11MaA Class Test #3 — 29th June, 2004

TOPICS: SERIES, PROBABILITY, COÖRDINATE GEOMETRY, APPLICATIONS OF DIFFERENTIATION, AND APPROXIMATE METHODS OF INTEGRATION

Instructions:

1. Time allowed is 90 minutes plus 5 minutes reading time.
2. All necessary working must be shown for full marks. (Marks will be deducted for careless or badly-arranged work.)
3. Use black or blue pen for written answers, but pencil for diagrams and graphs.
4. The value of each question is boxed in the left margin.
5. Board-approved calculators may be used

1. Find the next two terms and write down the general term in the following series:

[3] (a) $18 + 16 + 14 + 12 + \ldots$

[3] (b) $1024 - 512 + 256 - 128 + \ldots$

[3] (c) $\sqrt{3} + \sqrt{12} + \sqrt{27} + \ldots$

[3] (d) $1 + 0 + 1 + 4 + 9 + \ldots$

2. From a pack of cards the four aces are turned face down on a table and thoroughly mixed. A girl selects one card. What are her chances of selecting:

[1] (a) the ace of clubs?

[1] (b) a red ace?

[2] 3. (a) Which term of the series $7 + 12 + 17 + \ldots$ is 372?

[3] (b) Find the number of terms in an arithmetic series with $a = 5$, $d = 2$, and the last term 43.

[3] (c) Find the sum to 20 terms of the series whose nth term is $3n - 1$.

4. At a school assembly 62% of the students said they were in favour of a new design for the school hat. If the principal asked a student at random about the proposed design, what is the probability that this student was:

[1] (a) in favour of the new design?

[1] (b) not in favour?

[4] 5. Prove by the methods of coördinate geometry that the mid point of the hypotenuse of any right triangle is equidistant from the vertices.

6. Evaluate

[3] (a) $\displaystyle\sum_{n=4}^{14}(8-n)$

[3] (b) $\displaystyle\sum_{n=1}^{7}(-5)^{n-1}$

[3] 7. Find the domain for which the curve $y = x^3 - 8x^2 + 6x - 3$ is concave upwards.

8. A number is formed using all five digits 2, 3, 4, 5, 6. What is the probability that the number:

[1] (a) starts with 4?

[1] (b) is even?

[1] (c) is odd?

[1] (d) is greater than 30 000?

[1] (e) is divisible by 3?

[3] 9. (a) If the nth term of a series is given by $U_n = 5n - 23$, how many terms of the series are negative?

[3] (b) How many terms of the series $2 + 6 + 18 + 54 + \ldots$ are needed to give a sum of 728?

(c) Can there be a geometric series with

[2] (i) a limiting sum of $\frac{2}{3}$ and a first term of 4?

[2] (ii) a limiting sum of 4 and a first term of $\frac{2}{3}$?

10. A mortgage of \$450 000 is taken out over a thirty year term at 6·9% p.a. (reducible).

[1] (a) Show that the amount owing after the first monthly payment will be $\$452\,587{\cdot}50 - Q$, where Q is the monthly payment.

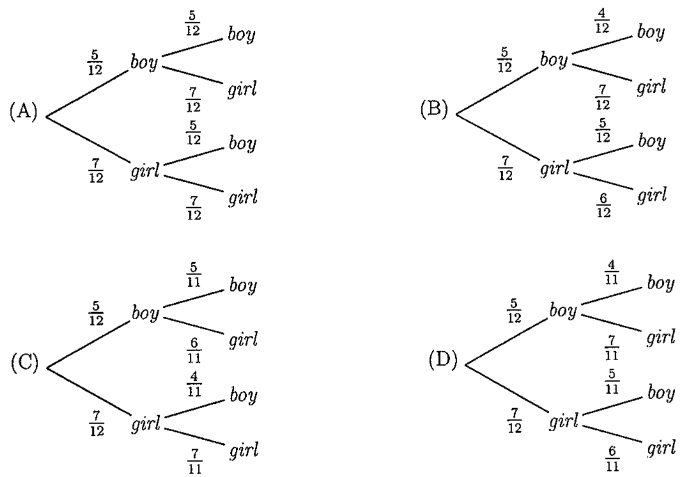
[4] (b) Find the value of Q.

[4] (c) After 10 years the partner begins to contribute equally (*i.e.*, they make a payment of $2Q$). How long will it now take to finish paying off the loan? Answer to the nearest month rounded up.

