SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

JUNE 2005

11A CLASS TEST # 4

Mathematics

General Instructions

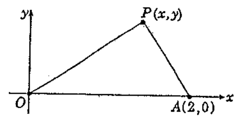
- Working time – 1 PERIOD
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators may be used
- *All* necessary working should be shown in every question.
- Marks may not be awarded for badly arranged or messy setting out of work.

(1) Find the values of m for which $12+4m-m^2>0$

(2) (i) Write down the discriminant of $3x^2+2x+k$

(ii) For what values of k does $3x^2+2x+k=0$ have real roots?

(3)



(i) Write down the gradient of AP in terms of x and y.

(ii) Show that the equation of the locus of all points P such that OP is perpendicular to AP is $x^2-2x+y^2=0$.

(iii) Deduce that the locus of all points P such that OP is perpendicular to AP is a circle.
Write down the centre and the radius of this circle.

(4) Consider the function $f(x)=x^4-4x^3$.

(i) Show that $f'(x)=4x^2(x-3)$.

(ii) Hence find the coordinates of the points where the tangents to $y=f(x)$ are parallel to the x axis.

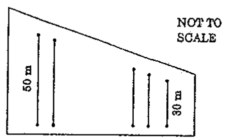
(5) At the beginning of 1991 Australia's population was 17 million.
At the beginning of 2004 the population was 20 million.
Assume that the population P satisfies (approximately) an equation of the form $P=A(3^{kt})$, where A and k are constants and t is measured in years from the beginning of 1991 ie $t=0$ is January 1st 1991.

(i) By substituting $t=0$ find the value of A.

(ii) By another suitable substitution, find the value of k to 5 significant figures.

(iii) Hence predict the year during which Australia's population will reach 30 million.

(6)



A simple instrument has many strings, attached as shown in the diagram.
The difference between the lengths of adjacent strings is a constant, so that the lengths of the strings are the terms of an arithmetic series.
The shortest string is 30 cm long and the longest string is 50 cm long.
The sum of all of the strings is 1240 cm.

(i) Find the number of strings.

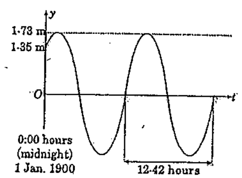
(ii) Find the difference in length between adjacent strings.

(7) $f(x)$ is defined as follows:

$$f(x)=\begin{cases} -x^2 & x\le 0 \\ 5x-4 & 0<x\le 1 \\ 4x^2-3x & 1<x\le 2 \\ 3x+4 & x\ge 2 \end{cases}$$

Without drawing $f(x)$, show that $f(x)$ is **not** continuous at $x = 0$.
Make use of appropriate notation ie $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^-} f(x)$

(8)



The diagram shows the tidal effect due to the Moon at Port Hedland on 1 January 1900. The water level can be approximated by a sine curve of the form $y = A\sin(at+b)$, where y is the water level in metres measured as on the diagram and t is the time in hours after 0:00 hours.

(i) Find the amplitude A.

(ii) Estimate b by letting $t = 0$.

(iii) Estimate a.

Put your calculator in **radians** mode!

(9) Evaluate $\lim_{x\to a}\dfrac{x^{3/2}-a^{3/2}}{x-a}$

[Hint: $y^{3/2}=\left(\sqrt{y}\right)^3$]

(10) The equation of a parabola is $x^2 = 8(y+3)$

(i) Write down the equation of the axis of symmetry of the parabola.

(ii) Write down the equation of the directrix of the parabola.

(11) (i) Show that $x=\dfrac{\pi}{3}$ is a solution of $\sin x=\dfrac{1}{2}\tan x$.

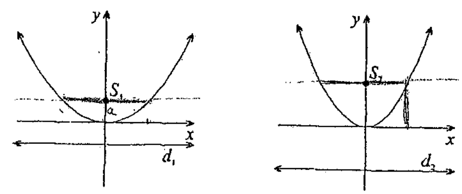
(ii) On the same set of axes, sketch the graphs of the functions $y=\sin x$ and $y=\dfrac{1}{2}\tan x$ for $-\pi\le x\le\pi$.

(iii) Hence find ALL the solutions of $\sin x=\dfrac{1}{2}\tan x$ for $-\dfrac{\pi}{2}<x<\dfrac{\pi}{2}$.

(iv) Use your graphs to solve $\sin x\le\dfrac{1}{2}\tan x$ for $-\dfrac{\pi}{2}<x<\dfrac{\pi}{2}$.

