SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

Yearly Examination 2008

Mathematics
Extension

General Instructions

- Working time – 75 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators may be used.
- All necessary working should be shown in every question.
- Answer in simplest exact form unless otherwise instructed.

Total Marks – 69

- All Questions may be attempted.
- Each Question is worth 23 marks, and should be handed up in a separate examination booklet.
- Full Marks may not be awarded for careless or poorly set out work.

Examiner – *A.M.Gainford*

Question 1. (23 Marks)

(a) Find the exact value of $\sec 210°$. **1**

(b) Solve for x: **6**

(i) $|2-x| \le 3$

(ii) $x^2 - 4 < 0$

(iii) $\dfrac{1}{2x-6} < 1$

(c) Find the remainder when the polynomial $P(x) = x^3 - 5x - 1$ is divided by $x-3$. **1**

(d) Give the general solution of the equation $2\cos\left(\theta + \dfrac{\pi}{6}\right) + 1 = 0$. **3**

(e) Use the substitution $t = \tan\left(\frac{\theta}{2}\right)$ to solve the equation $2\sin\theta + \cos\theta = 0$ for $-180° \le \theta \le 180°$. (Answer correct to the nearest minute.) **3**

(f) Express $\cos 3\theta$ as an expression in powers of $\cos\theta$ only. **3**

(g) Differentiate: **6**

(i) $x\sqrt{x} - \dfrac{1}{x}$

(ii) $(1-x^2)^4(1+x^2)^4$

(iii) $\dfrac{1+x}{1-x}$

Question 2. (23 Marks)

(a) Express $\cos x - \sin x$ in the form $R\cos(x+\alpha)$, where $R > 0$, α is acute. **2**

(b) (i) Show that $2\tan\theta\sec\theta = \frac{1}{1-\sin\theta} - \frac{1}{1+\sin\theta}$. **4**

(ii) Simplify $\frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x}$.

(c) If α, β, γ are the roots of the polynomial equation $3x^3 - 6x^2 + 3x + 1 = 0$, evaluate: **4**

(i) $\alpha\beta\gamma$

(ii) $\alpha\beta + \alpha\gamma + \beta\gamma$

(iii) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

(d) Given the polynomial $P(x) = x^3 + 6x^2 - x - 30$. **3**

(i) Use the factor theorem to find a zero of the polynomial.

(ii) Express $P(x)$ as a product of three linear factors.

(e) Draw neat sketches of the following functions: **6**

(i) $y = 1 - |x-1|$ (ii) $y = 2^{-x}$ (iii) $y = 1 - \sqrt{x}$

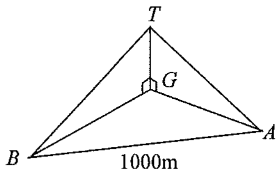
(f) Find all solutions of $\sin 2x = \sin x$ (x in radians). **4**

Question 3. (23 Marks)

(a) Solve $\left(x+\frac{1}{x}\right)^2 - 6\left(x+\frac{1}{x}\right) + 8 = 0$. **4**

(b) Calculate to the nearest minute the acute angle between the lines $x - 3y = 6$ and $2x + y = -5$. **2**

(c) A surveyor observes, from the top (T) of a vertical cliff a kilometer-post (A) on a straight flat road on the plain below on a bearing of $176°$T, and at an angle of depression of $4°$. He then observes the next kilometer-post (B) (one kilometer further along the road) on a bearing of $194.5°$T, and at an angle of depression of $3°$. **5**



(i) Copy the diagram to your answer booklet, and find $\theta = \angle AGB$.

(ii) Let $GT = h$ m, $AG = x$ m, $BG = y$ m, write an equation connecting x, y and θ.

(iii) Hence find the height of the cliff.

(d) Let $P(2ap, ap^2)$ be a point on the parabola $x^2 = 4ay$. **4**

(i) Write down the co-ordinates of the midpoint M of the interval joining P to the focus S.

(ii) Show that, as P moves along the curve, the locus of M is a parabola, and state its focal length.

