \theta d\theta$$

$$= \frac{1}{4} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 - \sin^2 \theta} \cdot \cos \theta d\theta$$

$$= \frac{1}{4} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 \theta d\theta$$

(since even)

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos^2 \theta d\theta$$

$$= \frac{1}{4} \int_0^{\frac{\pi}{2}} (1 + \cos 2\theta) d\theta$$

$$= \frac{1}{4} \left[\theta + \frac{1}{2} \sin 2\theta \right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{4} \left[\frac{\pi}{2} \right]$$

$$= \frac{\pi}{8}$$

OR

$$\text{let } u = x - \frac{1}{2}$$

$$\frac{du}{dx} = 1$$

$$dx = du$$

$$\text{when } x=1, u = \frac{1}{2}$$

$$x=0, u = -\frac{1}{2}$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{\frac{1}{4} - u^2} du$$

Area of semicircle
radius $\frac{1}{2}$.

$$= \frac{1}{2} \pi \left(\frac{1}{2}\right)^2$$

$$= \frac{\pi}{8}$$

$$b) i) \int \frac{1}{x \ln x} dx$$

$$= \int \frac{\left(\frac{1}{x}\right)}{\ln x} dx$$

$$= \ln(\ln x) + C$$

$$ii) \int x \ln x dx$$

$$u = \ln x \quad v = x$$

$$u' = \frac{1}{x} \quad v' = \frac{x^2}{2}$$

$$= \frac{x^2}{2} \ln x - \int \frac{x}{2} dx$$

$$= \frac{x^2}{2} \ln x - \frac{x^2}{4} + C$$

$$iii) \int \frac{x+1}{x^2+x+1} dx$$

$$= \frac{1}{2} \int \frac{2x+1}{x^2+x+1} dx + \frac{1}{2} \int \frac{dx}{x^2+x+1}$$

$$= \frac{1}{2} \ln(x^2+x+1) + \frac{1}{2} \int \frac{dx}{(x^2+x+\frac{1}{4})+\frac{3}{4}}$$

$$= \frac{1}{2} \ln(x^2+x+1) + \frac{1}{2} \int \frac{dx}{\frac{3}{4} + (x+\frac{1}{2})^2}$$

$$= \frac{1}{2} \ln(x^2+x+1) + \frac{1}{2} \cdot \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C$$

$$= \frac{1}{2} \ln(x^2+x+1) + \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

$$c) \int_0^{\frac{1}{2}} \ln \cos^{-1} x \, dx$$

$$u = \cos^{-1} x \quad v' = 1$$

$$u' = \frac{-1}{\sqrt{1-x^2}} \quad v = x$$

$$= \left[x \cos^{-1} x \right]_0^{\frac{1}{2}} + \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}}$$

$$\text{let } u = 1-x^2$$

$$\frac{du}{dx} = -2x$$

$$\frac{dx}{du} = \frac{du}{-2x}$$

$$\text{when } x = \frac{1}{2}, u = \frac{3}{4}$$

$$x = 0, u = 1$$

$$= \frac{1}{2} \cos^{-1} \left(\frac{1}{2} \right) - 0 + \int_1^{\frac{3}{4}} \frac{x}{\sqrt{u}} \cdot \frac{du}{-2x}$$

$$= \frac{1}{2} \cdot \frac{\pi}{3} = \frac{1}{2} \int_1^{\frac{3}{4}} u^{-\frac{1}{2}} du$$

$$= \frac{\pi}{6} = \frac{1}{2} \left[2u^{\frac{1}{2}} \right]_1^{\frac{3}{4}}$$

$$= \frac{\pi}{6} - \frac{1}{2} \left[2 \cdot \frac{\sqrt{3}}{2} - 2 \right]$$

$$= \frac{\pi}{6} + \frac{2-\sqrt{3}}{2}$$

equating coefficients of x^2

$$0 = 1 + B$$

$$B = -1$$

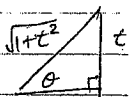
equating constants

$$10 = 1 + 3C$$

$$3C = 9$$

$$C = 3$$

$$ii) \int \frac{10x \, dx}{3+x^2}$$



$$t = \tan \theta$$

$$\frac{dt}{d\theta} = \sec^2 \theta$$

$$d\theta = \frac{dt}{\sec^2 \theta}$$

$$d\theta = \cos^2 \theta dt$$

$$d\theta = \frac{dt}{1+t^2}$$

$$= \int \frac{10}{3+t} \cdot \frac{dt}{1+t^2} \quad \text{using (i)}$$

$$= \int \frac{dt}{3+t} + \int \frac{-t+3}{1+t^2} dt$$

$$= \int \frac{dt}{3+t} - \frac{1}{2} \int \frac{2t}{1+t^2} dt + 3 \int \frac{dt}{1+t^2}$$

$$= \ln|3+t| - \frac{1}{2} \ln|1+t^2| + 3 \tan^{-1} t + C$$

$$= \ln|3+\tan \theta| - \frac{1}{2} \ln|1+\tan^2 \theta| + 3\theta + C$$

$$d) i) \frac{10}{(3+x)(1+x^2)} = \frac{A}{3+x} + \frac{Bx+C}{1+x^2}$$

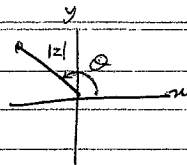
$$10 = A(1+x^2) + (Bx+C)(3+x)$$

$$\text{when } x = -3$$

$$10 = 10A$$

$$A = 1$$

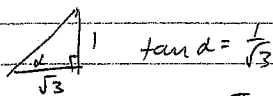
2) a) i)



