SYDNEY BOYS' HIGH SCHOOL

MOORE PARK, SURRY HILLS

HALF-YEARLY EXAMINATION May 2002

MATHEMATICS

EXTENSION 1

Time allowed — Ninety Minutes
Examiner: A.M.Gainford

DIRECTIONS TO CANDIDATES

- *ALL* questions may be attempted.
- All necessary working should be shown in every question. Full marks may not be awarded for careless or badly arranged work.
- Approved calculators may be used.
- Start each Section on a new page. Section A (Q1, Q2, Q3), Section B (Q4, Q5), Section C (Q6, Q7), Section D (Q8, Q9), Section E (Q10, Q11), Section F (Q12, Q13).
- If required, additional paper may be obtained from the Examination Supervisor upon request.

Section A

Marks

Question 1 6

(a) Express $\frac{7\pi}{9}$ radians in degrees.

(b) State the exact value of:

(i) $\sec 45^\circ$

(ii) $\tan 210^\circ$

(c) By expressing it in its simplest form, show that $\frac{1}{\sqrt{7}-2}-\frac{1}{\sqrt{7}+2}$ is rational.

Question 2 6

Factorise completely:

(a) $12x^2+5x-3$

(b) $2xy+6x-y-3$

(c) a^3-8

Question 3 6

On separate diagrams, sketch the graphs of the following, showing essential features:

(a) $y=x^2-1$

(b) $y=2^{-x}$

(c) $y=\sqrt{9-x^2}$

Section B

Question 4 6

For the points $A(1, 6)$ and $B(3, 8)$:

(a) Find the coordinates of M, the midpoint of AB.

(b) Find the equation of the line through M, perpendicular to AB.

(c) Write the equation of the line AB.

Question 5 6

(a) Show that the lines $y = 2x - 1$ and $2x - y + 3 = 0$ are parallel.

(b) Find the perpendicular (shortest) distance between the two lines in Part (a).

(c) By completing the square on x, or otherwise, find the minimum value of the quadratic expression $x^2 + 8x + 9$.

Section C

Question 6 6

Graph, on separate number lines, the solutions of:

(a) $6x^2 + 5x > 4$

(b) $|2x - 3| < |x + 5|$

(c) $\dfrac{4}{x-3} < 1$

(d) $\dfrac{1}{|x-2|} < 3$

Question 7 6

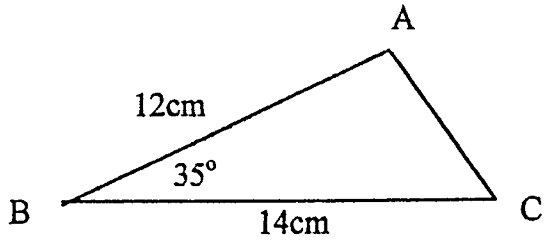
(a) Sketch on a cartesian diagram the locus of all points equidistant from the x- and y-axes.

(b) Write down an equation to represent the locus described above.

(c) A lighthouse keeper 120 m above sea level observes a ship at sea at an angle of depression of $89°07'$. Find to the nearest metre the horizontal distance of the ship from the lighthouse.

Section D

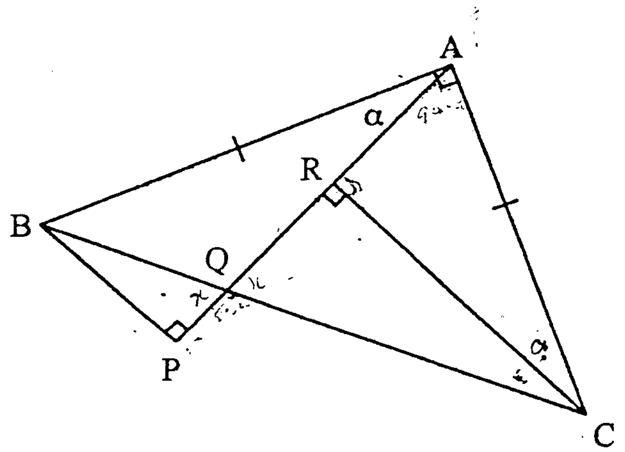
Question 8 6



(a) Given the triangle above, calculate the area of the figure, and the length of AC.

