Sydney Boys High School

MOORE PARK
SURRY HILLS

DECEMBER 2003

HSC Assessment Task #1

YEAR 11

Mathematics

General Instructions

- Reading time – 5 minutes.
- Working time 90 minutes.
- Write using black or blue pen.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may **NOT** be awarded for careless or badly arranged work.
- Start each question in a separate answer booklet.

Total Marks - 80 Marks

- Attempt Questions 1 to 5
- All questions are of equal value.

Examiner: *R. Boros*

Question 1: (16 marks) Marks

(a) Evaluate $\log_p 18$ given that $\log_p 3 = 0\cdot 4771$ and $\log_p 2 = 0\cdot 3010$. 2

(b) Write a single logarithm for $\log x - \log y + 2\log z$. 1

(c) For what value of n is the sum of n terms of $12 + 15 + 18 + \ldots$ equal to 441? 2

(d) Evaluate $\sum_{n=3}^{13} 2^n$ 2

(e) One card is drawn out from a set of cards numbered 1 to 20. Find the probability of drawing out an even number or a number less than 8. 2

(f) When 2 regular dice are thrown and the total on these dice are counted, find the probability of scoring a total greater than 7. 2

(g) A plant has a probability $0\cdot 7$ of producing a variegated leaf. If 3 plants are grown, find the probability of producing no plants with variegated leaves. 3

(h) A coin is tossed n times. Find an expression for the probability of throwing at least 1 tail. 2

Question 2: (16marks) START A NEW BOOKLET — Marks

(a) Simplify $\dfrac{(x^{m+1})^2 \times (x^3)^{n+1}}{x^{5m}}$. — 2

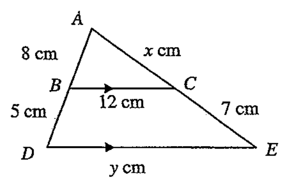
(b) Solve for x: $2^{x-1} = \dfrac{\sqrt{2}}{32}$ — 2

(c) Write in simplest form: $\dfrac{2^{n+2}+8}{2^{2n}+2^{n+1}}$ — 2

(d) Show that the points $A(6a,-2b)$, $B(2a,0)$ and $C(0,b)$ are collinear. — 2

(e) Prove that the points $A(3,5)$, $B(4,4)$, $C(1,1)$ and $D(0,2)$ are the vertices of a rectangle. — 4

(f) Prove that $\Delta ABC\ |||\ \Delta ADE$. Hence find the values of x and y. — 4



Question 3: (16 marks) START A NEW BOOKLET — Marks

(a) Find $\lim_{x\to 1} \dfrac{x^2+2x-3}{x-1}$ — 2

(b) (i) Find the gradient of the tangent to the curve $y = x^2+2x+1$ at the point (x, y).

(ii) Hence find the gradient of the tangent at the point $(\frac{1}{2}, 2\frac{1}{4})$. — 3

(iii) Find the angle which the tangent in (ii) makes with the positive direction of the x axis.

(c) Find the first derivative of:

(i) $y = \dfrac{-7}{x+1}$

(ii) $y = (x^2+x)^3$ — 5

(iii) $y = \dfrac{1}{\sqrt{3x^2+4}}$

(d) Find the gradient of the normal to the curve $y = 5x\sqrt{4-x}$ at the point $(3,15)$ — 3

(e) Find the maximum value of the function $y = x^2-4x+3$ in the domain $1 \le x \le 4$. — 3

Question 4: (16 marks) START A NEW BOOKLET Marks

(a) For the curve $y = 2x^3 - 3x^2 - 12x + 2$:

(i) Find all stationary points.

(ii) Determine the nature of the stationary points. 9

(iii) Find any points of inflexion.

(iv) Sketch the curve.

(b) Show that $y = \frac{5}{x}$ is always a decreasing function for all real $x \neq 0$. 2

(c) Draw a neat sketch of a continuous curve $y = f(x)$ which has the following features:

$f'(x) < 0$ for $0 \le x < 3$

$f'(3) = 0$

$f'(x) > 0$ for $3 < x < 7$ 3

$f'(7) = 0$ and

$f'(x) > 0$ for $7 < x \le 10$.