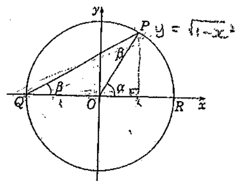
(12) (i) The *latus rectum* of a parabola is the focal chord parallel to the directrix.
The parabola $x^2 = 4ay$ has focal length $|a|$.
Show that the length of its latus rectum is $4|a|$.

(ii) A pupil drew the graphs of two different parabolas, shown below, using the same scale on both graphs and using the given foci and directrices.



The left hand graph is correct.
Explain why the right hand graph **must** be incorrect.

(13) In the diagram, Q is the point $(-1,0)$, R is the point $(1,0)$, and P is another point on the circle with centre O and radius 1.
Let $\angle POR = \alpha$ and $\angle PQR = \beta$ and let $\tan\beta = m$



(i) Explain why $\alpha = 2\beta$.

(ii) Show that the equation of the line PQ is $y = m(x+1)$.

(iii) Show that the x coordinates of P and Q are solutions of the equation
$$(1+m^2)x^2 + 2m^2x + m^2 - 1 = 0 \quad (*)$$

(iii) Using the *sum of the roots* of (*) show that the coordinates of P are given by
$$P\left(\frac{1-m^2}{1+m^2}, \frac{2m}{1+m^2}\right).$$

(iv) Hence deduce that $\tan 2\beta = \dfrac{2\tan\beta}{1-\tan^2\beta}$

(14) Betty decides to set up a trust fund for her grand son, Luis. She invests \$80 at the beginning of each month. The money is invested at 6% pa, compounded monthly. The trust fund matures at the end of the month of her final investment, 25 years after her first investment. This means that Betty makes 300 monthly investments.

(i) After 25 years, what will be the value of the first \$80 invested?

(ii) By writing a geometric series for the value of all Betty's investments, calculate the final value of Luis' trust fund.

(15) Solve $\log_5(3x-1) + \log_5 2x + 2 = 0$.
Leave solution(s) to 3 significant figures.

(16) A triangle ABC is right angled at B. D is the point on AC such that BD is perpendicular to AC. Let $\angle BAC = \theta$

(i) Draw a diagram showing this information

You are given that $6AD + BC = 5AC$

(iii) Show that $6\cos\theta + \tan\theta = 5\sec\theta$.

(iv) Deduce that $6\sin^2\theta - \sin\theta - 1 = 0$.

(iv) Find θ.

(17) Find the equation of the normal to the curve $y = \dfrac{x-2}{2x+1}$ at the point where the curve crosses the x axis.

(18) Helen sets up a prize fund with a single investment of \$1000 to provide her school with an annual prize valued at \$72. The fund accrues interest at a rate of 6% pa, compounded annually. The first prize is awarded one year after the investment is set up.

(i) Calculate the balance in the fund at the beginning of the second year.

(ii) Let $\$B_n$ be the balance in the fund at the end of n years (and after the nth prize has been awarded). Show that $B_n = 1200 - 200\times(1\cdot06)^n$.

(iii) At the end of the 10^{th} year (and after the 10^{th} prize has been awarded) it is decided to increase the prize value to \$90.
For how many more years can the prize fund be used to award the prize?

End of paper

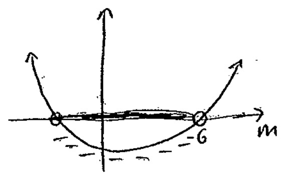
11A Class Test #4

Solutions

(1) $12+4m-m^2>0$

$\therefore m^2-4m-12<0$

$\therefore (m-6)(m+2)<0$



$\therefore -2<m<6$

2 (i) $\Delta = b^2-4ac$

$= 4-4\times3\times k$

$= 4(1-3k)$

(ii) real roots $\Delta \geq 0$

$\therefore 4(1-3k) \geq 0$

$\therefore k \leq 1/3$

3 (i) $m_{AP} = \dfrac{y-0}{x-2} = \dfrac{y}{x-2}$

(ii) $m_{OP} = \dfrac{y}{x}$

$OP \perp AP \Rightarrow m_{AP}\times m_{OP} = -1$

$\therefore \dfrac{y}{x}\times\dfrac{y}{x-2} = -1 \Rightarrow y^2 = -x(x-2)$

$\therefore y^2+x^2-2x = 0$

$\therefore x^2-2x+y^2=0$

(iii) $x^2-2x+1+y^2=1$

$\therefore (x-1)^2+y^2=1$

$\therefore$ P lies on a circle centre $(1,0)$ radius 1

[except $x=0,2$]

