



**SYDNEY BOYS HIGH
SCHOOL**
MOORE PARK, SURRY HILLS

2006

YEAR 11
HSC Task 1
Term 4

Mathematics

General Instructions

- Working time – 90 Minutes
- Reading time – 5 Minutes
- Write using black or blue pen
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may not be awarded for messy or badly arranged work.
- Hand in your answer booklets in 5 sections. Section A (Question 1), Section B (Question 2), Section C (Question 3), Section D (Question 4) and Section E (Question 5).

Total Marks – 76

- Attempt Questions 1 – 5.
- All questions are NOT of equal value.

Examiner: *A. Ward*

SECTION A

Question 1 (16 Marks)

Marks

- a) Differentiate with respect to x and simplify:
- | | |
|-----------------------------|---|
| (i) $y = x^5 - 1$ | 1 |
| (ii) $y = (3x^4 - 5)^7$ | 1 |
| (iii) $y = \frac{x+1}{3-x}$ | 2 |
- b) Write down the third and fourth terms of the series $12+6+\dots$ if it is:
- | | |
|--------------------------|---|
| (i) an arithmetic series | 1 |
| (ii) a geometric series | 1 |
- c) (i) Find, to 2 decimal places, the roots of:
 $2x^2 - 3x - 4 = 0$ 2
- (ii) Show that $2x^2 - 3x + 4 = 0$ has no real roots. 1
- d) Determine each of the following:
- | | |
|---|---|
| (i) $\int x^6 dx$ | 1 |
| (ii) $\int (x-1)(x+2) dx$ | 2 |
| (iii) $\int \frac{\sqrt{t}+1}{\sqrt{t}} dt$ | 2 |
- e) Three terms of an arithmetic series have the sum 21 and a product of 315.
Find the 3 numbers. 2

End of Section A

SECTION B – Start a new booklet

Marks

Question 2. (17 marks)

- a) Given that $ab^c = d$:
- (i) Find b in terms of a , c and d . 1
 - (ii) Find c in terms of a , b and d . 2
 - (iii) Calculate b , correct to 4 significant figures, when $a = 75.12$, $c = 1.142$ and $d = 61.94$. 1
- *b) Given that, $f(x) = a - 2x - x^2$ where a is a constant. Find:
- (i) the value for a for which the roots of the equation differ by 3. 2
 - (ii) the set of values of a for which $f(x) < 0$ for all values of x . 2
- c) At what points does the tangent to $f(x) = 2x^3 - 3x^2 + 1$, have slope 0. 2
- d) Evaluate: $\int_1^2 (3x^2 - 2x) dx$ 2
- e) For the function, $f(x) = 5x^3 - 7x^2 + 3x + 2$
- (i) Show that $f(x)$ passes through the point (1,3) 1
At this point, find:
 - (ii) the gradient. 1
 - (iii) the equation of the tangent in gradient-intercept form. 1
 - (iv) the equation of the normal in general form. 2

End of Section B

SECTION C – Start a new booklet

Marks

Question 3. (15 marks)

- a) Find the values of y which satisfy the equation:
- $$(8^y)^y \times \frac{1}{32^y} = 4 \quad 2$$
- b) A point P is equidistant from the x -axis and the point $F(0,2)$. Find the locus of the point P . 3
- c) Using first principles, find the derivative of the function $f(x) = x^2 + x$, (all working must be shown). 2
- d) Find the area bounded by $y = \sqrt{x} + 3$ and the x -axis for $1 \leq x \leq 4$. 2
- e) The equation $3x^2 - 6x + 8 = 0$ has roots α and β . Find an equation which has roots $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ 2
- f) A wall vase has one plane face, and its volume is equivalent to that generated when the area enclosed by $x = \frac{y^3}{64} + 1$, the y -axis and $y = 8$, is rotated through two right angles about the y -axis, the units being centimetres. Calculate its volume. 4