$$|z| = \sqrt{(-\sqrt{3})^2 + (1)^2}$$

$$= \sqrt{4}$$

$$= 2$$



$$\alpha = \frac{\pi}{6}$$

$$\theta = \frac{5\pi}{6}$$

$$\therefore -\sqrt{3} + i = 2 \left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right)$$

$$ii) (-\sqrt{3} + i)^{10} = \left[2 \left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right) \right]^{10}$$

$$= 2^{10} \left(\cos \frac{50\pi}{6} + i \sin \frac{50\pi}{6} \right)$$

$$= 1024 \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)$$

$$= 1024 \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i \right)$$

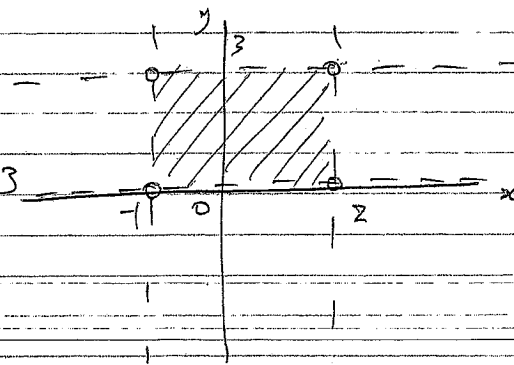
$$= 512 + 512\sqrt{3}i$$

b) i) let $z = x+iy$

$$-1 < \operatorname{Re}(z) < 2, \quad 0 < \operatorname{Im}(z) < 3$$

$$-1 < \operatorname{Re}(x+iy) < 2, \quad 0 < \operatorname{Im}(x+iy) < 3$$

$$-1 < x < 2, \quad 0 < y < 3$$



ii) let $z = x+iy$

$$z\bar{z} - (1-i)z - (1+i)\bar{z} < 2$$

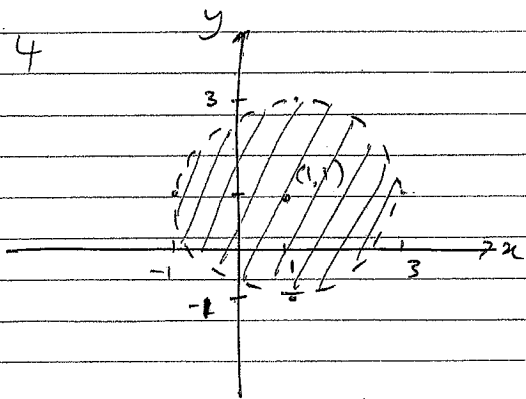
$$(x+iy)(x-iy) - (1-i)(x+iy) - (1+i)(x-iy) < 2$$

$$x^2+y^2 - (x+iy-ix+y) - (x-iy+ix+y) < 2$$

$$x^2+y^2 - x - iy + ix - y - x + iy - ix + y < 2$$

$$x^2 - 2x + 1 + y^2 - 2y + 1 < 2 + 1 + 1$$

$$(x-1)^2 + (y-1)^2 < 4$$



iii) $0 < \arg[(1-i)z] < \frac{\pi}{6}$

* When two complex numbers are multiplied together the argument of the resulting complex number is the sum of the individual arguments

$$\arg(1-i) = -\frac{\pi}{4}$$

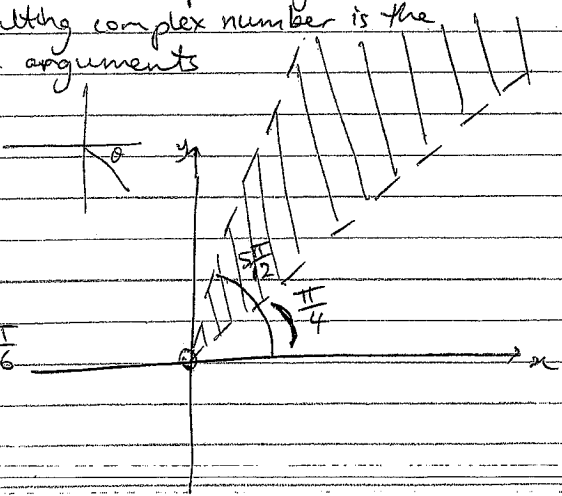
$$\text{let } z = r(\cos \theta + i \sin \theta)$$

$$\arg(z) = \theta$$

$$0 < \arg[(1-i)z] < \frac{\pi}{6}$$

$$0 < \theta - \frac{\pi}{4} < \frac{\pi}{6}$$

$$\frac{\pi}{12} < \theta < \frac{5\pi}{12}$$



c) i) let $\sqrt{-3+4i} = x+iy$ where x & y are real,

$$-3+4i = x^2-y^2+2xyi$$

equating real & imaginary parts

$$x^2-y^2 = -3 \quad \text{--- (1)}$$

$$2xy = 4 \quad \text{--- (2)}$$

$$\text{rearrange (2) } y = \frac{2}{x} \quad \text{--- (2a)}$$

sub (2a) into (1)

$$x^2 - \left(\frac{2}{x}\right)^2 = -3$$

$$x^2 - \frac{4}{x^2} = -3$$

$$x^4 - 4 = -3x^2$$

$$x^4 + 3x^2 - 4 = 0$$

$$(x^2+4)(x^2-1) = 0$$

$$x^2 = -4 \quad \cdot \quad x^2 = 1$$

But $x^2 \neq -4$ since $x = \pm 1$
 x is real.