(b) State the equation of the locus of a point moving such that it is always 2 units from the point (1, 0).

Question 9 8



In the figure $AB = AC$; $\angle BAC = \angle BPA = \angle CRA = 90^\circ$; $\angle BAP = \alpha$.
Prove that:

(a) $\angle ACR = \alpha$.

(b) Triangles ABP and CAR are congruent.

(c) Triangles BPQ and CRQ are similar.

(d) $\dfrac{PQ}{QR} = \dfrac{RA}{AP}$.

Section E

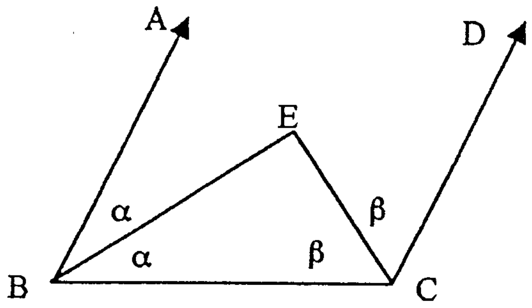
Question 10 6

(a) Show that $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$.

(b) Show that $2\cot\theta\ \mathrm{cosec}\,\theta = \dfrac{1}{1-\cos\theta} - \dfrac{1}{1+\cos\theta}$

Question 11 6

(a) Given that $AB \parallel CD$ and angles are as marked, find the measure of $\angle BEC$. (Give reasons)



(b) Find the equation of the line with gradient -1, which passes through the intersection of the lines $2x-5y+19=0$ and $2x+3y-5=0$.

Section F

Question 12 6

(a) If $\tan\theta = 2$, and $0<\theta<\dfrac{\pi}{2}$, find the exact value of $\sin\left(\theta+\dfrac{\pi}{4}\right)$.

(b) Two buoys, P and Q, are 1500 m apart. The bearing from P to Q is 058°T. A ship at R has P on a bearing of 322°T and Q on a bearing of 025°T.

(i) Sketch a diagram to represent this situation.

(ii) Calculate the distance of Q from R, to the nearest metre.

Question 13 6

(a) Given the function $f(x)=\sqrt{x^2-9}$:

(i) State the domain of $f(x)$.

(ii) State the range of $f(x)$.

(iii) Show that $f(x)$ is an even function.

(b) Show that in any triangle ABC,

$$\sin C = \sin A\cos B + \cos A\sin B.$$

5) a) gradient of $y = 2x - 1 = 2. \ (m_1)$

$2x - y + 3 = 0$
$y = 2x + 3$
gradient $= 2 \ (m_2)$

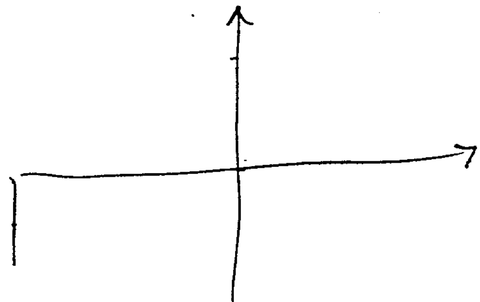
$\therefore$ these lines are parralell (both gradients are equal)

b) $y = 2x - 1$
let $x = 0$ $y = -1$
$\Rightarrow (0, -1)$

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} = \text{perpendicular distance.}$$

$$\frac{|2x - y + 3|}{\sqrt{(2)^2 + (-1)^2}} = \frac{|0 + 1 + 3|}{\sqrt{5}} = \boxed{\frac{4}{\sqrt{5}}}$$

c) $x^2 + 8x + 9.$
$x^2 + 8x + (\frac{8}{2})^2 = -9.$
$x + 8x + 16 = -9.$
$(x+4)^2 = -9$
$(x+4)^2 + 9 = 0$



minimum value equals 25.