End of Section C

SECTION D – Start a new booklet**Marks****Question 4. (12 marks)**

- a) (i) Tabulate, to 2 decimal places, the values of the function $f(x) = \sqrt{1+x^2}$, at unit intervals from $x=2$ to $x=5$ inclusive. 2
- (ii) Use these values to find an estimate, by the trapezoidal rule, of the area between $y = f(x)$ and the x -axis for $2 \leq x \leq 5$ to 3 decimal places.. 2
- b) A point P has x -coordinate a which is taken to be on the line $y = 3x - 9$.
- (i) If Q is the point $(1,4)$, show that $PQ^2 = 10a^2 - 80a + 170$ 2
- (ii) Find the value of a which will make PQ a minimum. 2
- (iii) N is a point on the line such that QN is perpendicular to the line. Find the co-ordinates of N . 2
- (iv) Find the equation of QN in general form. 2

End of Section D**SECTION E – Start a new booklet****Marks****Question 5. (16 marks)**

- a) Solve for x :
- (i) $\log_5 x + \log_2 8 = 0$ 1
- (ii) $\log_3 x + 3 \log_x 3 = 4$ 2
- b) Prove that, if the sum of the radii of two circles remains constant, the sum of the areas of the circles is least when the circles are equal. 3
- c) A prize fund is set up with an investment of \$2000, to provide a prize of \$150 each year. The fund accrues compound interest at 5% p.a. paid six monthly. The first prize is awarded 1 year after the initial investment, after interest is received.
- (i) Find the value of the fund immediately after the first years prize is drawn from the fund. 1
- (ii) Find the value of the fund immediately after the third prize is drawn from the fund. 2
- (iii) Find the number of prizes of the full \$150 which can be drawn from the fund. 3
- d) Three real, distinct and non-zero numbers a , b and c are such that a , b , c are in arithmetic series and a , c , b are in geometric series.
- (i) Find the numerical value of the common ratio of the geometric series. 2
- (ii) Hence, find an expression in terms of a for the sum to infinity of the geometric series whose first terms are a , c , b . 2

End of Section E**End of Examination Paper**

Question 1

a) i) $y = x^5 - 1$

$y' = 5x^4$

ii) $y = (3x^4 - 5)^7$

$y' = 7(3x^4 - 5)^6 \cdot 12x^3$
 $= 84x^3(3x^4 - 5)^6$

iii) $y = \frac{x+1}{3-x}$

$u = x+1 \quad v = 3-x$
 $u' = 1 \quad v' = -1 \quad y' = \frac{vu' - uv'}{v^2}$

$y' = \frac{(3-x)(1) - (x+1)(-1)}{(3-x)^2}$

$= \frac{4}{(3-x)^2}$

b) i) $a = 12, d = -6$

$\therefore T_3 = 0, T_4 = -6$

ii) $a = 12, r = \frac{1}{2}$

$\therefore T_3 = 3, T_4 = \frac{3}{2}$

c) i) $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$(2x^2 - 3x - 4 = 0)$

$x = \frac{3 \pm \sqrt{9 - 4(2)(-4)}}{2(2)}$

$x = \frac{3 \pm \sqrt{41}}{4}$

$x = -0.85, 2.35$ to 2 decimal places

ii) $\Delta = b^2 - 4ac \quad (2x^2 - 3x + 4 = 0)$

$\Delta = (-3)^2 - 4(2)(4)$

$\Delta = -23$

since $\Delta < 0$, $2x^2 - 3x + 4 = 0$ has no real roots.

d) i) $\int x^6 dx = \frac{x^7}{7} + C$

ii) $\int (x-1)(x+2) dx = \int (x^2 + x - 2) dx$
 $= \frac{x^3}{3} + \frac{x^2}{2} - 2x + C$

iii) $\int \frac{\sqrt{t}+1}{\sqrt{t}} dt = \int (1 + t^{-\frac{1}{2}}) dt$

$= t + \frac{t^{\frac{1}{2}}}{(\frac{1}{2})} + C$

$= t + 2\sqrt{t} + C$

e) let the three terms be $a-d, a, a+d$

$(a-d) + (a) + (a+d) = 21$

$3a = 21$

$a = 7$

$(a-d)(a)(a+d) = 315$

$a(a^2 - d^2) = 315$

sub in $a = 7$

$7(49 - d^2) = 315$

$49 - d^2 = 45$

$-d^2 = -4$

$d^2 = 4$

$d = \pm 2$

\therefore the three terms are 5, 7 & 9.

