SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2004

YEAR 11

HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK # 1

Mathematics Extension 1

General Instructions
- Working time – 90 minutes.
- Reading Time – 5 minutes.
- Write using black or blue pen.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may not be awarded for messy or badly arranged work

Total Marks –81
- Attempt all questions.
- All questions are **NOT** of equal value.

Examiner: *C. Kourtesis*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

Question 1 (15 marks) **Marks**

(a) Simplify $\frac{1}{5!}+\frac{1}{6!}$ 1

(b) In how many ways can a committee of 6 be chosen from a group of 10 people? 2

(c) Write down the general solution of 2

$$\cos\theta = \frac{1}{2}$$

(d) If α, β and γ are the roots of the cubic equation $2x^3 + 12x^2 - 6x + 1 = 0$ find the value of: 4

(i) $\alpha+\beta+\gamma$

(ii) $\alpha\beta\gamma$

(iii) $\alpha^{-1}+\beta^{-1}+\gamma^{-1}$

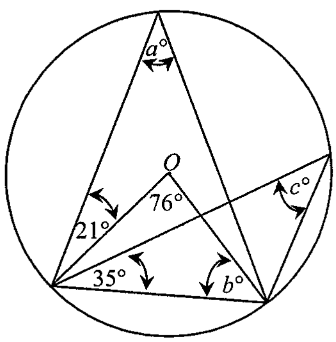
(e) The equation of a parabola is 3

$$y = x^2 - 6x + 7$$

Find the

(i) focal length
(ii) coordinates of the vertex

(f) 3



Find the values of *a*, *b* and *c*.
[There is no need to give reasons]

O is the centre of the circle

Question 2 [15 marks] **Marks**

(a) The polynomial $P(x) = x^3 - 6x^2 + \theta x - 4$ has $x = 1$ as a zero. 5

Find the:

(i) value of θ

(ii) other zeros of $P(x)$

(iii) values of x for which $P(x) < 0$

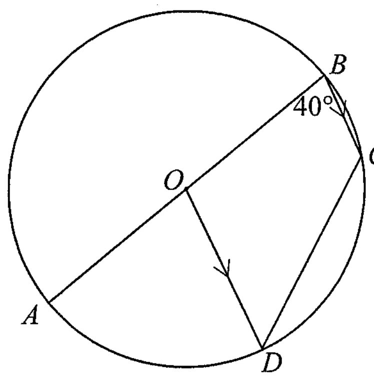
(b) The parametric coordinates of P, a point on a curve are given by $P(8t, 2t^2)$ where t is the parameter 3

Find:

(i) the Cartesian equation of the curve

(ii) the gradient of the tangent to the curve at the point where $t = 3$

(c)



AB is the diameter of a circle centre O. 3
BC and OD are parallel and $\angle OBC = 40°$. Find the size of $\angle OCD$ giving reasons

(d) Prove by Mathematical Induction that 4

$$1 + 3 + 5 + 7 + \ldots\ldots + (2n-1) = n^2$$

for positive integers n

Question 3 **[18 marks]** **Marks**

(a) T is the point $(1,-2)$ and the tangents from T to the parabola $x^2 = 12y$ touch the parabola at A and B. Write down the equation of the line AB 2

(b) In how many ways can 5 different books be placed in a row so that two specified books: 4

(i) occupy the end positions

(ii) must always be together?

(c)

(i) Write down the expansion of $\sin(x-\theta)$ 6

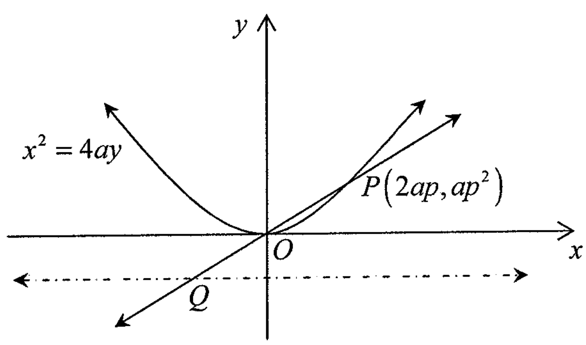
(ii) If $\sin x - \cos x = A\sin(x-\theta)$ where $A > 0$ and $0 < \theta < \frac{\pi}{2}$, show that $A = \sqrt{2}$ and $\theta = \frac{\pi}{4}$

(iii) Hence or otherwise solve the equation for **all** values of x.

$$\sin x - \cos x = 1$$

(iv) Find the maximum value of $\sin x - \cos x$

(d) 6



The diagram shows the parabola $x^2 = 4ay$. PO is produced to meet the directrix at Q.