sub into (2a)

$$\text{when } x = 1 \quad x = -1$$

$$y = 2 \quad y = -2$$

$\therefore 1+2i$ & $-1-2i$ are the square roots of $-3+4i$.
ie $\pm(1+2i)$ are the square roots of $-3+4i$

$$\text{ii) } x^2 - (4-2i)x + (6-8i) = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{4-2i \pm \sqrt{-(4-2i)^2 - 4(1)(6-8i)}}{2(1)}$$

$$x = \frac{4-2i \pm \sqrt{16-16i-4-24+32i}}{2}$$

$$x = \frac{4-2i \pm \sqrt{-12+16i}}{2}$$

$$x = \frac{4-2i \pm 2\sqrt{-3+4i}}{2}$$

$$x = 2-i \pm \sqrt{-3+4i}$$

from (i)

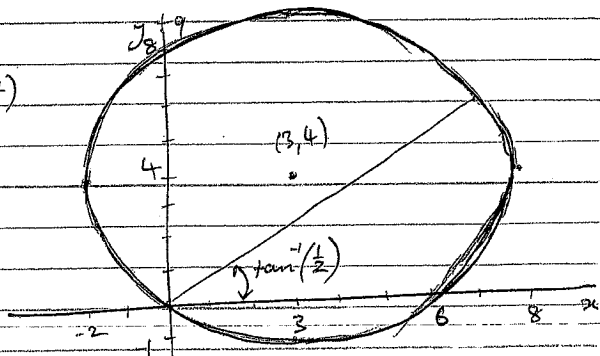
$$x = 2-i \pm (1+2i)$$

$$x = 2-i+1+2i \quad \text{or} \quad x = 2-i-1-2i$$

$$x = 3+i \quad \quad \quad x = 1-3i$$

\therefore the roots are $3+i$ & $1-3i$.

d) i) $|z - (3+4i)| = 5$
is a circle centre $(3,4)$
radius 5.



$$\text{ii) } \theta = \tan^{-1}\left(\frac{1}{2}\right) \quad (x-3)^2 + (y-4)^2 = 25 \quad \text{--- (2)}$$

$$\tan \theta = \frac{1}{2}$$

$$m = \frac{1}{2}$$

$$y = \frac{x}{2} \quad \text{--- (1)}$$

sub (1) into (2)

$$(x-3)^2 + \left(\frac{x}{2}-4\right)^2 = 25$$

$$x^2 - 6x + 9 + \frac{x^2}{4} - 4x + 16 = 25$$

$$4x^2 - 24x + 36 + x^2 - 16x + 64 = 100$$

$$5x^2 - 40x + 100 = 100$$

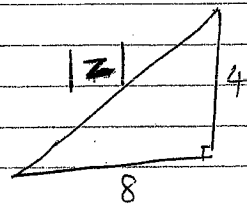
$$5x(x-8) = 0$$

$$x=0 \quad \quad x=8$$

sub into (1)

$$y=0 \quad \quad y=4$$

The line $y = \frac{x}{2}$ intersects the circle $(x-3)^2 + (y-4)^2 = 25$
at $(0,0)$ and $(8,4)$.



$$|z| = \sqrt{8^2 + 4^2}$$

$$= \sqrt{64+16}$$

$$= \sqrt{80}$$

$$= 4\sqrt{5}$$

QUESTION 3.

a, Let the polynomial be $x^3 - S_1x^2 + S_2x - S_3 = 0$ (A)

now $S_1 = \alpha + \beta + \gamma = 7$.

$S_3 = 5$

$S_1^2 = (\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

i.e. $S_1^2 = \alpha^2 + \beta^2 + \gamma^2 + 2S_2$.

$49 = 29 + 2S_2$

$\therefore S_2 = 10$.

\therefore (A) becomes $\boxed{x^3 - 7x^2 + 10x - 5 = 0}$

b, (i) $P(x) = x^4 + Ax^2 + B$ where A and B are positive

Clearly $P(x) \geq B \therefore P(x) > 0$ (x^4 and Ax^2 are ~~for~~ non-negative)
 $\therefore P(x) \neq 0 \therefore$ no real zeros.

NOTE* Those who treated $P(x)$ as a quadratic struggled to gain marks.

* Other approaches, commonly used, involved calculus to find that B is the minimum value.

(ii) Since the coefficients are real, by the conjugate root theorem, the four roots are $\pm ib$ & $\pm id$.

$\therefore x^4 + Ax^2 + B \equiv (x+ib)(x-ib)(x+id)(x-id)$
 $\equiv (x^2 + b^2)(x^2 + d^2)$
 $\equiv x^4 + (b^2 + d^2)x^2 + b^2d^2$

equating $b^2 + d^2 = A$
 $b^2d^2 = B$

now $b^4 + d^4 \equiv (b^2 + d^2)^2 - 2b^2d^2$

$\therefore \boxed{b^4 + d^4 = A^2 - 2B}$

NOTE. There were several other ways of doing this equation.

c, (i) $f(x) = x^3 - 3ax + b$

$f'(x) = 3x^2 - 3a$

For turning pts $f'(x) = 0$

i.e. $3(x^2 - a) = 0$

$x^2 = a$

$x = \pm\sqrt{a}$

\therefore turning pts exist if x is real

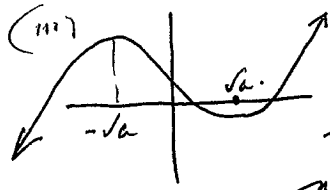
i.e. $a > 0$

(NB. to argue that x^2 is positive therefore $a > 0$ is not answering the question as asked)