Q2.

(a) (i) $ab^c = d$

$$b^c = \frac{d}{a}$$

$$\boxed{b = \left(\frac{d}{a}\right)^{\frac{1}{c}}} \text{ OR } \sqrt[c]{d/a}$$

(ii) $b^c = \frac{d}{a}$

$$\boxed{c = \log_b \frac{d}{a}}$$

(iii) $b = \left(\frac{61.94}{75.12}\right)^{\frac{1}{1.142}}$

$$\boxed{\approx 0.8446} \text{ (4.s.f.)}$$

(b) (i) Let the roots be $x, x-3$.

$$\text{now } x + x - 3 = -2 \quad (S_1 = -\frac{b}{a})$$

$$2x = 1$$

$$x = \frac{1}{2}$$

$$\text{now } S_2 = \frac{c}{a} = -a$$

$$\therefore a = -\left(\frac{1}{2} \times -\frac{5}{2}\right)$$

$$\boxed{a = \frac{5}{4}}$$

(ii) If $f(x) < 0$ the quadratic would be negative definite.

$$\therefore \Delta < 0$$

$$\text{ie } (-2)^2 - 4 \times 1 \times a < 0$$

$$4 - 4a < 0$$

$$4a < -4$$

$$\boxed{a < -1}$$

(c) $f(x) = 6x^2 - 6x$

Let $f(x) = 0$

$$6x(x-1) = 0$$

$$x = 0, 1$$

\therefore POINTS are (0, 1) and (1, 0)

NOTE If $x = 0$

$$f(0) = 0 - 0 + 1 = 1$$

$$\text{If } x = 1$$

$$f(1) = 2 - 3 + 1 = 0$$

(d) $\int_{-1}^2 (3x^2 - 2x) dx = \left[x^3 - x^2 \right]_{-1}^2$

$$= (8 - 4) - (1 - 1)$$
$$= 4 - 0$$
$$= 4$$

(NB Answer is not 6)

(e) (i) $f(1) = 5 - 7 + 3 + 2 = 3$

$$\therefore (1, 3) \text{ lies on } f(x) = 5x^3 - 7x^2 + 3x + 2$$

(ii) $f(x) = 5x^3 - 7x^2 + 3x + 2$

$$f'(x) = 15x^2 - 14x + 3$$

$$\therefore f'(1) = 15 - 14 + 3$$

$$= 4$$

\therefore gradient at (1, 3) is 4

(iii) $y - 3 = 4(x - 1)$

$$y - 3 = 4x - 4$$

$$\boxed{y = 4x - 1}$$

in "y = mx + b" form.

(iii) $y - 3 = -\frac{1}{4}(x - 1)$

$$4y - 12 = -x + 1$$

$$\boxed{x - 4y + 13 = 0}$$

in "general form"

Question 3 (15 Marks)

- (a) Find the values of y which satisfy the equation:

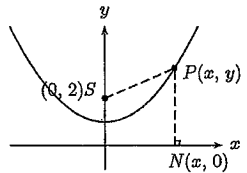
$$(8^y)^y \times \frac{1}{32^y} = 4$$

2

Solution: $2^{3y^2} \times 2^{-5y} = 2^2$,
 $3y^2 - 5y - 2 = 0$,
 $(3y + 1)(y - 2) = 0$,
 $\therefore y = 2, -\frac{1}{3}$.

- (b) A point P is equidistant from the x -axis and the point $F(0, 2)$. Find the locus of the point P .

3

Solution:  $PN^2 = PS^2$,
 $y^2 = x^2 + (y - 2)^2$,
 $y^2 = x^2 + y^2 - 4y + 4$,
 $\therefore x^2 = 4(y - 1)$.