~~6) a) $6x^2 + 5x > 4$~~
~~$6x^2 + 5x - 4 > 0$~~
~~$\frac{(6x+8)(6x-3)}{6} > 0$~~
~~$(3x+4)(2x-1) > 0.$~~
~~$x < -4/3$~~
~~$x > 1/2.$~~
~~$-4/3$ 0. 1/2.~~

~~b) $|2x - 3| < |x + 5|$~~
~~case 1 $2x - 3 < x + 5$~~
~~$x < 8.$~~
~~case 2 $-2x + 3 < x + 5$~~
~~$-2 < 3x$~~
~~$-2/3 < x$~~
~~$-2/3$ 8~~

1) a) $\frac{7\pi}{9} = 140°$. 1

c) $a^3 - 8$ 1

$= (a-2)(a^2+2a+4)$.

b) i) $\sec 45° = \frac{1}{\cos 45°} = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}$ 1

ii) $\tan 210° = \tan 30° = \frac{1}{\sqrt{3}}$ 1

(c)

2) a) $12x^2 + 5x - 3$. $\frac{-36}{5}$

$\frac{(12x-4)(12x+9)}{12}$

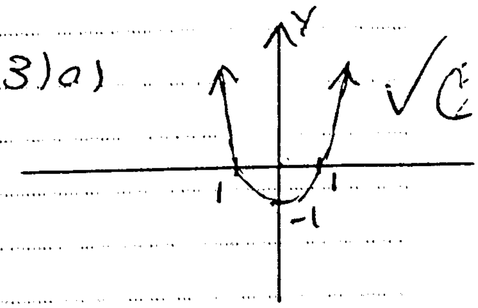
$= (3x-1)(4x+3)$ 2

b) $2xy + 6x - y - 3$.

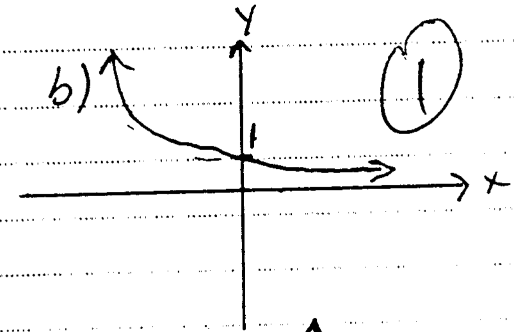
$= y(2x-1) + 3(2x-1)$

$= (y+3)(2x-1)$ 2

3) a)



b)



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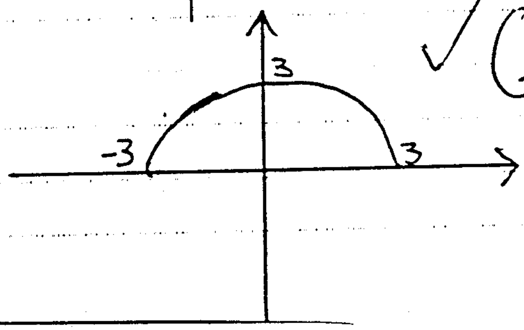
c)



2

4) A(1,6) B(3,8)

a) $\left(\frac{1+3}{2}, \frac{6+8}{2}\right)$

$= M(2, 7)$

b) $m_1 AB = \frac{8-6}{3-1} = \frac{2}{2} = 1$

$\therefore m_2 = -1$

$y = mx + b$

$y = -x + b$.

$7 = -2 + b$

$b = 9$ $\therefore$ equation $= y = -x + 9$ 3

c) gradient of $AB = 1$

$y = mx + b$ $m = 1$

$y = x + b$

$6 = 1 + b$

$b = 5$

$\therefore eq^n = y = x + 5$. 2

c) Prove $\triangle BPQ \,|||\, \triangle CRQ$

$C\hat{R}Q = B\hat{P}Q = 90^\circ$ (given)
$P\hat{Q}B = C\hat{Q}R$ (vertically opposite ∠'s) $\Rightarrow P\hat{B}Q = R\hat{C}Q$ (equal sum ∠ of Δ)
∴ $\triangle BPQ \,|||\, \triangle CRQ$ (equiangular)

d) ~~[illegible]~~ QR

$\triangle QCR \,|||\, \triangle CRA$ (equiangular)
$\triangle RCQ \,|||\, \triangle BPQ$ (equiangular)
∴ $\triangle CRA \,|||\, \triangle BPQ$.