(4) $f(x) = x^4-4x^3$

(i) $\therefore f'(x) = 4x^3-12x^2$

$= 4x^2(x-3)$

(ii) $f'(x)=0$

$\therefore 4x^2(x-3)=0$

$\therefore x=0,0,3$

$\therefore (0,0)$ and $(3,-27)$

(5) $P = A(3^{kt})$

(i) $\therefore t=0 \quad P = 17$ million

$\therefore A(3^0) = 17$ million

$\therefore A = 17\times10^6$

(ii) $t=13$ (2004)

$\therefore 20\times10^6 = 17\times10^6(3^{13k})$

$\therefore 3^{13k} = \dfrac{20}{17} \Rightarrow 13k\log_{10}3 = \log_{10}\left(\dfrac{20}{17}\right)$

$\therefore k = 0.011379$

5 (iii) $k = \dfrac{\log_{10}\left(\frac{20}{17}\right)}{13\log_{10}3}$

$30\times10^6 = 17\times10^6 \cdot 3^{kt}$

$3^{kt} = 30/17$

$\therefore kt\log_{10}3 = \log_{10}(30/17)$

$\therefore t = \dfrac{\log_{10}30/17}{k\log 3} \doteqdot 45.4$

In 2036 the population will be 30 million

(6) (i) $a = T_1 = 30$

$l = 50$

$S_n = 1240$

(i) $S_n = \dfrac{n}{2}(a+l)$

$\therefore 1240 = \dfrac{n}{2}(30+50)$

$\therefore n = \dfrac{1240}{40} = 31$

$\therefore$ 31 strings

(ii) $T_n = a+(n-1)d$

$T_{31} = 30+30d = 50$

$\therefore 30d = 20$

$d = 2/3$ cm.

$\therefore$ Each string is $2/3$ cm apart

(7) $f(0) = -(0)^2 = 0$

$\lim\limits_{x\to0^-} f(x) = \lim\limits_{x\to0^-}(-x^2) = 0$

$\lim\limits_{x\to0^+} f(x) = \lim\limits_{x\to0^+}(5x-4) = -4$

$\therefore \lim\limits_{x\to0^+} f(x) \neq \lim\limits_{x\to0^-} f(x)$

$\therefore \lim\limits_{x\to0} f(x)$ does NOT exist

$\therefore f(x)$ is NOT continuous at $x=0$

(8) (i) From the graph

$A = 1.73$

(ii) $t=0,\ y=1.35$

$\therefore 1.35 = 1.73\sin b$

$\therefore \sin b = \dfrac{1.35}{1.73}$

$\therefore b \doteqdot 0.895$

$\doteqdot 0.1$

(iii) Period $= 12.42\times2$

$= 24.82$

$\therefore \dfrac{2\pi}{a} = 24.82$

$\therefore a \doteqdot \dfrac{2\pi}{\cancel{24.82}\ 12.42}$

$\doteqdot \cancel{0.25}\ 0.51$

$\doteqdot \cancel{0.3}\ 0.5$

(9) $\lim_{x\to a} \dfrac{x^{3/2} - a^{3/2}}{x-a}$ (N.B $a > 0$)

METHOD 1 : $\dfrac{x^{3/2} - a^{3/2}}{x-a} = \dfrac{(\sqrt{x})^3 - (\sqrt{a})^3}{x-a} = \dfrac{(\sqrt{x}-\sqrt{a})(x+\sqrt{ax}+a)}{(x-a)}$

$= \dfrac{(\sqrt{x}-\sqrt{a})(x+\sqrt{ax}+a)}{x-a} \times \dfrac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}}$

$= \dfrac{\cancel{(x-a)}(x+\sqrt{ax}+a)}{\cancel{(x-a)}(\sqrt{x}+\sqrt{a})}$

$\therefore \lim_{x\to a} \dfrac{x^{3/2} - a^{3/2}}{x-a} = \dfrac{(a+\sqrt{a^2}+a)}{(\sqrt{a}+\sqrt{a})}$ $\boxed{\sqrt{a^2} = a \ \because a>0}$

$= \dfrac{3a}{2\sqrt{a}} = \dfrac{3}{2}\sqrt{a}$

METHOD 2 : $\dfrac{x^{3/2} - a^{3/2}}{x-a} = \dfrac{x\sqrt{x} - a\sqrt{a}}{x-a} \times \dfrac{x\sqrt{x}+a\sqrt{a}}{x\sqrt{x}+a\sqrt{a}}$

$= \dfrac{(x^3 - a^3)}{(x-a)(x\sqrt{x}+a\sqrt{a})}$

$= \dfrac{\cancel{(x-a)}(x^2+ax+a^2)}{\cancel{(x-a)}(x\sqrt{x}+a\sqrt{a})}$

$\therefore \lim_{x\to a} \dfrac{x^{3/2} - a^{3/2}}{x-a} = \dfrac{a^2+a^2+a^2}{2a\sqrt{a}} = \dfrac{3a^2}{2a\sqrt{a}} = \dfrac{3}{2}\sqrt{a}$