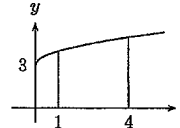
- (c) Using first principles, find the derivative of the function $f(x) = x^2 + x$ (all working must be shown).

2

Solution: $f'(x) = \lim_{h \rightarrow 0} \left\{ \frac{(x+h)^2 + (x+h) - x^2 - x}{h} \right\}$,
 $= \lim_{h \rightarrow 0} \left\{ \frac{x^2 + 2xh + h^2 + x + h - x^2 - x}{h} \right\}$,
 $= \lim_{h \rightarrow 0} \left\{ \frac{2xh + h^2 + h}{h} \right\}$,
 $= \lim_{h \rightarrow 0} \{2x + h + 1\}$,
 $= 2x + 1$.

- (d) Find the area bounded by $y = \sqrt{x} + 3$ and the x -axis for $1 \leq x \leq 4$.

2

Solution: 
$$\begin{aligned} \text{Area} &= \int_1^4 (x^{\frac{1}{2}} + 3) dx, \\ &= \left[\frac{2x^{\frac{3}{2}}}{3} + 3x \right]_1^4, \\ &= \frac{16}{3} + 12 - \left(\frac{2}{3} + 3 \right), \\ &= \frac{41}{3} \text{ or } 13\frac{2}{3}. \end{aligned}$$

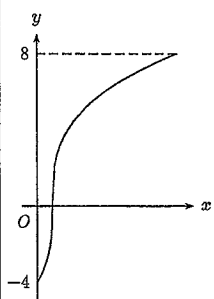
- (e) The equation $3x^2 - 6x + 8 = 0$ has roots α and β . Find an equation which has roots $\frac{1}{\alpha}$ and $\frac{1}{\beta}$.

2

Solution: $\alpha + \beta = 2$, $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta}$,
 $\alpha\beta = \frac{8}{3}$, $= \frac{3}{4}$,
 $\frac{1}{\alpha\beta} = \frac{3}{8}$.
 \therefore New equation: $x^2 - \frac{3x}{4} + \frac{3}{8} = 0$,
i.e. $8x^2 - 6x + 3 = 0$.

- (f) A wall vase has one plane face, and its volume is equivalent to that generated when the area enclosed by $x = \frac{y^3}{64} + 1$, the y -axis, and $y = 8$, is rotated through two right angles about the y -axis, the units being centimetres. Calculate its volume.

Solution: When $x = 0$, $y = -4$.



$$\begin{aligned} V &= \frac{1}{2}\pi \int_{-4}^8 x^2 dy, \\ &= \frac{\pi}{2} \int_{-4}^8 \left(\frac{y^6}{4096} + \frac{y^3}{32} + 1 \right) dy, \\ &= \frac{\pi}{2} \left[\frac{y^7}{7 \times 4096} + \frac{y^4}{4 \times 32} + y \right]_{-4}^8, \\ &= \frac{\pi}{2} \left\{ \frac{512}{7} + 32 + 8 - \left(-\frac{4}{7} + 2 - 4 \right) \right\}, \\ &= \frac{405\pi}{7}, \\ &\approx 181.76 \text{ cm}^3 \text{ (2 dec. pl.)} \end{aligned}$$

QUESTION 4

a (i)

$$f(x) = \sqrt{1+x^2}$$

x	2	3	4	5
$f(x)$	$\sqrt{5}$	$\sqrt{10}$	$\sqrt{17}$	$\sqrt{26}$
	2.24	3.16	4.12	5.10