~~BP~~ $\frac{PQ}{RQ} = \frac{AR}{RQ}$, $\frac{AR}{PQ} = \frac{RQ}{AR}$

Why?

$\Rightarrow PQ = AR$. $\Rightarrow AR = RQ$.

∴ $\frac{PQ}{QR} = \frac{RA}{AP}$.

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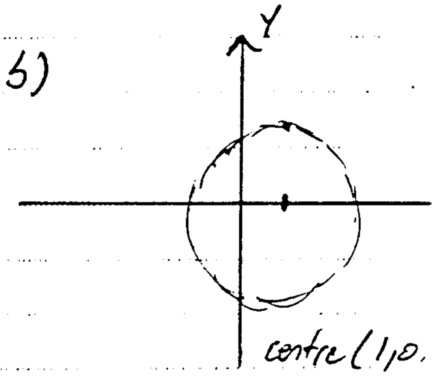
8)a) $\frac{1}{2} \times 12 \times 14 \times \sin 35 = \text{AREA}$
$84 \sin 35 = 48.2\text{cm}$ (1.d.p)

$AC^2 = 12^2 + 14^2 - 336\cos 35^\circ$
$AC^2 = 340 - 336\cos 35^\circ$
$(AC)^2 = 64.76$
∴ $AC = 8.05\text{cm}$ (2.d.p).

b)



centre (1,0.
radius 2

~~∴ $(x-1)^2 + ($~~

$\boxed{\therefore (x-1)^2 + y^2 = 4}$

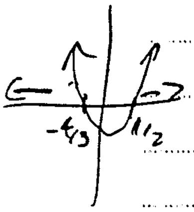
9) $C\hat{R}A = 90^\circ$ (~~co~~ supplementary adjacent ∠'s)
$R\hat{A}C = 90 - a$ (right angle $\triangle B\hat{A}C$)
$C\hat{R}A + R\hat{A}C + A\hat{C}R = 180^\circ$ (sum of ∠'s in Δ).
$90 + 90 - a + A\hat{C}R = 180^\circ$
$180 - a + A\hat{C}R = 180^\circ$
∴ $A\hat{C}R = a$.

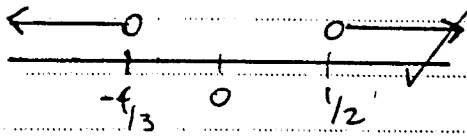
b) $B\hat{A}P = R\hat{C}A = a$ (proven above)
$B\hat{P}A = C\hat{R}A = 90^\circ$ (given and proven above).
$AC = BA$ (given)
∴ $\triangle ABP$ ~~[illegible] congruent~~ and $\triangle CAR$ are congruent (AAS test)