$f(\sqrt{a}) = a\sqrt{a} - 3a\sqrt{a} + b$
 $= b - 2a\sqrt{a}$

$f(-\sqrt{a}) = -a\sqrt{a} + 3a\sqrt{a} + b$
 $= b + 2a\sqrt{a}$

\therefore turning points at $(\sqrt{a}, b - 2a\sqrt{a})$ and $(-\sqrt{a}, b + 2a\sqrt{a})$



For three distinct real zeros $f(a)$ and $f(-a)$ need to be opposite in sign

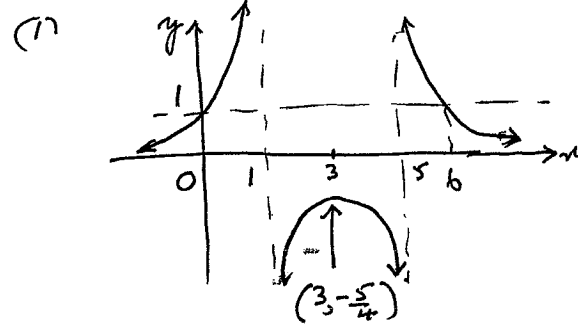
$$\text{i.e. } f(a) \times f(-a) < 0.$$

$$(b + 2a^2a)(b - 2a^2a) < 0.$$

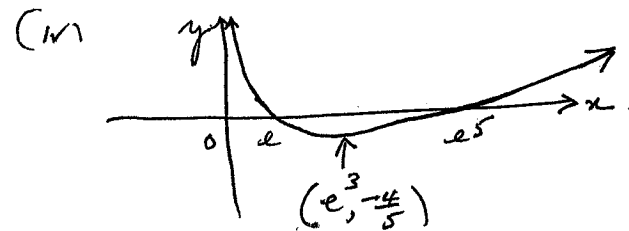
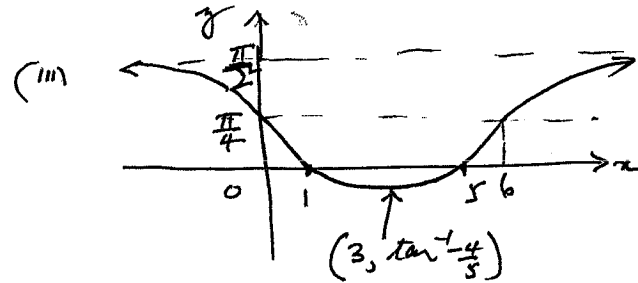
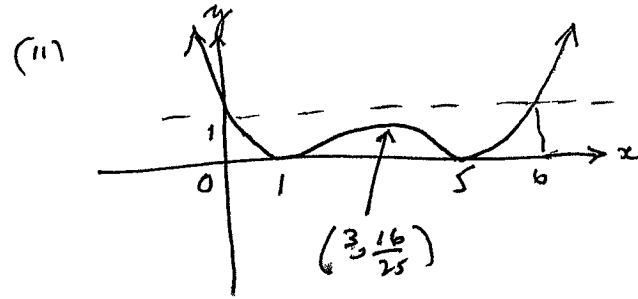
$$b^2 - 4a^3 < 0$$

$$\therefore \boxed{b^2 < 4a^3}$$

QUESTION 4.



NB
Answers required
zeros, turning
pts, intercepts
and asymptotes



(b) (i) now $f(x) = O(x) + E(x)$ — (1)

& $f(-x) = O(-x) + E(-x)$ (NB

ie $f(-x) = -O(x) + E(x)$ — (2) $\left. \begin{array}{l} O(-x) = -O(x) \\ \& E(-x) = E(x) \end{array} \right\}$

(1) - (2)

$f(x) - f(-x) = 2O(x)$

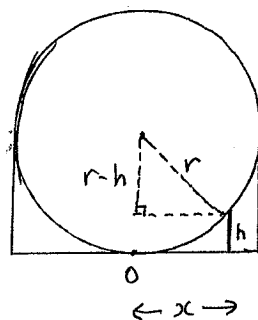
$\therefore \left[O(x) = \frac{f(x) - f(-x)}{2} \right]$

(ii) If $f(x) = e^x$

$O(x) = \left[\frac{e^x - e^{-x}}{2} \right]$ OR. $\frac{e^x - 1}{2e^x}$

QUESTION 5

(a) (i)



By Pythagoras' Theorem

$(r-h)^2 + x^2 = r^2$

ie $h^2 - 2rh + x^2 = 0$

As a quadratic in h

$\Rightarrow h = \frac{2r \pm \sqrt{4r^2 - 4x^2}}{2}$

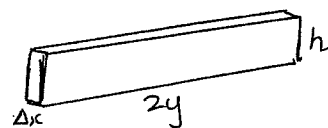
$h = r \pm \sqrt{r^2 - x^2}$

Since $h \leq r$

only $h = r - \sqrt{r^2 - x^2}$

applies

(ii) $\Delta V = \text{vol. of slice}$



$\Delta V = A \Delta x$

$\Delta V = 2yh \Delta x$

$= 2\sqrt{r^2 - x^2} [r - \sqrt{r^2 - x^2}] \Delta x$

ie $\Delta V = [2r\sqrt{r^2 - x^2} - 2(r^2 - x^2)] \Delta x$

(iii) Vol. of half-clamp V

$V = 2 \int_0^r (2r\sqrt{r^2 - x^2} - 2r^2 + 2x^2) dx$

$= \left\{ 2r \cdot \frac{1}{4} \pi r^2 - [2r^2 x]_0^r + \left[\frac{2x^3}{3} \right]_0^r \right\}$