(i) Show that the equation of the tangent at P has equation
$$y - px + ap^2 = 0$$

(ii) Find the coordinates of the point Q.

(iii) Prove that QS is parallel to the tangent at P.
[S is the focus]

Question 4 **[18 marks]** **Marks**

(a) 17 people sit at a round table. In how many ways can they be seated if: 4

(i) there are no restrictions

(ii) two particular people cannot sit together

(b) If $t = \tan\frac{\theta}{2}$ express in terms of t 3

$$\cos\theta + \sin^2\left(\frac{\theta}{2}\right)$$

(c) When a polynomial $P(x)$ is divided by $x^2 - 3x + 2$ the remainder is $4x - 7$. Find the remainder when $P(x)$ is divided by $(x-1)$. 2

(d) A polynomial $P(x)$ has the following properties: 4

(i) $P(x)$ is odd and has a factor of $(x-5)^2$.

(ii) The curve of $y = P(x)$ passes through $(1,1152)$.

Find the polynomial, $P(x)$ of least degree that satisfies the above, expressing your answer in factorized form.

(e) $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are points on the parabola $x^2 = 4ay$ with parameters p and q respectively. 5

(i) Find the coordinates of M the midpoint of PQ.

(ii) If PQ subtends a right angle at the vertex show that $pq + 4 = 0$.

(iii) Find the locus of M.

Question 5 **[15 marks]** **Marks**

(a) How many arrangements are there of the letters in the word 2

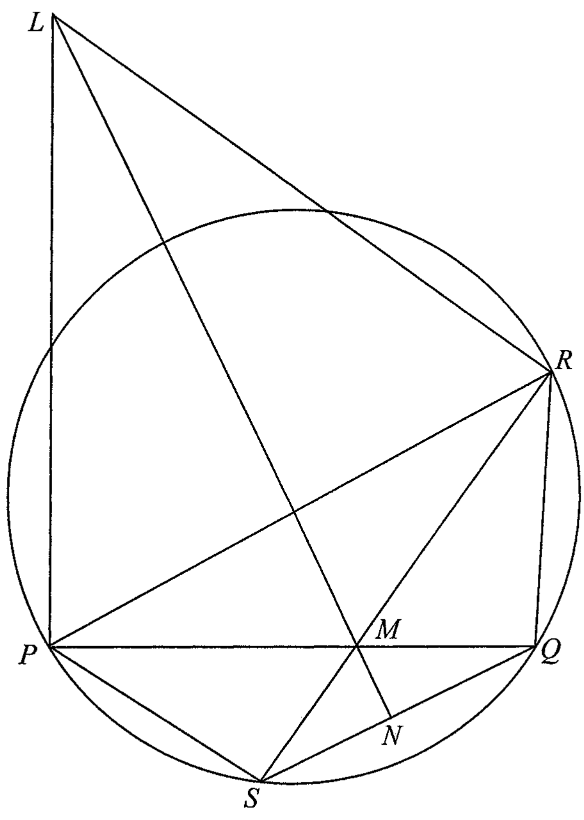
ANDAMOOKA

(b) From the letters of the word PROBLEMS how many different words consisting of 5 letters are possible if they include P, do not begin with P and the letter M is to be excluded? 3

(c) Prove by Mathematical Induction that 4

$$\sin(n\pi+\theta)=(-1)^n\sin\theta \text{ for } 0<\theta<\frac{\pi}{2} \text{ for positive integers } n$$

(d) PQ and RS are two chords of a circle which intersect at M inside the circle. MN is the perpendicular from M to SQ. L is the point on NM produced such that LP is perpendicular to PQ 6



(i) Copy the diagram

(ii) Show that $\Delta PML \,|||\, \Delta NMQ$

(iii) Hence show that $LR \perp RS$

End of paper

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Mathematics Extension 1

Sample Solutions

Question	Marker
1	Mr Kidd
2	Mr Choy
3	Mr Parker
4	Mr Dunne
5	Mr Bigelow

Question 1

Q1 (a) $\frac{1}{5!}+\frac{1}{6!} = \frac{6}{6!}+\frac{1}{6!}$

$= \frac{7}{6!}$ or $\frac{7}{720}$

(b) ${}^{10}C_6 = 210$

(c) $\theta = 2n\pi \pm \frac{\pi}{3}$ or $360^\circ n \pm 60^\circ$

(d)(i) $\alpha+\beta+\gamma = -\frac{b}{a} = -\frac{12}{2} = -6$

(ii) $\alpha\beta\gamma = \frac{-d}{a}$

$= -\frac{1}{2}$

(iii) $\alpha^{-1}+\beta^{-1}+\gamma^{-1} = \frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$