$6x^2+5x-4>0$

$\frac{(6x+8)(6x-3)}{6}>0$

$(3x+4)(2x-1)>0$





$1\frac{1}{2}$

b) $|2x-3|<|x+5|$

Case 1 $2x-3<x+5$

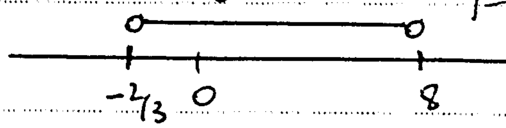
$x<8$

Case 2 $-2x+3<x+5$

$-2<3x$

$-\frac{2}{3}<x$

$1\frac{1}{2}$



c) $\frac{4}{x-3}<1 \times (x-3)^2$

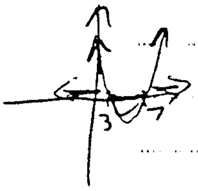
$4(x-3)<(x-3)^2$

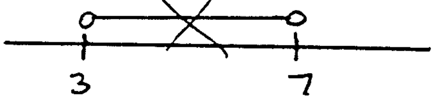
$4x-12<x^2-6x+9$

$0<x^2-10x+21$

$0<(x-7)(x-3)$

$3<x<7$ $\times$ 1





d) $\frac{1}{|x-2|}<3$

$x-2) < 3(x-2)^2$ not a reliable method

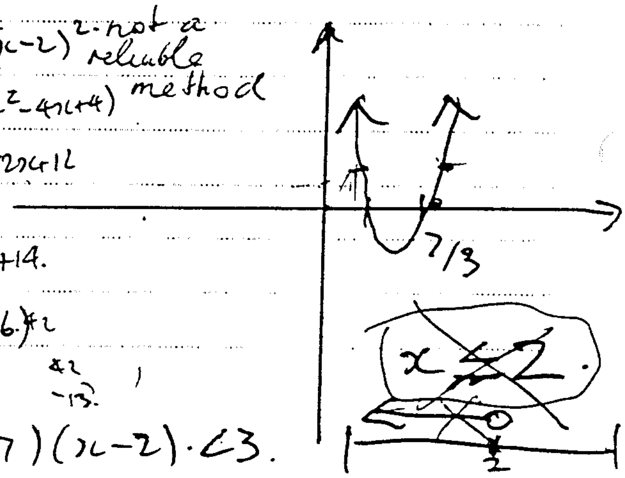
$x-2<3(x^2-4x+4)$

$x-2<3x^2-12x+12$

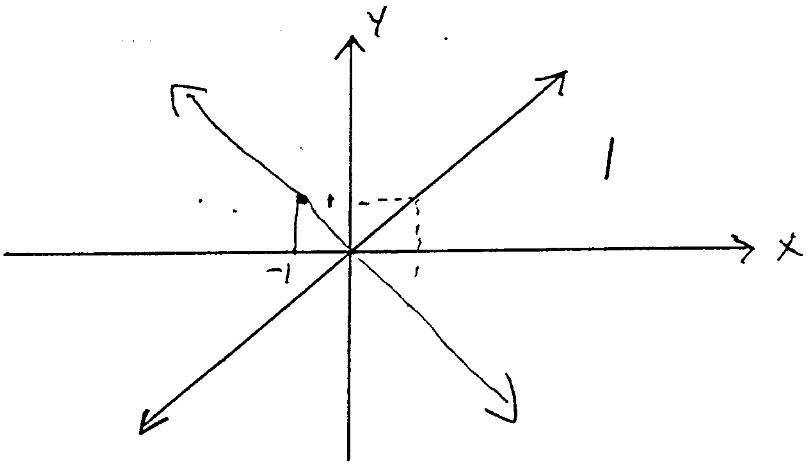
$3x^2-13x+14$

$\frac{(3x-7)(3x-6)}{3}$

$(3x-7)(x-2)<3$



7) a)



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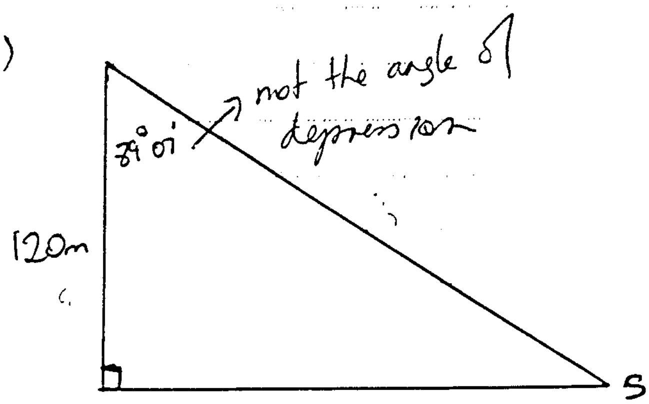
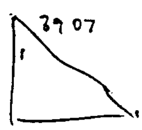
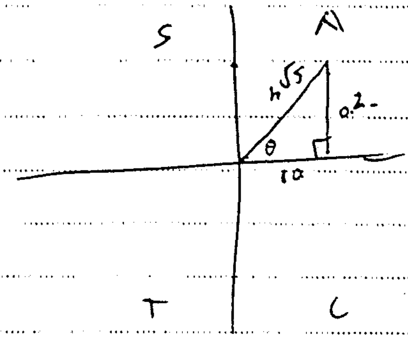
b) $y=x$ 1

c)