(d) For a certain curve $y'' = x^2(x-1)^3(x-3)$, for what values of x is the curve concave up? 2

Question 5: (16 marks) START A NEW BOOKLET Marks

(a) Solve for x (correct to 2 decimal places): $2^x = 3^{x-1}$. 2

(b) If $x^2 + y^2 = 7xy$, show that $\log(x+y) = \log 3 + \frac{1}{2}\log x + \frac{1}{2}\log y$. 2

(c) A ball is dropped from a height of 1 metre and bounces to $\frac{2}{3}$ of its height on each bounce. What is the total distance travelled by the ball? 3

(d) A sum of \$3 000 is invested at the beginning of each year in a superannuation fund. At the end of 35 years, how much money is available if the money invested earns interest at the rate of 6% per annum (compounded annually). 4

(e) A sum of \$75 000 is borrowed at an interest rate of 12% per annum, monthly reducible. If the money is repaid at regular monthly intervals over 10 years, find the amount of each payment. 5

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SAMPLE SOLUTIONS

16

Question 1

(a) $\log_p 18 = \log_p 3^2 + \log_p 2$
$= 2\log_p 3 + \log_p 2$
$= 1.2552$ **2**

(1)(b) $\log x - \log y + 2\log z$
$= \log\left(\frac{xz^2}{y}\right)$.

(c) $S_n = \frac{n}{2}\left(2a + (n-1)d\right)$
$\therefore 441 = \frac{n}{2}\left[24 + (n-1)3\right]$
$\therefore 441 = \frac{3n}{2}(n+7)$.
$\Longrightarrow n^2 + 7n - 294 = 0$
$\therefore (n+21)(n-14) = 0$
$\therefore n = 14$ **2**

(d) $2^3 + 2^4 + \cdots + 2^{13}$
$n = 11$, $a = 2^3$; $r = 3$.
$\therefore S_{11} = 8(2^{11} - 1)$
$= 8 \times 2047$
$= 16376$. **2**

(e) $P(\text{even}) = \frac{1}{2}$.
$P(x < 8) = \frac{7}{20}$
$\therefore P(E)$ or $P(x<8)$
$= \frac{1}{2} + \frac{7}{20} - \frac{3}{20}$.
$= \frac{14}{20}$
$= 7/10$ **2**

(f) $S = 6 \times 6 = 36$
sum > 7.
(2,6) (6,2) (3,5) (5,3)
(3,6) (6,3) (4,4), (4,5)
(5,4), (4,6) (6,4) (5,5)
(5,6) (6,5), (6,6).
$\therefore P(\text{sum} > 7) = \frac{15}{36}$
$= \frac{5}{12}$. **2**

(g) $(0.3)^3 = 0.027$
(3).

(h)
P(at least 1 tail)
= 1 − Pr(no tail)
$= 1 - \frac{1}{2^n}$ **2**

[2] 2. (a) $\dfrac{x^{2m+2} \times x^{3n+3}}{x^{5m}} = \dfrac{x^{2m+3n+5}}{x^{5m}},$

$= x^{3n-3m+5}.$

Or (if you realised it was a typo)—

$\dfrac{(x^{m+1})^2 \times (x^3)^{m+1}}{x^{5m}} = \dfrac{x^{2m+2} \times x^{3m+3}}{x^{5m}},$

$= x^5.$

[2] (b) $2^{x-1} = 2^{1/2} \times 2^{-5},$

$x - 1 = -4\frac{1}{2},$

$x = -3\frac{1}{2}.$

[2] (c) $\dfrac{2^{n+2} + 2^3}{2^{2n} + 2^{n+1}} = \dfrac{2^2(2^n+2)}{2^n(2^n+2)},$

$= 2^{2-n}.$

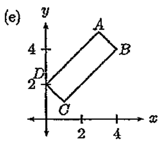
[2] (d) Slope $AB = \dfrac{0 - {}^-2b}{2a - 6a},$ $\quad$ Slope $BC = \dfrac{b - 0}{0 - 2a},$

$= \dfrac{2b}{-4a},$ $\quad$ $= -\dfrac{b}{2a},$

$= -\dfrac{b}{2a}.$ $\quad$ $=$ Slope AB.

As B is common, ABC is a straight line.