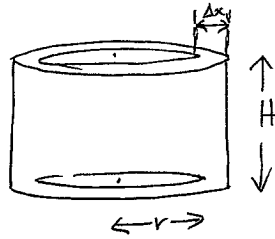
$= 2 \left\{ \frac{\pi}{2} r^3 - 2r^3 + \frac{2r^3}{3} \right\}$

$V = \left\{ \left(\frac{\pi}{2} - \frac{4}{3} \right) r^3 \right\} \text{ units}^3$

$V = \pi r^3 - \frac{8}{3} r^3$

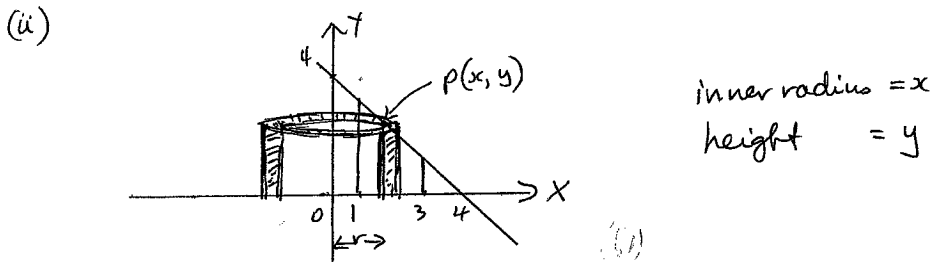
\therefore Vol. of whole clamp (one piece)

(5)(b) inner radius of shell = r
 outer radius of shell = $r + \Delta r$



(i) $\Delta V = \text{Vol. of shell}$
 $= [\pi(r + \Delta r)^2 - \pi r^2] H$
 $= \pi [r^2 + 2r\Delta r + (\Delta r)^2 - r^2] H$

$\Delta V = 2\pi r H \Delta r$ since $(\Delta r)^2$ is sufficiently small enough to be neglected.



$\Delta V = 2\pi r H \Delta r$
 $\Delta V = 2\pi x y \Delta x$ where $x = r, y = H$

$\therefore V = 2\pi \int_1^3 x y dx$
 $= 2\pi \int_1^3 x(4-x) dx$
 $= 2\pi \left[2x^2 - \frac{x^3}{3} \right]_1^3$
 $= 2\pi \left[(18 - 9) - \left(2 - \frac{1}{3}\right) \right]$
 $= 2\pi \times \frac{22}{3} = \frac{44\pi}{3} \text{ units}^3$

Question 6

(a) let $S(-2,0)$ $S(2,0)$ $T(x,y)$

(i) $TS' + TS = 6$

$\sqrt{(x+2)^2 + y^2} + \sqrt{(x-2)^2 + y^2} = 6$

Squaring, rearranging, squaring

$\Rightarrow \frac{x^2}{9} + \frac{y^2}{5} = 1$

(b) Auxiliary circle of radius 3

(ii) centro $(0,0)$ is

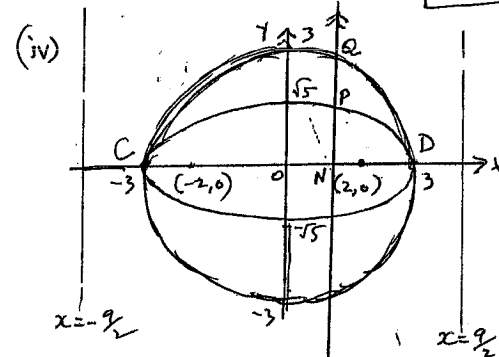
$x^2 + y^2 = 9$

(c) $b^2 = a^2(1 - e^2)$

(iii) $5 = 9(1 - e^2) \Rightarrow e = \frac{2}{3}$

Foci $(\pm ae, 0)$ i.e. $(\pm 2, 0)$

Directrices $x = \pm \frac{a}{e} \Rightarrow \frac{\pm 9}{2} = x$



(v) $N(3\cos\theta, 0)$

$P(3\cos\theta, \sqrt{5}\sin\theta)$

$Q(3\cos\theta, 3\sin\theta)$

(vi) Tangent at P is

$\frac{x\cos\theta}{3} + \frac{y\sqrt{5}\sin\theta}{5\sin\theta} = 1$ (A)

Tangent at Q

grad. is $2x + 2yy' = 0$

$y' = -\frac{x}{y}$

at Q grad is $-\frac{3\cos\theta}{3\sin\theta} = -\cot\theta$

Eqⁿ is $y - 3\sin\theta = -\cot\theta(x - 3\cos\theta)$

(vii) $\Rightarrow y\sin\theta + x\cos\theta = 3$ (B)

(x) Solving (A) and (B)

$5x\cos\theta + 3\sqrt{5}y\sin\theta = 15$ (A)

$x\cos\theta + y\sin\theta = 3$ (B)

$5 \times (B) \Rightarrow 5x\cos\theta + 5y\sin\theta = 15$ (C)

$(A) - (C) \Rightarrow (3\sqrt{5} - 5)y\sin\theta = 0$

$3\sqrt{5} - 5 \neq 0, \sin\theta \neq 0$ except at C, D

$\Rightarrow y = 0$ $x = \frac{3}{\cos\theta}$ $\cos\theta \neq 0$

$\therefore \forall x$ point R $(\frac{3}{\cos\theta}, 0)$ lies on the x-axis.