12) a)



$\sin\theta = \frac{2}{\sqrt{5}}$

$\Rightarrow \sin(\theta + 45^\circ) =$

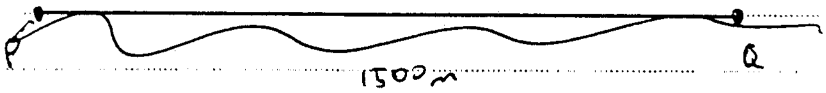
$\boxed{\frac{3}{\sqrt{10}}}$

$\sqrt{\frac{9}{10}} = \frac{3}{\sqrt{10}}$

$= \frac{3}{\sqrt{10}}$

slow marking

b)



b) $2\cot\theta \operatorname{cosec}\theta = \frac{1}{1-\cos\theta} - \frac{1}{1+\cos\theta}$.

LHS. $\frac{2}{\tan\theta} \times \frac{1}{\sin\theta} = \frac{2\cos\theta}{\sin\theta} \times \frac{1}{\sin\theta} = \frac{2\cos\theta}{\sin^2\theta} = \frac{2\cos\theta}{1-\cos^2\theta}$ ✓

$= \frac{2\cos\theta}{(1-\cos\theta)(1+\cos\theta)}$ ~~$\frac{}{1-\cos\theta}$ $\frac{}{1+\cos\theta}$~~

$\Downarrow$ ✓

$= \frac{(1+\cos\theta)-(1-\cos\theta)}{(1-\cos\theta)(1+\cos\theta)} = \frac{1}{1-\cos\theta} - \frac{1}{1+\cos\theta}$ ✓

$\therefore$ LHS = RHS

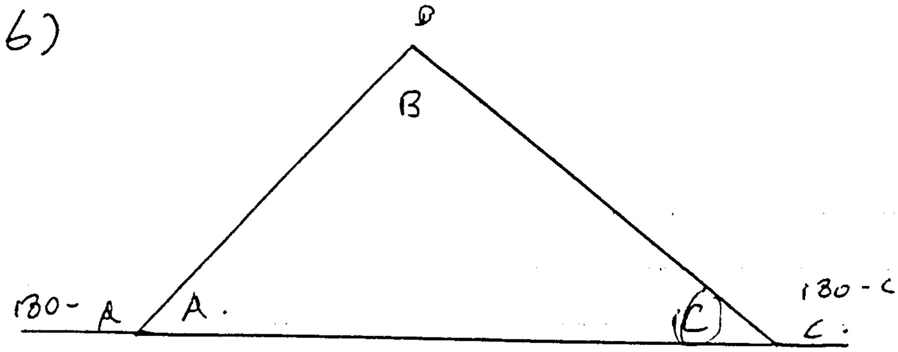
11) a) $2a + 2\beta = 180°$ ✓ (cointerior ∠'s, parallel lines AB + CD)
$\Rightarrow a + \beta = 90°$. ✓
In $\triangle BEC$. $a + \beta + B\hat{E}C = 180°$ (a ∠ sum of $\triangle$) ✓
$\Rightarrow B\hat{E}C = 90°$.

b) $2x - 5y + 19 = 0$
$2x + 3y - 5 = 0$.
$\Rightarrow -8y + 24 = 0$
$-8y = -24$
$y = 3$
$\Rightarrow x = -2$. ✓ $\Rightarrow (-2, 3)$. $m_1 = -1$

$y = mx + b$
$y = -x + b$.
$3 = 2 + b$. ✓
$b = 1$ $\Rightarrow eq^n = y = -x + 1$

$\Rightarrow y + x - 1 = 0$. ✓

6)