[4] (e)



$m_{AB} = -1,$ $\quad$ so $m_{AB} \times m_{BC} = -1,$

$m_{BC} = 1,$ $\quad$ $m_{BC} \times m_{CD} = -1,$

$m_{CD} = -1,$ $\quad$ $m_{CD} \times m_{DA} = -1,$

$m_{DA} = 1.$ $\quad$ $m_{DA} \times m_{AB} = -1.$

$\therefore$ All vertices are right angles and $ABCD$ is a rectangle.

[4] (f) $\widehat{A}$ is common,

$A\widehat{B}C = A\widehat{D}E$ (corresponding angles, $BC /\!/ DE$),

$\therefore\ \triangle ABC /\!/\!/ \triangle ADE$ (equiangular).

$\dfrac{x}{x+7} = \dfrac{8}{13},$ $\quad$ $\dfrac{y}{12} = \dfrac{13}{8},$

$13x = 8x + 56,$ $\quad$ $2y = 39,$

$5x = 56,$ $\quad$ $y = 19\frac{1}{2}$

$x = 11\frac{1}{5}$

Question 3

(a) $\lim_{x\to1} \dfrac{x^2+2x-3}{x-1}$

$= \lim_{x\to1} \dfrac{(x-1)(x+3)}{x-1}$

$= \lim_{x\to1} (x+3)$

$= 4$ [2]

(b) $y = x^2 + 2x + 1$

(i) $y' = 2x + 2$ [1]

(ii) At $(\frac{1}{2}, 2\frac{1}{4})$ $y' = 2(\frac{1}{2}) + 2$

$= 3$

$\therefore$ Gradient $= 3$ [1]

(iii) $m = 3 = \tan\alpha$

where α is the angle of inclination

$\therefore \alpha = \tan^{-1} 3$

$\doteqdot 71°34'$ [1]

(c) (i) $y = \dfrac{-7}{x+1}$

$= -7(x+1)^{-1}$

$y' = 7(x+1)^{-2} \times 1$

$= \dfrac{7}{(x+1)^2}$ [1]

(ii) $y = (x^2+x)^3$

$y' = 3(x^2+x)^2 \cdot \dfrac{d}{dx}(x^2+x)$

$= 3(x^2+x)^2 \times (2x+1)$

$= (6x+3)(x^2+x)^2$ [1]

$= 3x^2(2x+1)(x+1)^2$

(iii) $y = \dfrac{1}{\sqrt{3x^2+4}}$

$y' = \dfrac{-1}{(\sqrt{3x^2+4})^2} \times \dfrac{d}{dx}\sqrt{3x^2+4}$

$= \dfrac{-1}{3x^2+4} \cdot \dfrac{1}{2\sqrt{3x^2+4}} \cdot \dfrac{d}{dx}(3x^2+4)$

$= \dfrac{-6x}{2(3x^2+4)\sqrt{3x^2+4}}$

$= \dfrac{-3x}{(3x^2+4)^{3/2}}$ [2]

(d) $y = 5x\sqrt{4-x}$

$y' = 5\sqrt{4-x} + 5x \cdot \dfrac{-1}{2\sqrt{4-x}}$

$= 5\left[\sqrt{4-x} - \dfrac{x}{2\sqrt{4-x}}\right]$

At $x=3$ Grad. of $y = 5\left[1 - \dfrac{3}{2\times1}\right]$

$= 5\left[-\frac{1}{2}\right]$

$= -\frac{5}{2}$

$\therefore$ Grad. of Normal $= \dfrac{2}{5}$ [3]

(e) $y = x^2 - 4x + 3$

Function has a minimum.

$\therefore$ Max Value at boundary.

$y(1) = 1 - 4 + 3$

$= 0$

$y(4) = 16 - 16 + 3$

$= 3$

$\therefore$ Max Value $= 3$ [3]

QUESTION 4

(a) $y = 2x^3 - 3x^2 - 12x + 2$

(i) Stat. pts where $y' = 0$

$\therefore y' = 6x^2 - 6x - 12$

ie $y' = 6(x-2)(x+1) = 0$ 3

when $x = 2$ and $x = -1$.