(p) $ON \cdot OR = 3\cos\theta \cdot \frac{3}{\cos\theta} = 9$

\therefore independent of P and Q.

2006 Mathematics Extension 2 Trial HSC: Solutions Part D

7. (a) (i) Show that $z^5 + 1 = (z + 1)(z^4 - z^3 + z^2 - z + 1)$.

1

Solution: R.H.S. = $z(z^4 - z^3 + z^2 - z + 1) + (z^4 - z^3 + z^2 - z + 1)$
 = $(z^5 - z^4 + z^3 - z^2 + z) + (z^4 - z^3 + z^2 - z + 1)$
 = $z^5 + 1$
 = L.H.S.

(ii) If z is a solution to $z^5 + 1 = 0$ where $z \neq -1$, prove that $1 + z^2 + z^4 = z + z^3$.

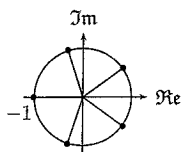
1

Solution: $(z + 1)(z^4 - z^3 + z^2 - z + 1) = 0$ but $z \neq -1$,
 $\therefore z^4 - z^3 + z^2 - z + 1 = 0$.
 Hence $1 + z^2 + z^4 = z + z^3$.

(iii) Hence show that $\cos \frac{\pi}{5} + \cos \frac{3\pi}{5} = \frac{1}{2}$.

3

Solution:



From the diagram

if $z^5 = -1$

$z = \text{cis } \pm \frac{\pi}{5}, \text{cis } \pm \frac{3\pi}{5}, -1$

Method 1: We take $z = \text{cis } \frac{\pi}{5}$.

$1 + z^2 + z^4 = z + z^3$ from (ii),

$\frac{1}{z^2} + 1 + z^2 = \frac{1}{z} + z$,

$2 \cos \frac{2\pi}{5} + 1 = 2 \cos \frac{\pi}{5}$,

$2 \cos \frac{\pi}{5} - 2 \cos \frac{2\pi}{5} = 1$,

$2 \cos \frac{\pi}{5} + 2 \cos \frac{3\pi}{5} = 1$,

$\therefore \cos \frac{\pi}{5} + \cos \frac{3\pi}{5} = \frac{1}{2}$.

Method 2: We consider the roots of $z^4 - z^3 + z^2 - z + 1 = 0$ from (ii), taken one-at-a-time,

$\text{cis } \frac{\pi}{5} + \text{cis } \frac{-\pi}{5} + \text{cis } \frac{3\pi}{5} + \text{cis } \frac{-3\pi}{5} = 1$.

But $z + \bar{z} = 2\Re(z)$,

so $\text{cis } \frac{\pi}{5} + \text{cis } \frac{-\pi}{5} = 2 \cos \frac{\pi}{5}$ etc.

$\therefore 2 \cos \frac{\pi}{5} + 2 \cos \frac{3\pi}{5} = 1$,

and $\cos \frac{\pi}{5} + \cos \frac{3\pi}{5} = \frac{1}{2}$.

(b) For integer values of k where $k = 0, 1, 2, \dots$ define I_k as follows:

$$I_k = \int_0^{\frac{\pi}{2}} \cos^k x \, dx$$

(i) Express I_{k+2} in terms of k and I_k .

2

Solution:

$u = \cos^{k+1} x$ $v' = \cos x$
 $u' = (k+1)(-\sin x) \cos^k x$ $v = \sin x$

$I_{k+2} = \int_0^{\frac{\pi}{2}} \cos^{k+1} x \cdot \cos x \, dx$,
 = $[\sin x \cos^{k+1} x]_0^{\frac{\pi}{2}} + (k+1) \int_0^{\frac{\pi}{2}} \sin^2 x \cdot \cos^k x \, dx$,

= $0 + (k+1) \int_0^{\frac{\pi}{2}} (\cos^k x - \cos^{k+2} x) \, dx$,

= $(k+1)I_k - (k+1)I_{k+2}$,

$(k+2)I_{k+2} = (k+1)I_k$,

$I_{k+2} = \frac{(k+1)}{(k+2)} I_k$.

(ii) Hence find an expression for I_{2n} , where $n = 0, 1, 2, \dots$

2

Solution: $I_0 = \int_0^{\frac{\pi}{2}} dx = I_{2 \times 0}$ i.e. $n = 0$,
 = $\frac{\pi}{2}$.

$I_2 = I_{0+2} = I_{2 \times 1}$ i.e. $n = 1$,

= $\frac{1}{2} I_0$,

= $\frac{1}{2} \cdot \frac{\pi}{2}$.

$I_4 = I_{2+2} = I_{2 \times 2}$ i.e. $n = 2$,

= $\frac{3}{4} I_2$,

= $\frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$.

$I_6 = I_{4+2} = I_{2 \times 3}$ i.e. $n = 3$,

= $\frac{5}{6} I_4$,

= $\frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$,

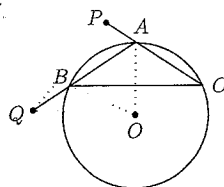
= $\frac{6}{6} \cdot \frac{5}{4} \cdot \frac{4}{4} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$,

= $\frac{1}{2^6} \cdot \frac{6!}{(3!)^2} \cdot \frac{\pi}{2}$.

$I_{2n} = \frac{1}{2^{2n}} \cdot \frac{(2n)!}{(n!)^2} \cdot \frac{\pi}{2}$.