$= \frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}$

$= \frac{\frac{-6}{2}}{-\frac{1}{2}}$

$= 6$

(e) $y = x^2-6x+7$

ie $x^2-6x+9 = y-7+9$

$(x-3)^2 = y+2$

$(x-3)^2 = 4\times\frac{1}{4}(y+2)$

$\therefore$ Focal length $= \frac{1}{4}$

Vertex is $(3,-2)$

(f) $a = \frac{76}{2} = 38^\circ$

$b = \frac{180-76}{2}$

$= 52^\circ$

$c = a = 38^\circ$.

Question 2

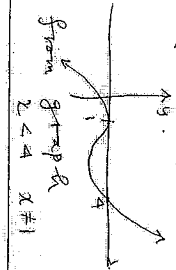
(a)
$P(x) = x^3 - 6x^2 + \theta x - 4$
$P(1) = 0$
$1 - 6 + \theta - 4 = 0$
$\theta - 9 = 0$

(i) $\therefore \theta = 9$.

(ii)
$$\begin{array}{r|l} & x^2 - 5x + 4 \\ \hline x-1 & x^3 - 6x^2 + 9x - 4 \\ & -(x^3 - x^2) \\ \hline & -5x^2 + 9x - 4 \\ & -(-5x^2 + 5x) \\ \hline & 4x - 4 \\ & -(4x-4) \\ \hline & 0 \end{array}$$

$\therefore P(x) = (x-1)^2(x-4)$.
$x = 1, 4$.

(iii) $P(x) < 0$
$(x-1)^2(x-4) < 0$



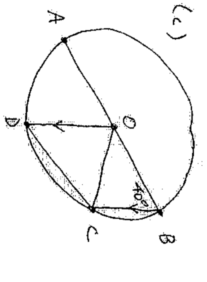
from graph
$x < 4 \quad x \neq 1$

(b) $x = 8t$, $y = 2t^2$
$y = 2\left(\frac{x}{8}\right)^2 = \frac{x^2}{32}$

(i) $\therefore x^2 = 32y$.

(ii) $\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$
$= 4t/8 = t/2$.
$\therefore \frac{dy}{dx}\Big|_{t=3} = \frac{3}{2}$.

(c)



Join OC.
$\triangle OBC$ isosceles
($OB = OC$, radii)
$\therefore \angle BCO = 40^\circ$
Also, $\angle COD = 40^\circ$
(alt. $\angle$s, $BC \parallel OD$).
but $\triangle ODC$ is also isosceles.
$OC = OD$ (radii)
$\Rightarrow \angle OCD = 70^\circ$
($\angle$ sum of a triangle ODC).

(d) Let $S(n)$ be the proposition that
$1 + 3 + 5 + \dots + (2n-1) = n^2$.
For $n = 1$, $LHS = 1$
$RHS = 1^2 = 1$
$\therefore LHS = RHS \Rightarrow S(1)$ is true.
Assume $S(k)$ is true
i.e.
$1 + 3 + 5 + \dots + (2k-1) = k^2$
Consider $n = k+1$
$1 + 3 + \dots + (2k-1) + (2k+1)$
$= k^2 + (2k+1)$
$= (k+1)^2$
Since the statement is true for $n=1$ and true for $n = k+1$ when true for $n = k$ ($k \in Z^+$) $\therefore S(n)$ is true for all $n \geq 1$.

