SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

SEPTEMBER 2004

YEAR 11

PRELIMINARY HIGH SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension

General Instructions

- Reading Time – 5 Minutes
- Working time – One and a half hours
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators may be used.
- All necessary working should be shown in every question.

Total Marks – 72

- Attempt all questions.
- All questions are of equal value.
- Each question is to be answered in a separate booklet.

Examiner: *A.M.Gainford*

Question 1. (18 Marks)

(a) Show that $\frac{1}{3-\sqrt{2}}+\frac{1}{3+\sqrt{2}}$ is a rational number. **2**

(b) Solve for x: **6**

(i) $|2x-1|=5$

(ii) $x^2 \geq 1$

(iii) $\frac{1}{x-1}<2$

(c) Find the remainder when the polynomial $P(x)=2x^3-3x^2+x-4$ is divided by $x-2$. **1**

(d) Simplify $\frac{x^3-1}{x^2-2x+1}$. **2**

(e) If $\tan\theta=2$, and $0<\theta<\frac{\pi}{2}$, find the exact value of $\sin\left(\theta+\frac{\pi}{4}\right)$. **2**

(f) Find the vertex and focus of the parabola $y=\frac{1}{4}\left(x^2-2x+9\right)$. **2**

(g) Show that for all θ: **3**

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta .$$

Question 2. (18 Marks)

(a) Differentiate: **8**

(i) $1+2x-4x^2-x^3$

(ii) $\sqrt{1-x^2}$

(iii) $(x-1)^4(3x+1)$

(iv) $\frac{2}{x^3-1}$

(b) (i) Express $\sin x-\sqrt{3}\cos x$ in the form $A\sin(x-\alpha)$, where $A>0$ and $0<\alpha<\frac{\pi}{2}$. **2**

(ii) Find the general solution to the equation $\sin x-\sqrt{3}\cos x=\frac{2}{\sqrt{2}}$. **2**

(c) Solve $(x-1)^2 < 4(x-1)$, and graph the solution on the number line. **2**

(d) Sketch the graph of $y = \cos x + \sin 2x$ in the domain $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. **2**

(e) Given the polynomial $P(x) = x^3 - 19x - 30$. **2**

(i) Use the factor theorem to find a zero of the polynomial.

(ii) Express $P(x)$ as a product of three linear factors.

Question 3. (18 Marks)

(a) (i) Express the decimal $0 \cdot 1\dot{5}\dot{4}$ as a common fraction in lowest terms. **2**

(ii) Find $\log_2 74$ correct to three decimal places.

(b) Draw neat sketches of the following functions, showing their principle features: **6**

(i) $y = |x+1|$ (ii) $y = 2^{-x}$ (iii) $y = \sqrt{9-x^2}$

(c) Given the function $f(x) = \dfrac{x}{x^2+1}$ **6**

(i) Find $f(-1)$.

(ii) Show that $f(x)$ is odd.

(iii) Find x such that $f(x) = 0$.

(iv) State the domain and range of $f(x)$.

(v) Sketch the function.

(d) Of the three roots of the cubic equation $x^3 - 15x + 4 = 0$, two are reciprocals. **2**

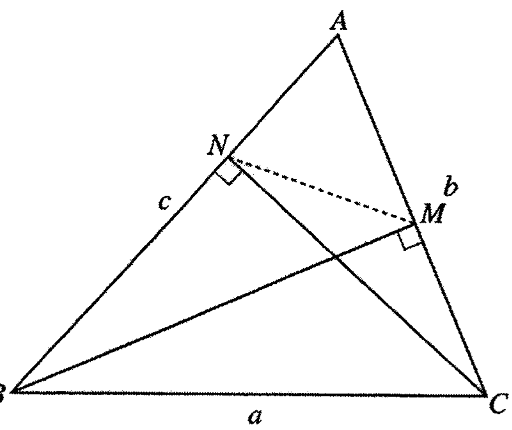
(i) Find the other root.

(ii) Find the reciprocal roots.

(e) Find the distance between the parallel lines $4x + 3y = 12$ and $4x + 3y = 5$. **2**

Question 4. (18 Marks)

(a) 4



Triangle ABC has sides of length a, b, c as shown.
BM is perpendicular to AC and CN is perpendicular to AB.

(i) Show that $AM = c\cos A$ and $AN = b\cos A$.

(ii) Hence, using the cosine rule, prove that $MN = a\cos A$.

(b) Let $P(2ap, ap^2)$ be a point on the parabola $x^2 = 4ay$. 4

(i) Write down the equation of the tangent at P.