[3] 11. If $\frac{3}{4}$ of the boys in a junior class use a backpack, and $\frac{2}{5}$ are not in full uniform, find the probability that a boy chosen by lot would be in uniform and using a backpack.

12. From 5 boys and 7 girls, two students will be chosen at random to work together on a project.

[1] (a) Which of the following probability trees could be used to determine the probability of choosing a boy and a girl?



(b) Briefly explain why you rejected the other three.

[2] (i) I rejected ... because ...

[2] (ii) I rejected ... because ...

[2] (iii) I rejected ... because ...

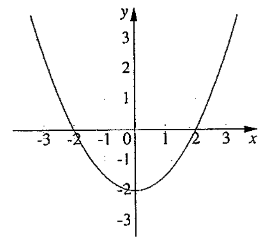
[4] 13. According to post office regulations, the sum of the dimensions (length, width, and height) of a parcel must not exceed two metres. What is the maximum allowed volume of a parcel with a length twice its width?

[5] 14. Find the stationary points on the graph of the function:

$$f(x) = (x-1)^2(x+1)$$

For each point, determine whether it is a maximum, a minimum, or a point of inflexion. Sketch the curve, showing the points where it cuts the axes and the stationary points.

[4] 15. The figure below shows the graph of the gradient function $p'(x)$. Draw a neat sketch of a possible primitive function $p(x)$.



[5] 16. Given that $\frac{d^2x}{dt^2} = 12t - 2$, that $\frac{dx}{dt} = 0$ when $t = 0$ and that $x = 4$ when $t = 1$, find x when $t = 5$.

End of the paper

11MaA Test #3 29 June 2004 solutions

[3] 1. (a) $+10+8+\quad+(20-2n)+\ldots$

[3] (b) $+64-32+\quad+(-1)^{n-1}2^{11-n}+\ldots$

[3] (c) $+4\sqrt{3}+5\sqrt{3}+\quad+n\sqrt{3}+\ldots$

[3] (d) $+16+25+\quad+(n-2)^2+\ldots$

[1] 2. (a) $\dfrac{1}{4}$.

[1] (b) $\dfrac{1}{2}$.

[2] 3. (a)
$$\begin{aligned} U_n &= 2+5n, \\ 372 &= 2+5n, \\ 5n &= 370, \\ n &= 74, \quad \text{i.e. the } 74^{\text{th}} \text{ term.} \end{aligned}$$

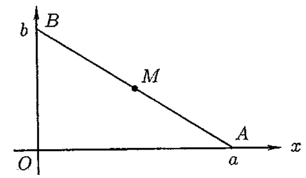
[3] (b)
$$\begin{aligned} U_n &= a+(n-1)d, \\ 43 &= 5+2(n-1), \\ 38 &= 2n-2, \\ 2n &= 40, \\ n &= 20, \quad \text{i.e. there are 20 terms.} \end{aligned}$$

[3] (c)
$$\begin{aligned} a &= 2, \\ d &= 5-2, \\ &= 3, \\ S_{20} &= 10(4+19\times 3), \\ &= 610. \end{aligned}$$

[1] 4. (a) $\dfrac{62}{100}=\dfrac{31}{50}$.

[1] (b) $\dfrac{100-62}{100}=\dfrac{19}{50}$.

5.



In ΔABO, $M=\left(\frac{a}{2},\ \frac{b}{2}\right)$
$$\begin{aligned} \overline{MO} &= \sqrt{\tfrac{a^2}{4}+\tfrac{b^2}{4}}, \\ &= \frac{\sqrt{a^2+b^2}}{2}. \\ \overline{MA} &= \sqrt{\left(a-\tfrac{a}{2}\right)^2+\tfrac{b^2}{4}}, \\ &= \frac{\sqrt{a^2+b^2}}{2} \\ \overline{MB} &= \sqrt{\tfrac{a^2}{4}+\left(\tfrac{b}{2}-b\right)^2}, \\ &= \frac{\sqrt{a^2+b^2}}{2} \end{aligned}$$

$\therefore\ \overline{MO}=\overline{MA}=\overline{MB}$.