(c) In $\triangle ABC$, in the diagram on the right, $AB = AC$.

Produce CA to P and AB to Q so that $AP = BQ$.



(i) Show that $\angle OAP = \angle OBQ$.

Solution: In $\triangle AOC, AOB$,
 $AC = AB$ (data),
 $OA = OB = OC$ (equal radii),
 $\therefore \triangle AOC \equiv \triangle AOB$ (SSS),
 $\angle OAC = \angle OBA$ (corresp. \angle s in congruent \triangle s),
 $\angle OAP + \angle OAC = 180^\circ$ ($= \angle PAC$),
 $\angle OBQ + \angle OBA = 180^\circ$ ($= \angle ABQ$),
 $\therefore \angle OAP = \angle OBQ$.

(ii) Prove that A, P, Q , and O , the centre of the circle, ABC are concyclic.

Solution: In $\triangle OAP, OBQ$,
 $AP = BQ$ (data),
 $\angle OAP = \angle OBQ$ (shown above),
 $OA = OB$ (equal radii),
 $\therefore \triangle OAP \equiv \triangle OBQ$ (SAS),
 $\angle APO = \angle BQO$ (corresp. \angle s in congruent \triangle s),
 $\angle BQO = \angle AQO$ (common),
 $\therefore \angle APO = \angle AQO$
 $\therefore A, P, Q$, and O are concyclic
 (equal angles subtended by the chord AO).

3

3

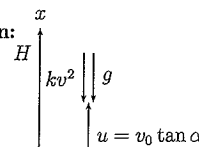
8. (a) A particle is projected vertically upwards in a resisting medium where the resistance varies as the square of the velocity and k is the constant of variation. If the velocity of projection is $v_0 \tan \alpha$,

(i) Show that the maximum height, H , reached is given by:

$$H = \frac{1}{2k} \ln \left(\frac{g + kv_0^2 \tan^2 \alpha}{g} \right)$$

3

Solution:

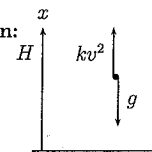


$$\begin{aligned} \ddot{x} &= -kv^2 - g \\ v \frac{dv}{dx} &= -(kv^2 + g) \\ \frac{dv}{dx} &= \frac{v}{kv^2 + g} \\ \int_0^H dx &= -\frac{1}{2k} \int_u^v \frac{2kv \, dv}{kv^2 + g} \\ 0 - H &= -\frac{1}{2k} \left[\ln(kv^2 + g) \right]_u^v \\ \text{Now, replacing } u &\text{ with } v_0 \tan \alpha, H = \frac{1}{2k} \ln \left(\frac{g + kv_0^2 \tan^2 \alpha}{g} \right). \end{aligned}$$

(ii) Show that the particle returns to the point of projection with velocity $v_0 \sin \alpha$ given that v_0 is the terminal velocity.

4

Solution:



$$\begin{aligned} \ddot{x} &= kv^2 - g \\ v \frac{dv}{dx} &= kv^2 - g \\ \text{At terminal velocity, } \ddot{x} &= 0, \\ \therefore kv_0^2 &= g, \\ \text{i.e. } v_0^2 &= \frac{g}{k} \text{ or } v_0 = \sqrt{\frac{g}{k}}. \end{aligned}$$

$$\begin{aligned} \int_H^0 dx &= -\frac{1}{2k} \int_0^{-v} \frac{2kv \, dv}{kv^2 - g}, \\ 0 - H &= \frac{1}{2k} \left[\ln(kv^2 - g) \right]_0^{-v}, \\ -H &= \frac{1}{2k} \ln \left(\frac{kv^2 - g}{-g} \right), \end{aligned}$$

Now, equating H s,

$$\begin{aligned} \frac{1}{2k} \ln \left(\frac{g + kv_0^2 \tan^2 \alpha}{g} \right) &= \frac{1}{2k} \ln \left(\frac{g}{g - kv^2} \right), \\ \text{So, } \frac{g + kv_0^2 \tan^2 \alpha}{g} &= \frac{g}{g - kv^2}, \end{aligned}$$

$$1 + \frac{kv_0^2}{g} \tan^2 \alpha = \frac{1}{1 - \frac{k}{g}v^2},$$

$$1 + \tan^2 \alpha = \frac{1}{1 - \frac{v^2}{v_0^2}},$$

$$\frac{\sec^2 \alpha}{\frac{v_0^2}{v_0^2 - v^2}} = \frac{1}{v_0^2 - v^2},$$

$$v_0^2 - v^2 = v_0^2 \cos^2 \alpha,$$

$$= v_0^2 - v_0^2 \sin^2 \alpha,$$

$$v^2 = v_0^2 \sin^2 \alpha,$$

$$v = v_0 \sin \alpha.$$

(iii) Show that the time of ascent is $\frac{v_0 \alpha}{g}$

3

Solution: From part (i),

$$\ddot{x} = -(g + kv^2),$$

$$\frac{dv}{dt} = -(g + kv^2).$$

$$\int_0^T dt = - \int_u^0 \frac{dv}{g + kv^2},$$

$$= \frac{1}{k} \int_0^{v_0 \tan \alpha} \frac{dv}{v_0^2 + v^2}, \text{ using } \frac{g}{k} = v_0^2 \text{ from (ii),}$$

$$T = \frac{1}{k} \cdot \frac{1}{v_0} \left[\tan^{-1} \frac{v}{v_0} \right]_0^{v_0 \tan \alpha},$$

$$= \frac{1}{kv_0} \tan^{-1} \tan \alpha,$$

\therefore time of ascent,

$$T = \frac{\alpha}{kv_0}, \text{ but } k = \frac{g}{v_0^2},$$

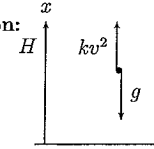
$$= \frac{\alpha}{v_0} \cdot \frac{v_0^2}{g},$$

$$= \frac{v_0 \alpha}{g}.$$

(iv) Show that the time of descent is $\frac{v_0}{g} \ln(\sec \alpha + \tan \alpha)$.