Question 3

(a) Chord of contact $xx_0 = 2a(y + y_0)$

$x^2 = 12y \Rightarrow a = 3$

$T\left(\underset{x_0}{1}, \underset{y_0}{-2}\right)$

$\therefore x \times 1 = 2 \times 3(y-2) \Rightarrow x = 6(y-2)$

$\therefore x - 6y + 12 = 0$

(b) (i) ■□□□■

Place the two books at either end, so there are 3! ways of arranging the books in between. The end books can then be switched.
Total = $3! \times 2 = 12$

(ii) □■■□□

Place the two books together, so there are 4 objects to arrange in 4! Ways. Then the books can be switched.
Total = $4! \times 2 = 48$

(c) (i) $\sin(x - \theta) = \sin x \cos\theta - \cos x \sin\theta$

(ii) $\sin x - \cos x = A\sin(x - \theta)$

$= (A\cos\theta)\sin x - (A\sin\theta)\cos x$

$\therefore A\cos\theta = 1, A\sin\theta = 1$

$\therefore A = \sqrt{1^2 + 1^2} = \sqrt{2} \quad (\because A > 0)$

$\tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4} \quad \left(\because 0 < \theta < \frac{\pi}{2}\right)$

(iii) $\sin x - \cos x = 1 \Rightarrow \sqrt{2}\sin\left(x - \frac{\pi}{4}\right) = 1$

$\therefore \sin\left(x - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$\boxed{\sin\theta = c \Rightarrow \theta = n\pi + (-1)^n \sin^{-1} c}$

$\therefore x - \frac{\pi}{4} = n\pi + (-1)^n \frac{\pi}{4}$

$\therefore x = n\pi + (-1)^n \frac{\pi}{4} + \frac{\pi}{4} = n\pi + \frac{\pi}{4}\left(1 + (-1)^n\right)$

$$x=\begin{cases} n\pi & n \text{ odd} \\ n\pi+\dfrac{\pi}{2} & n \text{ even}\end{cases}$$

Alternatively:

$$\sin\left(x-\frac{\pi}{4}\right)=\cos\left(\frac{3\pi}{4}-x\right)=\cos\left(x-\frac{3\pi}{4}\right)$$

$$\therefore x-\frac{3\pi}{4}=2n\pi\pm\frac{\pi}{4}$$

$$\boxed{\cos\theta=c\Rightarrow\theta=2n\pi\pm\cos^{-1}c}$$

$$x=2n\pi+\pi,2n\pi+\frac{\pi}{2}$$

$$\therefore x=(2n+1)\pi,2n\pi+\frac{\pi}{2}$$

(iv) $\max(\sin x-\cos x)=\max\left(\sqrt{2}\sin\left(x-\frac{\pi}{4}\right)\right)=\sqrt{2}$

(d) (i) $y=\dfrac{x^2}{4a}$

$$\frac{dy}{dx}=\frac{x}{2a}$$

$$m_P=\left.\frac{dy}{dx}\right|_{x=2ap}=\frac{2ap}{2a}=p$$

$$\therefore y-ap^2=p(x-2ap)$$

$$\therefore y=px-2ap^2+ap^2=px-ap^2$$

Alternatively:

$x=2at, y=at^2$

$$\frac{dx}{dt}=2a,\frac{dy}{dt}=2at$$

$$\therefore\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{2at}{2a}=t$$

$$\therefore m_P=\left.\frac{dy}{dx}\right|_{t=p}=p$$

(ii) The directrix is $y=-a$

$$m_{OP}=\frac{ap^2-0}{2ap-0}=\frac{p}{2}$$

$$\therefore y-ap^2=\frac{p}{2}(x-2ap)=\left(\frac{p}{2}\right)x-ap^2$$

$$\therefore y=\left(\frac{p}{2}\right)x$$

Q: sub $y=-a\Rightarrow x=-\dfrac{2a}{p}$

$$Q\left(-\frac{2a}{p},-a\right)$$

(iii) Focus is $(0,a)$

$$m_{QS}=\frac{a-(-a)}{0-\left(-\frac{2a}{p}\right)}=\frac{2a}{2a/p}=2a\times\frac{p}{2a}=p$$

$\therefore QS \parallel$ tangent at P.

Question 4

a i) $16!$ Standard Bookwork.

ii) If 2 people MUST be seated together

Consider them as joined A & B or B & A

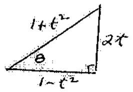
This can be done in $2 \times 15!$ ways

Hence 2 people not together $16! - 2 \times 15!$

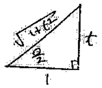
$= 15!\,(16-2)$

$= 14 \times 15!$

b)