[3] 6. (a)
$$\begin{aligned} \sum_{n=4}^{14}(8-n) &= 4+3+2+1+0-1-2-3-4-5-6, \\ &= -11. \end{aligned}$$

[3] (b)
$$\begin{aligned} \sum_{n=1}^{7}(-5)^{n-1} &= 1-5+25-125+625-3125+15625, \\ &= 13021. \end{aligned}$$
$$\begin{aligned} \text{Or } \sum_{n=1}^{7}(-5)^{n-1} &= \frac{1\times\left((-5^7-1\right)}{-5-1}, \\ &= 13021. \end{aligned}$$

[3] 7.
$$\begin{aligned} y &= x^3-8x^2+6x-3, \\ y' &= 3x^2-16x+6, \\ y'' &= 6x-16. \end{aligned}$$
Concave upwards when $6x-16>0$, *i.e.* $x>\frac{8}{3}$

[1] 8. (a) $\frac{1}{5}$.

[1] (b) $\frac{3}{5}$.

[1] (c) $\frac{2}{5}$.

[1] (d) $\frac{4}{5}$.

[1] (e) 0. Note that $2+3+4+5+6=20\neq 3n,\ n\in\mathbb{Z}$.

[3] 9. (a) $5n - 23 < 0$,
$5n < 23$,
$n < 4\frac{3}{5}$.
$\therefore$ The first 4 terms are negative.

[3] (b) $a = 2,\ r = 3.$ $728 = \dfrac{2(3^n - 1)}{3 - 1}$,
$3^n = 729$,
$n = 6$, *i.e.* 6 terms are needed.

[2] (c) (i) $\frac{2}{3} = \dfrac{4}{1 - r}$, $\quad S_\infty = \dfrac{a}{1-r},\ |r| < 1$
$12 = 2 - 2r$,
$r = -5$.
$|-5| > 1$, $\therefore$ No, not possible.

[2] (ii) $4 = \dfrac{2}{3} \times \dfrac{1}{1-r}$,
$12 - 12r = 2$,
$-12r = -10$,
$r = \frac{5}{6}$.
< 1, $\therefore$ Yes, possible.

[1] 10. (a) Owing after 1 month $= PR - Q$,
$= \$450\,000\left(1 + \dfrac{6{\cdot}9}{1200}\right) - Q$,
$= \$450\,000 \times 1{\cdot}00575 - Q$,
$= \$452\,587{\cdot}50 - Q$.

[4] (b) Owing after 2 months $= (PR - Q)R - Q$,
Owing after 3 months $= ((PR - Q)R - Q)R - Q$,
$= PR^3 - QR^2 - QR - Q$,
$= PR^3 - Q(R^2 + R + 1)$,
Owing after n months $= PR^n - \dfrac{Q(R^n - 1)}{R - 1}$.
So, after 30 years, $0 = PR^{360} - \dfrac{Q(R^{360} - 1)}{R - 1}$.
i.e. $Q = \dfrac{PR^{360}(R - 1)}{R^{360} - 1}$,
$= \dfrac{\$450\,000 \times 1{\cdot}00575^{360} \times 0{\cdot}00575}{1{\cdot}00575^{360} - 1}$,
$= \$2\,963{\cdot}70$ (to the nearest cent.)

[4] (c) Owing after 10 years $= \$450\,000 \times 1{\cdot}00575^{120} - \dfrac{\$2\,963{\cdot}70(1{\cdot}00575^{120} - 1)}{0{\cdot}00575}$,
$= \$385\,242{\cdot}50$ (or $\$385\,242{\cdot}60$ if working with whole cents).
For the residue, when paid off,
$PR^n(R - 1) = 2Q(R^n - 1)$,
$R^n(PR - P) = R^n(2Q) - 2Q$,
$R^n = \dfrac{-2Q}{PR - P - 2Q}$,
$n \log R = \log \dfrac{2Q}{2Q + P - PR}$,
$n = \dfrac{\log\left(\dfrac{2 \times 2963{\cdot}7}{2 \times 2963{\cdot}7 + 385\,242{\cdot}6 - 385\,242{\cdot}6 \times 1{\cdot}00575}\right)}{\log 1{\cdot}00575}$,
$= \dfrac{1{\cdot}596710968}{1{\cdot}00575}$,
$= 81{\cdot}61563957$ (by calculator).
$\therefore$ It will take 82 more months (or a further 6 years, 10 months).

[3] 11. $P(\text{Uniform} \cap \text{backpack}) = \dfrac{3}{4} \times \dfrac{3}{5}$,
$= \dfrac{9}{20}$.

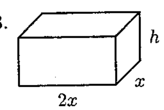
[1] 12. (a) D

[2] (b) I rejected A as both numerator and denominator remained unchanged after the first selection, *i.e.*, selection with replacement which is absurd in this case.