$\therefore (2, -18)$ and $(-1, 9)$

(ii) $y'' = 12x - 6$

At $x = 2$, $y'' = 18 > 0$

$\therefore$ MIN $(2, -18)$ 1

At $x = -1$, $y'' = -18 < 0$

$\therefore$ MAX $(-1, 9)$ 1

(iii) For P.O.I. $f''(x) = 0$ and there must be a change of concavity.

ie $f''(x)$ must change sign.

Now $f''(x) = 0$ when $x = \frac{1}{2}$

and

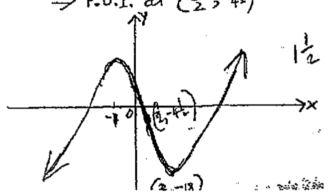
x	<	$\frac{1}{2}$	>
$f''(x)$	−	0	+

1

$\therefore$ at $x = \frac{1}{2}$ curve changes concavity from c.d. to c.u. 1

$\Rightarrow$ P.O.I. at $(\frac{1}{2}, -4\frac{1}{2})$

(iv) $1\frac{1}{2}$



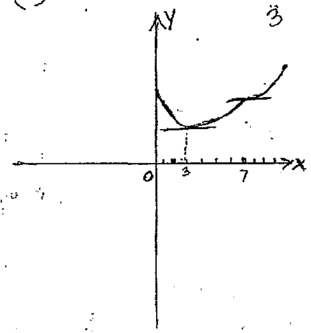
(b) $y = \frac{5}{x}$ 2

$\frac{dy}{dx} = \frac{-5}{x^2} < 0$

for all real x where $x \neq 0$

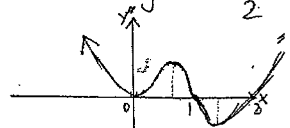
$\therefore y = \frac{5}{x}$ is decreasing for these values of x.

(c) 3



(d) $y'' = x^2(x-1)^3(x-3)$.

c.u. when $y'' > 0$ 2



$\therefore$ c.u. when

$x < 1$ and $x > 3$

Note: $x \neq 0$

Question 5

(a)

$2^x = 3^{x-1}$

$\log_{10} 2^x = \log_{10} 3^{x-1}$

$x \log_{10} 2 = (x-1)\log_{10} 3$

$x \log_{10} 2 - x\log_{10} 3 = -\log_{10} 3$

$x(\log_{10} 2 - \log_{10} 3) = \log_{10} 3^{-1} = \log_{10} 1/3$

$x(\log_{10} 2/3) = \log_{10} 1/3$

$x = \dfrac{\log_{10} 1/3}{\log_{10} 2/3} \approx 2 \cdot 71$

(b)

$x^2 + y^2 = 7xy \Rightarrow x^2 + y^2 + 2xy = 9xy$

$x^2 + y^2 + 2xy = (x+y)^2 = 9xy$

$\therefore \log(x+y)^2 = \log 9xy$

$\therefore 2\log(x+y) = \log 9 + \log x + \log y = \log 3^2 + \log x + \log y$

$\therefore 2\log(x+y) = 2\log 3 + \log x + \log y$

$\therefore \log(x+y) = \log 3 + \frac{1}{2}\log x + \frac{1}{2}\log y$

QED

(c) Distance

$= 1 + 2\times(\frac{2}{3}\times 1) + 2\times(\frac{2}{3}\times\frac{2}{3}) + 2\times(\frac{2}{3}\times(\frac{2}{3})^2) + \cdots$

$= 1 + 2\times\left[\frac{2}{3} + (\frac{2}{3})^2 + (\frac{2}{3})^3 + \ldots\right]$

$= 1 + 2\times\dfrac{\frac{2}{3}}{1-\frac{2}{3}} = 1 + 2\times\dfrac{\frac{2}{3}}{\frac{1}{3}} = 1 + 2\times 2 = 5$

(d) The first \$3000 would earn $3000(1\cdot06)^{35}$, the next \$3000 would earn $3000(1\cdot06)^{34}$ and so on until the start of the 35th year where the last \$3000 would earn $3000(1\cdot06)$.