(ii) Let θ be the acute angle between the tangent at P and the line SP, which joins P with the focus S.

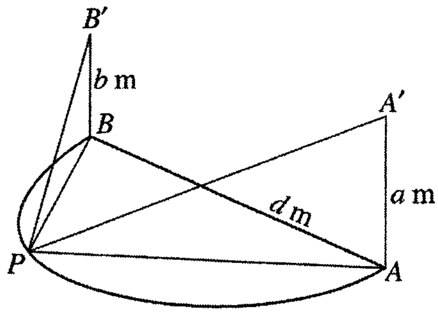
Show that $\tan\theta = \dfrac{1}{|p|}$.

(iii) Explain the situation at the one point where this angle is not acute.

(c) Show that $\cot\theta + \tan\theta = 2\operatorname{cosec}2\theta$. 2

(d) The point $P(0,4)$ divides the interval from (a, b) to (b, a) in ratio 3 : 1. 2
Find the values of a and b.

(e) APB is a horizontal semicircle, diameter d m. **6**
At A and B are vertical posts of height a m and b m respectively.
From P, the angle of elevation of the tops of both posts is θ.
The angle APB is a right angle.



(i) Prove that $d^2 = \dfrac{a^2}{\tan^2\theta} + \dfrac{b^2}{\tan^2\theta}$.

(ii) From B, the angle of elevation of A' is α, and from A, the angle of elevation of B' is β.

Prove that $\tan^2\alpha + \tan^2\beta = \tan^2\theta$.

End of the paper.

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SEPTEMBER 2004

PHSC Examination

YEAR 11

Mathematics Extension

Sample Solutions

Question	Marker
1	PSP
2	Mr Choy
3	Mr Hespe
4	Mr Bigelow

Question 1

(a) $$\frac{1}{3-\sqrt{2}}+\frac{1}{3+\sqrt{2}}=\frac{3+\sqrt{2}+3-\sqrt{2}}{\left(3-\sqrt{2}\right)\left(3+\sqrt{2}\right)}$$

$$=\frac{6}{9-2}$$

$$=\frac{6}{7}$$

$\frac{6}{7}\in\mathbb{Q}\Rightarrow\frac{1}{3-\sqrt{2}}+\frac{1}{3+\sqrt{2}}$ is a rational number

QED

(b) (i) $|2x-1|=5$

$\therefore 2x-1=5$ or $2x-1=-5$

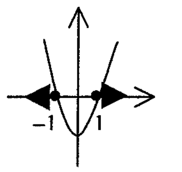
$\therefore 2x=6,-4$

$\therefore x=3,-2$

(ii) $x^2\geq 1\Rightarrow x^2-1\geq 0$

$\therefore (x-1)(x+1)\geq 0$

$\therefore x\leq -1, x\geq 1$



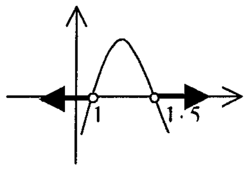
(iii) $\frac{1}{x-1}<2\Rightarrow\frac{1}{x-1}-2<0$

$\therefore \frac{1-2(x-1)}{x-1}<0\Rightarrow\frac{1-2x+2}{x-1}<0$

$\therefore \frac{3-2x}{x-1}<0 \quad \left[\times(x-1)^2\right]$

$\therefore (x-1)(3-2x)<0$

$\therefore x<1, x>\frac{3}{2}$



(c) $P(x)=2x^3-3x^2+x-4$

By the Remainder Theorem:

Remainder $=P(2)=16-12+2-4=2$

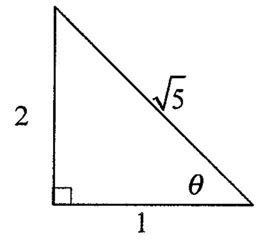
(d) $$\frac{x^3-1}{x^2-2x+1}=\frac{(x-1)\left(x^2+x+1\right)}{(x-1)^2}=\frac{x^2+x+1}{x-1}$$

(e) $\sin\left(\theta+\frac{\pi}{4}\right)=\sin\theta\cos\frac{\pi}{4}+\cos\theta\sin\frac{\pi}{4}$