$\cancel{\hat{B}+\hat{C} = 180 - A}$

$\hat{A}+\hat{B} = 180 - \hat{C}$

$\sin(A+B) = \sin A \cos B + \cos A \sin B.$

$\Rightarrow \sin A \cos B + \cos A \sin B = 180 - \hat{C}$

$\cancel{\cos(180-\hat{C})}$

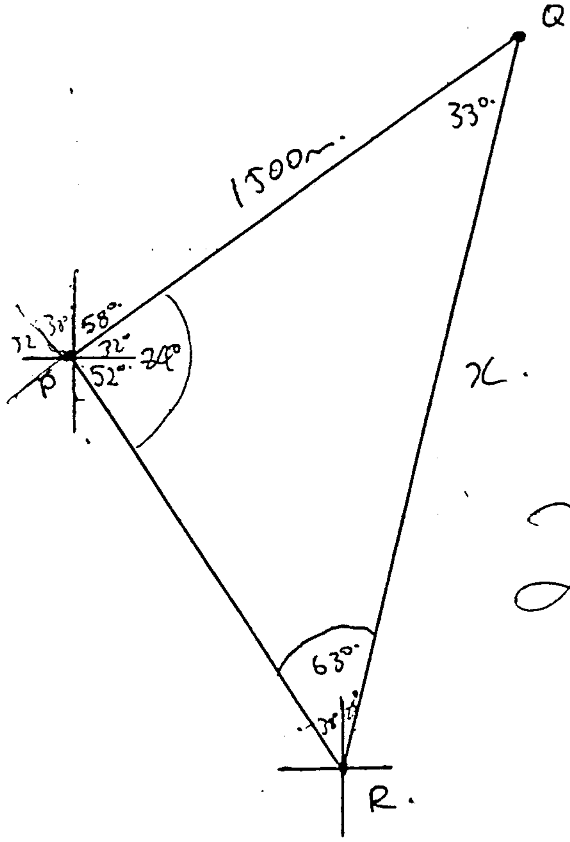
$\cos(180-\hat{C})$ why $\cos(180-C)$

$= \sin \hat{C}$

$\therefore \sin A \cos B + \cos A \sin B = \sin C.$

1/2

6) i)



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ii)

$$\frac{\sin 84^\circ}{x} = \frac{\sin 63^\circ}{1500}$$

$y^2 = (x-1)(x+3)$

$$x = \frac{\sin 84 \times 1500}{\sin 63^\circ}$$

2

$x^2 - 9 = y^2$

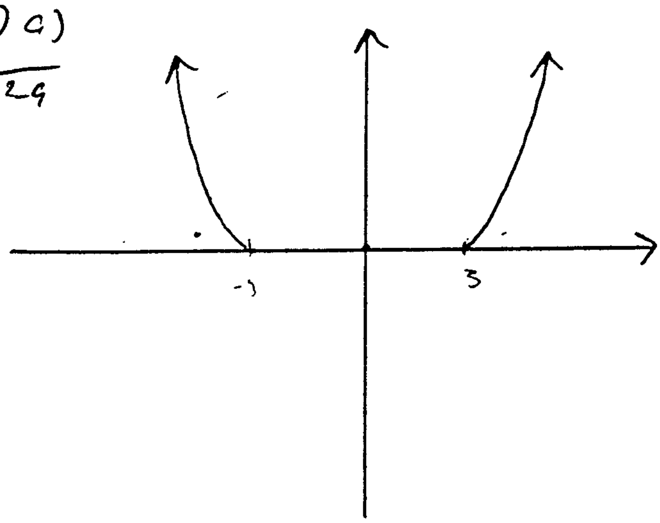
$y = \sqrt{x^2-9}$

$x = 1674m.$

$y^2 = x^2 = -9$

13) a)

$(x) = \sqrt{x^2-9}$



i) $x \geq 3$
$x \leq -3.$

3

ii) $y \geq 0.$

1

iii) $f(x) = \sqrt{x^2-9}.$

$f(-x) = \sqrt{(-x)^2-9}.$

$= \sqrt{x^2-9}.$

$\therefore f(x)$ is EVEN.