$\cos\theta = \frac{1-t^2}{1+t^2}$



$\tan\frac{\theta}{2} = t$ Hence $\sin\frac{\theta}{2} = \frac{t}{\sqrt{1+t^2}}$

$\sin^2\frac{\theta}{2} = \frac{t^2}{1+t^2}$

$\cos\theta + \sin^2\frac{\theta}{2} = \frac{1-t^2}{1+t^2} + \frac{t^2}{1+t^2} = \frac{1}{1+t^2}$

c) $P(x) = (x^2-3x+2)\,Q(x) + (4x-7)$

$= (x-1)(x-2)\,Q(x) + (4x-7)$

$P(1) = 0 + 4 - 7$

$= -3$

d) If $P(x)$ is odd then curve passes through origin

ie x is a factor

If $(x-5)^2$ is a factor then $(x+5)^2$ must also be a factor $P(-5) = -P(5) = 0$

Hence $P(x) = kx\,(x-5)^2(x+5)^2$

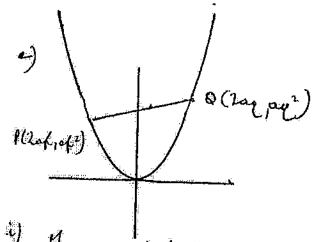
If $P(1) = 1152$

$1152 = k \times 16 \times 36$

$k = 2$

Hence $P(x) = 2x\,(x-5)^2(x+5)^2$

e)



i) M

$x = \frac{2ap+2aq}{2}$

$= a(p+q)$ — ①

$y = \frac{ap^2+aq^2}{2}$

$= \frac{a}{2}(p^2+q^2)$ — ②

ii) Gradient $OP = \frac{ap^2}{2ap} = \frac{p}{2}$

Gradient $OQ = \frac{q}{2}$

Since $\perp$ $\frac{p}{2} \cdot \frac{q}{2} = -1$

$pq + 4 = 0$

iii) From ① $\frac{x^2}{a^2} = p^2 + 2pq +$

$= p^2 + q^2 - 8$

$p^2 + q^2 = \frac{x^2}{a^2} + 8$

From ②

$p^2 + q^2 = \frac{2y}{a}$

Hence $\frac{2y}{a} = \frac{x^2}{a^2} + 8$

$y = \frac{x^2}{2a} + 4a$

Question 5

SOLUTIONS.

(a) $\dfrac{9!}{3! \times 2!} = \boxed{30240.}$

(b) Place first letter ie 6 ways. then place P, ie 4 ways. then place the other three letters ie. $5 \times 4 \times 3$ ways.

$\therefore\ 6 \times 4 \times 5 \times 4 \times 3 = \boxed{1440.}$

(c) when $n=1$. $LHS = \sin(\pi + \theta) = -\sin\theta$

$RHS = (-1)^1 \sin\theta = -\sin\theta$.

$\therefore$ true when $n=1$.

when $n=k$. $\sin(k\pi+\theta) = (-1)^k \sin\theta$.

Assuming above to be true, show that it is also true when $n=k+1$.

ie $\sin((k+1)\pi+\theta) = (-1)^{k+1}\sin\theta$.

$LHS = \sin((k+1)\pi+\theta)$

$= \sin((k\pi+\theta)+\pi)$

$= -\sin(k\pi+\theta)$ (using the identity $\sin(A+\pi) = -\sin A$)

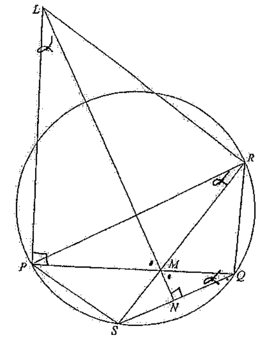
$= -1 \times (-1)^k \sin\theta$

$= (-1)^{k+1}\sin\theta$

$= RHS$.

Having assumed true for $n=k$ we proved true for $n=k+1$. Now since it is true for $n=1$ it is true for $n=2$ etc and hence true for all positive integers.

(d). (i)



(ii) $\angle LPM = \angle MNQ = 90^\circ$ (data)

$\angle LMP = \angle NMQ$ (vertically opposite)

$\therefore \triangle PML \parallel\!| \triangle NMQ$ (equiangular)

(iii) $\angle PLM = \angle NQM = \alpha$ (corresponding angles of similar triangles)

$\angle NQM = \angle SRP = \alpha$ (angles in the same segment standing on the same arc are equal).

$\therefore \angle PLM = \angle PRM = \alpha$.

$\therefore$ PMRL is a cyclic quadrilateral. (The interval PM subtends equal angles at the points L and R)

$\therefore \angle LRS = 90^\circ$ (opposite angles of a cyclic quadrilateral are supplementary)

$\therefore LR \perp RS$.