[2] (c) I rejected B because although the numerator was replaced correctly, the denominator remained unchanged.

[2] (d) I replaced C because the wrong numerator was reduced after each selection (boy for girl, girl for boy).

[4] 13.



$3x + h = 2$, (considering the maximum case)
$h = 2 - 3x$.
$\therefore$ Volume, $V = 2x^2(2 - 3x)$,
$= 4x^2 - 6x^3$.
Now, $\dfrac{dV}{dx} = 8x - 18x^2$,
$= 0$ when $x = 0\ \frac{4}{9}$ m.
$\dfrac{d^2V}{dx^2} = 8 - 36x$,
> 0 when $x = 0$ m,
< 0 when $x = \frac{4}{9}$ m.
Maximum volume $= 2\left(\frac{4}{9}\right)^2\left(2 - 3\left(\frac{4}{9}\right)\right)$ m^3,
$= \frac{64}{243}$ m^3,
$= 0{\cdot}2634$ m^3 nearest cm^3.

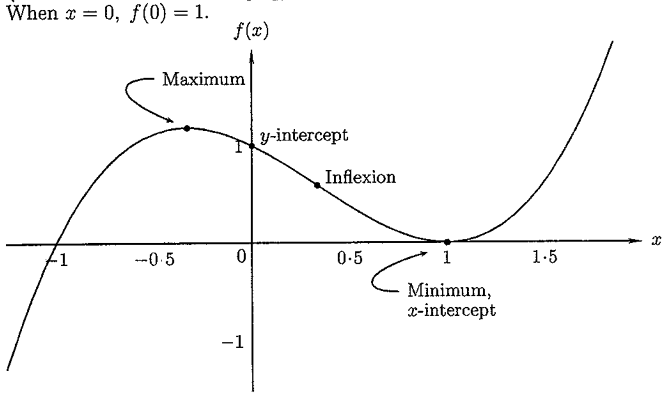
[5] 14.

$$\begin{aligned}
f(x) &= (x-1)^2(x+1).\\
f'(x) &= (x-1)^2 + 2(x-1)(x+1),\\
&= (x-1)(2x+2+x-1),\\
&= (x-1)(3x+1).\\
f''(x) &= 3x+1+3(x-1),\\
&= 6x-2.
\end{aligned}$$

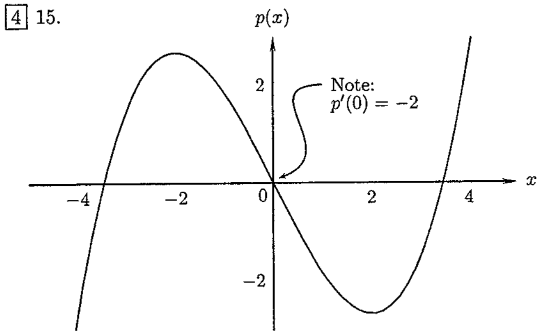
Now,

$$\begin{aligned}
f'(x) &= 0 \text{ when } x = 1,\ -\tfrac{1}{3},\\
f''(1) &= 4,\\
f''(-\tfrac{1}{3}) &= -4,\\
f''(x) &= 0 \text{ when } x = \tfrac{1}{3}.
\end{aligned}$$

$\therefore$ Minimum at $(1,\ 0)$,
Maximum at $(-\frac{1}{3},\ \frac{32}{27})$,
(also note an inflexion at $(\frac{1}{3},\ \frac{16}{27})$),
When $x = 0,\ f(0) = 1$.



[4] 15.



Also note that the vertical position of the x-axis is not important. Moving it up or down corresponds to different values of the constant, C, in the integration. The slope at $p(0)$ *is* important and can be read off the original graph.

[5] 16.

$$\begin{aligned}
\frac{d^2x}{dt^2} &= 12t - 2,\\
\frac{dx}{dt} &= 6t^2 - 2t + c,\\
0 &= 0 + c \text{ (when } t = 0).\\
\therefore\ \frac{dx}{dt} &= 6t^2 - 2t.\\
x &= 2t^3 - t^2 + k,\\
4 &= 2 - 1 + k \text{ (when } t = 1).\\
\therefore\ x &= 2t^3 - t^2 + 3.
\end{aligned}$$

Now, when $t = 5$,

$$\begin{aligned}
x &= 2 \times 125 - 25 + 3,\\
&= 228.
\end{aligned}$$