(ii)

$$A = \frac{h}{2} \{ f(2) + f(5) + 2[f(3) + f(4)] \}$$

$$h = \frac{b-a}{n} = \frac{5-2}{3} = 1$$

$$A = \frac{1}{2} [2.24 + 5.10 + 2(3.16 + 4.12)]$$

$$= 10.950 \text{ (3 dp.)}$$

b (i)

$$P(a, 3a-9) \quad Q(1, 4)$$

$$PQ^2 = \sqrt{(a-1)^2 + (3a-9-4)^2}$$

$$PQ^2 = a^2 - 2a + 1 + 9a^2 - 78a + 169$$

$$PQ^2 = 10a^2 - 80a + 170$$

(ii)

$$(PQ^2)' = 20a - 80$$

turn pt. $20a = 80, a = 4$

$$(PQ^2)'' = 20$$

$$(PQ^2)''' = 0$$

\therefore min value of $PQ = 4$

(iii)

min value when line \perp to PQ

$\therefore N$ is $(4, 3)$

(iv)

$$y - 3 = -\frac{1}{3}(x - 4)$$

$$3y - 9 = -x + 4$$

$$x + 3y - 13 = 0$$

SECTION E QUESTION 5

(a) (i) $\log_5 x + \log_2 8 = 0$
 $\log_5 x = -3$
 $x = 5^{-3} = \frac{1}{125}$

(ii) $\log_3 x + \frac{\log_3 27}{\log_3 x} = 4$
 $(\log_3 x)^2 - 4 \log_3 x + 3 = 0$
 $\log_3 x = 3$ or $\log_3 x = 1$
 $x = 27$ or $x = 3$

b) $r_1 + r_2 = k$ — (*)

$S = \pi r_1^2 + \pi r_2^2$
 $= \pi (r_1^2 + r_2^2)$
 $= \pi [r_1^2 + (k - r_1)^2]$
 $S = \pi [2r_1^2 - 2kr_1 + k^2]$

$\frac{dS}{dr_1} = \pi [4r_1 - 2k] = 0$

$\Rightarrow r_1 = \frac{k}{2}$

$\therefore r_2 = \frac{k}{2}$ from *

$\frac{d^2 S}{dr_1^2} = 4\pi > 0$ MIN.

(c) V_n = value at end of n years (after n^{th} prize).

(i) $V_1 = 2000(1.025)^2 - 150$

$V_1 = \$1951.25$

(ii) $V_1 = 2000(1.025)^2 - 150$

$V_2 = [2000(1.025)^2 - 150](1.025)^2 - 150$

$V_2 = 2000(1.025)^4 - 150[1 + 1.025^2]$

$V_3 = \{2000(1.025)^4 - 150[1 + 1.025^2]\}(1.025^2 - 150)$

$V_3 = 2000(1.025)^6 - 150[1 + 1.025^2 + 1.025^4]$

$V_3 = \$1846.77$

(iii) $V_n = 2000(1.025)^{2n} - 150[1 + 1.025^2 + 1.025^4 + \dots + 1.025^{2(n-1)}]$

Let $V_n = 0$

$\frac{2000(1.025)^{2n}}{150} = \frac{1(1.025^2)^n - 1}{0.050625}$

$0.675(1.025)^{2n} = 1.025^{2n} - 1$

$1 = [1.025^{2n}] \cdot (0.325)$

$n = \frac{-1 \cdot \log(0.325)}{2 \cdot \log(1.025)} \approx 22$

$\therefore 22$ prizes

(d)

(i) a, b, c A.S.

a, c, b G.S.

$\Rightarrow b - a = c - b$

$b = \frac{a+c}{2}$

$r = \frac{c}{a} = \frac{b}{c}$

$c^2 = ab$

$\Rightarrow c^2 = a \left(\frac{a+c}{2}\right)$

$2c^2 = a^2 + ac$

$2c^2 - ac - a^2 = 0$

$(\div a^2) \quad 2\left(\frac{c}{a}\right)^2 - \left(\frac{c}{a}\right) - 1 = 0$

$\Rightarrow \frac{c}{a} = \frac{1 \pm \sqrt{1 - 4(2)(-1)}}{4}$

Common ratio $\frac{c}{a} = 1$ or $-\frac{1}{2}$

However only $-\frac{1}{2}$ applies $\therefore r = -\frac{1}{2}$

(ii) $S = \frac{a}{1-r}$

$= \frac{a}{1 - (-\frac{1}{2})}$

$S = \frac{2}{3}a$