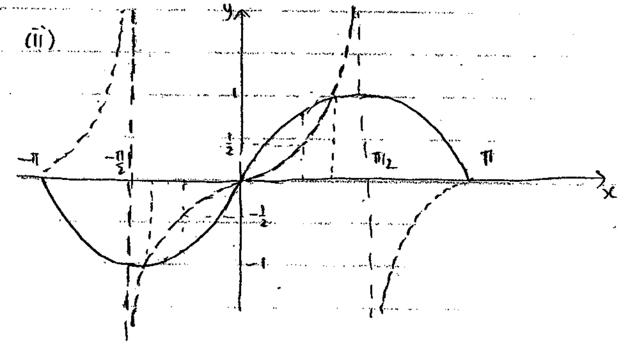
(10) $x^2 = 8(y+3)$
$= 4a(y+3)$

(i) vertex is $(0,-3)$
$\therefore$ axis of symmetry is $\boxed{x=0}$

(ii) $a=2$
$\therefore$ directrix is $y=-3-2$
$\therefore \boxed{y=-5}$

(11) (i) LHS $= \sin \pi/3 = \dfrac{\sqrt{3}}{2}$ RHS $= \dfrac{1}{2}\tan \pi/3 = \dfrac{1}{2}(\sqrt{3}) = \dfrac{\sqrt{3}}{2}$

(ii)



(iii) $\sin x$ and $\frac{1}{2}\tan x$ are odd $\therefore$ $\sin x = \frac{1}{2}\tan x$
$\Rightarrow x = 0, \pm \pi/3$

(iv) $\sin x \le \frac{1}{2}\tan x \Rightarrow \pi/3 \le x < \pi/2$ and $-\pi/3 \le x \le 0$

(12) (i) The focus is at $(0,a)$
$\therefore y=a \Rightarrow x^2 = 4a(a) = 4a^2$
$\therefore x = \pm 2|a|$
$\therefore$ distance $= 4|a|$

(ii) In the left graph the latus rectum is clearly NOT $4|a|$.

(13) (i) By the exterior angle theorem for Δ

$$\alpha = \beta + \beta = 2\beta$$

(ii) $m_{PQ} = \tan\beta = m$

$Q(-1, 0)$

$\therefore y - 0 = m(x+1)$

$\therefore y = m(x+1)$

(iii) $y = m(x+1)$ — (1)

$x^2 + y^2 = 1$ — (2)

sub (1) $\Rightarrow$ (2)

$\therefore x^2 + m^2(x+1)^2 = 1$

$\therefore x^2 + m^2(x^2+2x+1) = 1$

$\therefore x^2 + m^2x^2 + 2m^2x + m^2 - 1 = 0$

$\therefore (1+m^2)x^2 + 2m^2x + m^2 - 1 = 0$

(iv) $\alpha + \beta = -\dfrac{2m^2}{1+m^2}$

where α is the x-coord of Q and β is the x-coordinate of P

$\therefore \beta - 1 = \dfrac{-2m^2}{1+m^2}$

$\therefore \beta = 1 - \dfrac{2m^2}{1+m^2} = \dfrac{1+m^2-2m^2}{1+m^2}$

$= \dfrac{1-m^2}{1+m^2}$ — (3)

sub (3) $\Rightarrow$ (1)

$\therefore y = m\left(\dfrac{1-m^2}{1+m^2} + 1\right) = m\left(\dfrac{1-m^2+1+m^2}{1+m^2}\right)$

$= \dfrac{2m}{1+m^2}$

$\therefore P\left(\dfrac{1-m^2}{1+m^2}, \dfrac{2m}{1+m^2}\right)$

(13) (iv) $\tan\alpha = \dfrac{y_P}{x_P}$

$\therefore \tan 2\beta = \dfrac{2m/1+m^2}{(1-m^2)/1+m^2} = \dfrac{2m}{1-m^2} = \dfrac{2\tan\beta}{1-\tan^2\beta}$

(14) 6% pa = 0.5% per month = 0.005

25 years = 300 months

(i) $80(1.005)^{300} = \$357.20$

(ii) 2nd \$80 earns $80(1.005)^{299}$, until the 300th \$80 earns $80(1.005)$

$\therefore$ Total $= 80(1.005)^{300} + \cdots + 80(1.005)$

$= 80(1.005 + 1.005^2 + \cdots + 1.005^{300})$

$= 80 \times S_{300}$ where $a = 1.005$, $r = 1.005$

$= 80 \times \dfrac{1.005(1.005^{300} - 1)}{0.005}$ $\quad S_n = \dfrac{a(r^n - 1)}{r-1}$