$=\frac{1}{\sqrt{2}}(\sin\theta+\cos\theta)$

$=\frac{1}{\sqrt{2}}\left(\frac{2}{\sqrt{5}}+\frac{1}{\sqrt{5}}\right)$

$=\frac{3}{\sqrt{10}}=\frac{3\sqrt{10}}{10}$



(f) $y=\frac{1}{4}\left(x^2-2x+9\right)$

$\therefore 4y=x^2-2x+9 \Rightarrow x^2-2x+1+8$

$\therefore (x-1)^2=4y-8=4(y-2)$

$\therefore a=1$

Vertex $(1,2)$, Focus $(1,3)$

(g) $\text{LHS}=\cos 3\theta$

$=\cos(2\theta+\theta)$

$=\cos 2\theta\cos\theta-\sin 2\theta\sin\theta$

$=\left(2\cos^2\theta-1\right)\cos\theta-(2\sin\theta\cos\theta)\sin\theta$

$=2\cos^3\theta-\cos\theta-2\cos\theta\sin^2\theta$

$=2\cos^3\theta-\cos\theta-2\cos\theta\left(1-\cos^2\theta\right)$

$=2\cos^3\theta-\cos\theta-2\cos\theta+2\cos^3\theta$

$=4\cos^3-3\cos\theta$

$=\text{RHS}$

QED

Question 2

Question (2)

(a) (i) $\frac{d}{dx}(1+2x-4x^2-x^3)$

$= \boxed{2-8x-3x^2}$ 2

(ii) $\frac{d}{dx}(1-x^2)^{\frac{1}{2}}$

$= \frac{1}{2}(1-x^2)^{-1/2}(-2x)$

$= \boxed{-x(1-x^2)^{-1/2}}$ 2

(iii) $\frac{d}{dx}(x-1)^4(3x+1)$

$= (3x+1)4(x-1)^3 + (x-1)^4 \cdot 3$

$= (x-1)^3(12x+4+3x-3)$

$= \boxed{(x-1)^3(15x+1)}$ 2

(iv) $\frac{d}{dx}\left(\frac{2}{x^3-1}\right) = 2\frac{d}{dx}(x^3-1)^{-1}$

$= -2(x^3-1)^{-2}(3x^2)$

$= \boxed{-6x^2(x^3-1)^{-2}}$ 2

(b) (i) $\sin x - \sqrt{3}\cos x = A\sin(x-\alpha)$

$A(\sin x\cos\alpha - \cos x\sin\alpha)$

$\therefore A\cos\alpha = 1$

$A\sin\alpha = \sqrt{3}$

$\therefore \tan\alpha = \sqrt{3}, \; \alpha = \frac{\pi}{3}$

$A^2 = 1+3 = 4, \; A = 2$

$\therefore \boxed{\sin x - \sqrt{3}\cos x = 2\sin(x-\frac{\pi}{3})}$ 2

(ii) $2\sin(x-\frac{\pi}{3}) = \frac{2}{\sqrt{2}}$

$\therefore \sin(x-\frac{\pi}{3}) = \sin\frac{\pi}{4}$

$x - \frac{\pi}{3} = n\pi + (-1)^n\frac{\pi}{4}$

$\therefore \boxed{x = (n\pi + \frac{\pi}{3}) + (-1)^n\frac{\pi}{4}}$

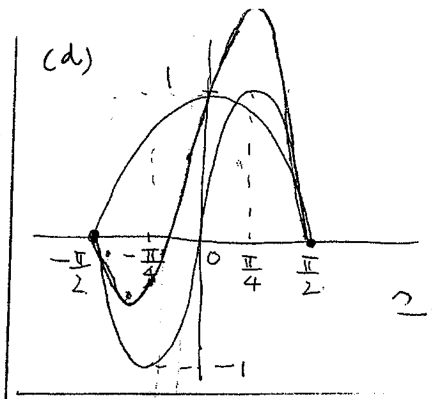
$n\pi + (-1)^n\frac{\pi}{4} + \frac{\pi}{3}$ 2

(c) $(x-1)[(x-1)-4] < 0$

$(x-1)(x-5) < 0$ 2

$\therefore \boxed{1 < x < 5}$

(d)



2

(e) $P(x) = x^3 - 19x - 30$

(i) $P(-2) = -8 + 38 - 30 = 0$

$\therefore (x+2)$ is a factor. 1

$$\begin{array}{r} x^2 - 2x - 15 \\ x+2 \,\overline{)\, x^3 - 19x - 30} \\ -(x^3 + 2x^2) \\ \hline -2x^2 - 19x - 30 \\ -(-2x^2 - 4x) \\ \hline -15x - 30 \\ -(-15x - 30) \\ \hline 0 \end{array}$$

1

$\therefore$ (ii) $\boxed{P(x) = (x+2)(x+3)(x-5)}$

Question 3

3. (a) (i) Express the decimal $0{\cdot}1\dot{5}\dot{4}$ as a common fraction in lowest terms.