So the total investment is worth

$S = 3000(1\cdot06)^{35} + 3000(1\cdot06)^{34} + \cdots + 3000(1\cdot06)$

$S = 3000\left[(1\cdot06) + 3000(1\cdot06)^2 + \cdots + 3000(1\cdot06)^{35}\right]$

$= 3000 \times \dfrac{1\cdot06\left[(1\cdot06)^{35} - 1\right]}{1\cdot06 - 1}$

$= \$354362\cdot60$

(e) Let \$$M$ be the monthly repayment, let \$$A_n$ be the amount owing after n months.

Question 5

(a) $2^x = 3^{x-1}$

$\log_{10} 2^x = \log_{10} 3^{x-1}$

$x\log_{10} 2 = (x-1)\log_{10} 3$

$x\log_{10} 2 - x\log_{10} 3 = -\log_{10} 3$

$x(\log_{10} 2 - \log_{10} 3) = \log_{10} 3^{-1} = \log_{10} 1/3$

$x(\log_{10} 2/3) = \log_{10} 1/3$

$x = \dfrac{\log_{10} 1/3}{\log_{10} 2/3} \approx 2\cdot 71$

(b) $x^2 + y^2 = 7xy \Rightarrow x^2 + y^2 + 2xy = 9xy$

$x^2 + y^2 + 2xy = (x+y)^2 = 9xy$

$\therefore \log(x+y)^2 = \log 9xy$

$\therefore 2\log(x+y) = \log 9 + \log x + \log y = \log 3^2 + \log x + \log y$

$\therefore 2\log(x+y) = 2\log 3 + \log x + \log y$

$\therefore \log(x+y) = \log 3 + \tfrac{1}{2}\log x + \tfrac{1}{2}\log y$

QED

(c) Distance $= 1 + 2\times(\tfrac{2}{3}\times 1) + 2\times(\tfrac{2}{3}\times\tfrac{2}{3}) + 2\times(\tfrac{2}{3}\times(\tfrac{2}{3})^2) + \cdots$

$= 1 + 2\times\left[\tfrac{2}{3} + \left(\tfrac{2}{3}\right)^2 + \left(\tfrac{2}{3}\right)^3 + \ldots\right]$

$= 1 + 2\times\dfrac{\frac{2}{3}}{1-\frac{2}{3}} = 1 + 2\times\dfrac{\frac{2}{3}}{\frac{1}{3}} = 1 + 2\times 2 = 5$

(d) The first \$3000 would earn $3000(1\cdot 06)^{35}$, the next \$3000 would earn $3000(1\cdot 06)^{34}$ and so on until the start of the 35[th] year where the last \$3000 would earn $3000(1\cdot 06)$.

So the total investment is worth

$S = 3000(1\cdot 06)^{35} + 3000(1\cdot 06)^{34} + \cdots + 3000(1\cdot 06)$

$S = 3000\left[(1\cdot 06) + 3000(1\cdot 06)^2 + \cdots + 3000(1\cdot 06)^{35}\right]$

$= 3000\times\dfrac{1\cdot 06\left[(1\cdot 06)^{35} - 1\right]}{1\cdot 06 - 1}$

$= \$354362\cdot 60$

(e) Let \$$M$ be the monthly repayment, let \$$A_n$ be the amount owing after n months.

12% pa = 1% per month, 10 years = 120 months

$A_1 = 75000(1\cdot 01) - M$

$A_2 = A_1(1\cdot 01) - M$

$= 75000(1\cdot 01)^2 - M(1 + 1\cdot 01)$

$A_3 = A_2(1\cdot 01) - M$

$= 75000(1\cdot 01)^3 - M(1 + 1\cdot 01 + 1\cdot 01^2)$

$A_n = 75000(1\cdot 01)^n - M(1 + 1\cdot 01 + \ldots + 1\cdot 01^{n-1})$

$A_{120} = 75000(1\cdot 01)^{120} - M(1 + 1\cdot 01 + \ldots + 1\cdot 01^{119})$

Let $S_{120} = 1 + 1\cdot 01 + \ldots + 1\cdot 01^{119}$

$= \dfrac{1\cdot 01^{120} - 1}{1\cdot 01 - 1} = 100(1\cdot 01^{120} - 1)$

$A_{120} = 0$

$\Rightarrow M = \dfrac{75000(1\cdot 01)^{120}}{S_{120}} = 1076\cdot 03$

So the monthly repayment is $\$1076\cdot 03$