$= \$55\,716.71$

(15) $\log_5(3x-1) + \log_5 2x + 2 = 0$ N.B $x > \frac{1}{3}$ or $x > 0$ $\therefore \boxed{x > \frac{1}{3}}$

$\therefore \log_5[(3x-1)(2x)] = -2$

$\therefore (3x-1)(2x) = 5^{-2} = \frac{1}{25}$

$\therefore 25(3x-1)(2x) - 1 = 0$

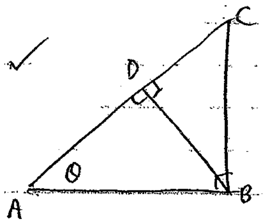
$\therefore 150x^2 - 50x - 1 = 0$

$\therefore x = \frac{50 \pm \sqrt{3100}}{300} = \frac{50 \pm 10\sqrt{31}}{300} = \frac{5 \pm \sqrt{31}}{30}$

$x \doteqdot 0.352, -0.284$

$\therefore \boxed{x \doteqdot 0.352}$

(16) (i)



$0° < \theta < 90°$
$\therefore \cos\theta \neq 0$

(ii) $6AD + BC = 5AC$

$AD = AB\cos\theta$

$BC = AB\tan\theta$

$AC = \frac{AB}{\cos\theta} = AB\sec\theta$

$\therefore 6AB\cos\theta + AB\tan\theta = 5AB\sec\theta$

$\therefore 6\cos\theta + \tan\theta = 5\sec\theta$ $(\because AB \neq 0)$

(iii) $6\cos\theta + \frac{\sin\theta}{\cos\theta} = \frac{5}{\cos\theta}$

$\therefore 6\cos^2\theta + \sin\theta = 5$

$\therefore 6(1-\sin^2\theta) + \sin\theta = 5$ $(\cos^2\theta + \sin^2\theta = 1)$

$\therefore 6 - 6\sin^2\theta + \sin\theta = 5$

$\therefore 6\sin^2\theta - \sin\theta - 1 = 0$

(iv) $\sin\theta = \frac{1 \pm \sqrt{25}}{12}$

$= \frac{1 \pm 5}{12}$

$\therefore \sin\theta = \frac{1}{2}, -\frac{1}{3}$

$\therefore \sin\theta = \frac{1}{2}$

$\therefore \boxed{\theta = 30°}$ $0° < \theta < 90$

(17) $y = \frac{x-2}{2x+1}$ $u = x-2$ $v = 2x+1$
$u' = 1$ $v' = 2$

$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(2x+1)(1) - (x-2)(2)}{(2x+1)^2}$

$= \frac{2x+1 - 2x+4}{(2x+1)^2}$

$= \frac{5}{(2x+1)^2}$

at $x = 2$, $y = 0$

$\frac{dy}{dx} = \frac{1}{5}$

$\therefore m_\perp$ = gradient of normal $= -5$

$\therefore y - 0 = -5(x-2)$

$\therefore \boxed{y = -5x + 10}$

(18) (i) $B_1 = 1000(1.06) - 72$

$B_2 = B_1(1.06) - 72$

$= 1000(1.06)^2 - 72(1+1.06)$

$= 1000(1.06)^2 - 72(2.06)$

$= \$975.28$

(ii) $B_n = 1000(1.06)^n - 72(1+1.06+\dots+1.1$

$= 1000(1.06)^n - \frac{72 \times (1.06^n - 1)}{0.06}$

$= 1000(1.06)^n - 1200(1.06^n - 1)$

$= (1000)(1.06)^n - 1200(1.06)^n + 120$

$= 1200 - 200(1.06)^n$

(iii) $B_{10} = 1200 - 200(1.06)^{10}$

$\doteqdot \$841.83$ (store in memory)

Let b_n = amount now remaining

$b_1 = B_{10}(1.06)' - 90$

$b_2 = b_1(1.06) - 90$

$\therefore b_n = B_{10}(1.06)^n - \frac{90(1.06^n - 1)}{0.06}$

$= B_{10}(1.06)^n - 1500(1.06^n - 1)$

$b_n = 0 \Rightarrow$

$(1.06)^n[B_{10} - 1500] = -1500$

$\therefore (1.06)^n = \frac{1500}{1500 - B_{10}}$

$\therefore n\log_{10}1.06 = \log_{10}\left(\frac{1500}{1500 - B_{10}}\right.$

$\therefore n \doteqdot 14.1$

$\therefore$ only 14 more years.