Solution:
$$\begin{aligned} x &= 0{\cdot}1\dot{5}\dot{4}, \\ 100x &= 15{\cdot}4\dot{5}\dot{4}, \\ 99x &= 15{\cdot}3, \\ x &= \frac{153}{990}, \\ &= \frac{17}{110}. \end{aligned}$$

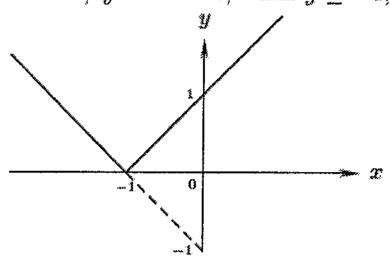
(ii) Find $\log_2 74$ correct to three decimal places.

Solution:
$$\begin{aligned} \log_2 74 &= \frac{\log 74}{\log 2}, \\ &\approx 6{\cdot}209 \quad [6{\cdot}20945336562 \text{ on calculator}]. \end{aligned}$$

(b) Draw neat sketches of the following functions, showing their principal features:

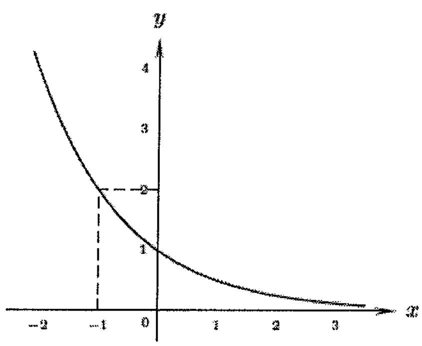
(i) $y = |x+1|$

Solution: When $x < -1,\ y = -x - 1$; when $y \geq -1,\ y = x + 1$.



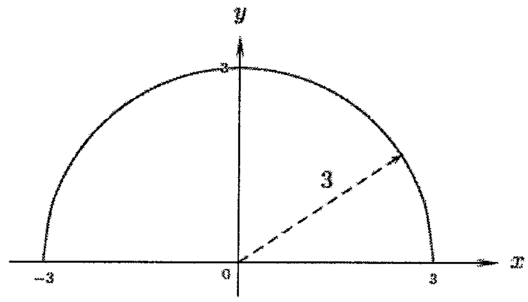
(ii) $y = 2^{-x}$

Solution:



(iii) $y = \sqrt{9 - x^2}$

Solution:



(c) Given the function $f(x) = \dfrac{x}{x^2 + 1}$

(i) Find $f(-1)$

Solution: $$\begin{aligned} f(-1) &= \frac{-1}{(-1)^2 + 1}, \\ &= -\tfrac{1}{2}. \end{aligned}$$

(ii) Show that $f(x)$ is odd.

Solution: $$\begin{aligned} f(-x) &= \frac{-x}{(-x)^2 + 1}, \\ &= -\frac{x}{x^2 + 1}, \\ &= -f(x). \end{aligned}$$
$\therefore\ f(x)$ is odd.

(iii) Find x such that $f(x) = 0$.

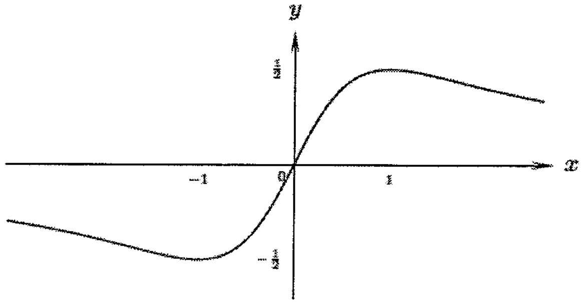
Solution: $$\begin{aligned} \frac{x}{x^2 + 1} &= 0, \\ \therefore\ x &= 0. \end{aligned}$$

(iv) State the domain and range of $f(x)$.

Solution: Domain: $x \in \mathbb{R}$, or all real x.
Now, putting $y = f(x)$ and rearranging,
$$\begin{aligned} yx^2 + y &= x, \\ yx^2 - x + y &= 0, \\ \Delta &= 1 - 4y^2\ \geq 0 \text{ for real values of x}, \\ i.e.\ \tfrac{1}{4} &= y^2. \end{aligned}$$
And thus the range is $-\frac{1}{2} \leq f(x) \leq \frac{1}{2}$.

(v) Sketch the function.

Solution:



(d) Of the three roots of the cubic equation $x^3 - 15x + 4 = 0$, two are reciprocals.