2

Solution:



$$\ddot{x} = kv^2 - g,$$

$$\frac{dv}{dt} = kv^2 - g.$$

$$\int_0^T dt = \int_0^{-v_0 \sin \alpha} \frac{dv}{kv^2 - g},$$

$$= \frac{1}{k} \int_0^{-v_0 \sin \alpha} \frac{dv}{v^2 - v_0^2}.$$

$$\frac{1}{v^2 - v_0^2} \equiv \frac{A}{v + v_0} + \frac{B}{v - v_0},$$

$$1 \equiv A(v - v_0) + B(v + v_0).$$

$$\text{Put } v = v_0, \quad B = \frac{1}{2v_0},$$

$$\text{put } v = -v_0, \quad A = -\frac{1}{2v_0}.$$

$$\text{Then, } T = \frac{1}{2kv_0} \int_0^{-v_0 \sin \alpha} \left(\frac{1}{v - v_0} - \frac{1}{v + v_0} \right) dv,$$

$$= \frac{1}{2kv_0} \left[\ln(v - v_0) - \ln(v + v_0) \right]_0^{-v_0 \sin \alpha},$$

$$= \frac{1}{2kv_0} \left\{ \ln \left(\frac{-v_0 \sin \alpha - v_0}{-v_0} \right) - \ln \left(\frac{v_0 - v_0 \sin \alpha}{v_0} \right) \right\},$$

$$= \frac{1}{2kv_0} \ln \left\{ \left(\frac{\sin \alpha + 1}{1 - \sin \alpha} \right) \cdot \left(\frac{1 + \sin \alpha}{1 + \sin \alpha} \right) \right\},$$

$$= \frac{1}{2kv_0} \ln \left\{ \frac{(\sin \alpha + 1)^2}{\cos^2 \alpha} \right\},$$

$$= \frac{1}{kv_0} \ln \sqrt{(\sec \alpha + \tan \alpha)^2},$$

$$= \frac{1}{kv_0} \ln(\sec \alpha + \tan \alpha).$$

(b) Prove by induction that, for all integers n where $n > 1$, that

3

$$\frac{4^n}{n+1} < \frac{(2n)!}{(n!)^2}$$

Solution: Test for $n = 2$:

$$\begin{aligned} \text{L.H.S.} &= \frac{4^2}{3}, & \text{R.H.S.} &= \frac{4!}{(2!)^2}, \\ &= 5\frac{1}{3}, & &= 6. \end{aligned}$$

\therefore True for $n = 2$.

Now, assume true for $n = k \geq 2$,

$$\text{i.e. } \frac{4^k}{k+1} < \frac{(2k)!}{(k!)^2}$$

Then test for $n = k + 1$,

$$\text{i.e. } \frac{4^{k+1}}{k+2} < \frac{(2k+2)!}{((k+1)!)^2}$$

$$\begin{aligned} \text{L.H.S.} &= \frac{4 \cdot 4^k}{k+1} \times \frac{k+1}{k+2}, \\ &< \frac{(2k)!}{(k!)^2} \times \frac{4(k+1)}{k+2} \text{ by the assumption,} \\ &< \frac{(2k+2)!}{((k+1)!)^2} \times \frac{(k+1)^2}{(2k+2)(2k+1)} \times \frac{4(k+1)}{(k+2)}, \\ &< \frac{(2k+2)!}{((k+1)!)^2} \times \frac{(2k^2+4k+2)}{(2k^2+5k+2)}, \\ &< \frac{(2k+2)!}{((k+1)!)^2} \times \left(1 - \frac{k}{2k^2+5k+2}\right), \\ &< \frac{(2k+2)!}{((k+1)!)^2} \text{ as } \left(1 - \frac{k}{2k^2+5k+2}\right) < 1. \end{aligned}$$

$$\begin{aligned} \text{Alternatively, R.H.S.} &= \frac{(2k)!}{(k!)^2} \times \frac{(2k+2)(2k+1)}{(k+1)^2}, \\ &> \frac{4^k}{k+1} \times \frac{2(2k+1)}{(k+1)} \text{ by the assumption,} \\ &> \frac{4^{k+1}}{k+2} \times \frac{4(k+1)}{(k+2)} \times \frac{2(2k+1)}{(k+1)}, \\ &> \frac{4^{k+1}}{k+2} \times \frac{(2k^2+5k+2)}{(2k^2+4k+2)}, \\ &> \frac{4^{k+1}}{k+2} \times \left(1 + \frac{k}{2k^2+4k+2}\right), \\ &> \frac{4^{k+1}}{k+2} \text{ as } \left(1 + \frac{k}{2k^2+4k+2}\right) > 1. \end{aligned}$$

\therefore The statement is true for $n = k + 1$ if true for $n = k$.

As true for $n = 2$, so true for $n = 3, 4, \dots$ and so on for all natural numbers $n > 1$.