(i) Find the other root.

Solution: Let the roots be α, $\dfrac{1}{\alpha}$, β, then

$$\begin{aligned} \alpha \times \frac{1}{\alpha} \times \beta &= -4 \text{ (product of roots)}, \\ \textit{i.e. } \beta &= -4. \end{aligned}$$

(ii) Find the reciprocal roots.

Solution:

$$\begin{aligned} \alpha + \frac{1}{\alpha} - 4 &= 0 \text{ (sum of roots)}, \\ \alpha^2 - 4\alpha + 1 &= 0, \\ \alpha &= \frac{4 \pm \sqrt{16-4}}{2}, \\ &= 2 \pm \sqrt{3}. \end{aligned}$$

i.e. the reciprocal roots are $2 \pm \sqrt{3}$.

(e) Find the distance between the parallel lines $4x + 3y = 12$ and $4x + 3y = 5$.

Solution: One point on $4x + 3y = 12$ is $(0,\ 4)$.

$$\begin{aligned} \therefore \text{ Distance} &= \frac{|0 \times 4 + 4 \times 3 - 5|}{\sqrt{16+9}}, \\ &= \frac{7}{5}. \end{aligned}$$

Question 4

(a) (i) $\frac{AM}{c} = \cos A$

$\therefore AM = c\cos A.$

$\frac{AN}{b} = \cos A.$

$\therefore AN = b\cos A.$

(ii) $MN^2 = AN^2 + AM^2 - 2AN.AM.\cos A.$

$= b^2\cos^2 A + c^2\cos^2 A - 2bc\cos^2 A.\cos A.$

$= \cos^2 A\,(b^2 + c^2 - 2bc\cos A)$ (NB. $a^2 = b^2 + c^2 - 2bc\cos A$ from Cosine Rule.)

$= \cos^2 A.\,a^2$

$\therefore MN = a\cos A.$

(b) (i) $y - px + ap^2 = 0.$

(ii) $m_{SP} = \frac{ap^2 - a}{2ap}$

$= \frac{a(p^2-1)}{2ap}$

$= \frac{p^2-1}{2p}$

$\tan\theta = \left| \frac{\frac{p^2-1}{2p} - p}{1 + \frac{p^2-1}{2p} \times p} \right|$

$= \left| \frac{\frac{p^2-1-2p^2}{2p}}{\frac{2+p^2-1}{2}} \right|$

$= \left| \frac{-(1+p^2)}{p(1+p^2)} \right|$

$= \left| \frac{-1}{p} \right|$

$= \frac{1}{|p|}.$

(iii) $\tan\theta$ is undefined where $p=0$. At this point $(0,0)$ the angle is $90°$.

(c) LHS $= \cot\theta + \tan\theta$

$$= \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta}$$

$$= \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{1}{\sin\theta\cos\theta}$$

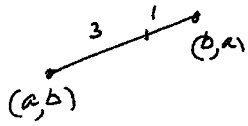
$$= \frac{2}{2\sin\theta\cos\theta}$$

$$= \frac{2}{\sin 2\theta}$$

$$= 2\,\mathrm{cosec}\,2\theta$$

= RHS.

(d)



$$\frac{3b+a}{4} = 0 \Rightarrow 3b+a=0 \quad \text{— (1).}$$

$$\frac{3a+b}{4} = 4 \Rightarrow 3a+b=16 \quad \text{— (2)}$$

Solving (1) + (2) simultaneously.

From (1) $a = -3b$.

Substitute in (2)

$$-9b + b = 16$$
$$-8b = 16$$
$$b = -2.$$

Sub in (1)

$$-6 + a = 0$$
$$a = 6.$$

$\therefore$ $a = 6$, $b = -2$.

(e) (i) $\frac{a}{AP} = \tan\theta$ $\therefore$ $AP = \frac{a}{\tan\theta}$ also $\frac{b}{BP} = \tan\theta$ $\therefore BP = \frac{b}{\tan\theta}$.

By Pythagoras $d^2 = AP^2 + BP^2$

$$d^2 = \frac{a^2}{\tan^2\theta} + \frac{b^2}{\tan^2\theta} \quad \text{— (A)}$$

(ii) LHS $= \tan^2\alpha + \tan^2\beta = \frac{a^2}{d^2} + \frac{b^2}{d^2}$

$$= \frac{a^2+b^2}{d^2}$$

(From (A) $a^2+b^2 = d^2\tan^2\theta$.)

$$= \frac{d^2\tan^2\theta}{d^2}$$

$$= \tan^2\theta$$

= RHS.