(e) Find the value of the constants a and b if $x^2 + 3x - 4$ is a factor of the polynomial $P(x) = x^3 + x^2 + ax + b$. **2**

(f) The point $P(3, 3\frac{1}{2})$ divides the interval AB externally in ratio 3:1. If B is (1, 3), find the co-ordinates of A. **2**

(g) Consider the cubic curve $y = x^3 - x$. **4**

(i) Write down the equation of the tangent to the curve at the point where $x = a$, in terms of a.

(ii) Find, in terms of a, the x co-ordinate of the point where this tangent meets the curve again.

This is the end of the paper.

QUESTION 1

(a) $-\frac{\sqrt{3}}{2}$

(b)(i) $2-x \le 3$ or $-2+x \le 3$

$-x \le 1 \quad x \le 5$

$-1 \le x \le 5$

(ii) $(x+2)(x-2) < 0$

$-2 < x < 2$

(iii) $2x-6 < 4x^2-24x+36$

$4x^2-26x+42 > 0$

$2x^2-13x+21 > 0$

$(2x-7)(x-3) > 0$

$x < 3$ or $x > 3\frac{1}{2}$

(c) $P(3) = 27-15-1 = 11$

$R = 11$

(d) $\cos\left(\theta+\frac{\pi}{6}\right) = -\frac{1}{2}$

$\theta+\frac{\pi}{6} = 120, 360-120, 360+120, 2\times360-120$

$-120, -360+120, -360-120, 2\times360+120$

$\theta+\frac{\pi}{6} = 2n\pi + \frac{2\pi}{3}$ or $2n\pi - \frac{2\pi}{3}$

$\theta = 2n\pi + \frac{\pi}{2}$ or $2n\pi - \frac{5\pi}{6}$

(e) $\frac{2(2t)}{1+t^2} + \frac{1-t^2}{1+t^2} = 0$

$4t+1-t^2 = 0$

$t^2-4t-1 = 0$

$t = \frac{4 \pm \sqrt{20}}{2}$

$\tan\frac{\theta}{2} = 2+\sqrt{5}$ or $2-\sqrt{5}$

$\frac{\theta}{2} = 76.717, -13.282$

$\theta = 153°26'$ or $-26°34'$

(f) $\cos 3\theta = \cos(2\theta+\theta)$

$= \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$

$= (2\cos^2\theta-1)\cos\theta - 2\sin\theta\cos\theta\sin\theta$

$= 2\cos^3\theta - \cos\theta - 2(1-\cos^2\theta)\cos\theta$

$= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$

$= 4\cos^3\theta - 3\cos\theta$

(g)(i) $y = x^{\frac{3}{2}} - x^{-1}$

$y' = \frac{3}{2}x^{\frac{1}{2}} - -x^{-2}$

$= \frac{3}{2}\sqrt{x} + \frac{1}{x^2}$

(ii) $(1-x^2)^4 \times 4(1+x^2)^3 \times 2x +$

$(1+x^2)^4 \times 4(1-x^2)^3 \times -2x$

$8x(1-x^2)^4(1+x^2)^3 - 8x(1+x^2)^4(1-x^2)^3$

(iii) $\frac{(1-x)\times 1 - (1+x)\times -1}{(1-x)^2}$

$\frac{2}{(1-x)^2}$

YR 11 Yearly exam 2008 Continuous extension

2 (a) $1\cos x - 1\sin x = \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x\right)$

$R = \sqrt{1^2+(-1)^2}$

$= \sqrt{2}$

$= R(\cos x\cos\alpha - \sin x\sin\alpha)$

$R = \sqrt{2}$.

and $\cos\alpha = \frac{1}{\sqrt{2}}$, $\sin\alpha = \frac{1}{\sqrt{2}}$ $\Big\}$ $\alpha = 45°$.

2

So $\cos x - \sin x = \sqrt{2}\cos(x+45°)$.

(b) (i) RHS $\frac{1}{1-\sin\theta} - \frac{1}{1+\sin\theta}$

$\frac{1+\sin\theta-(1-\sin\theta)}{(1-\sin\theta)(1+\sin\theta)}$

$= \frac{2\sin\theta}{1-\sin^2\theta}$ Using $S^2+C^2=1$

$= \frac{2\sin\theta}{\cos^2\theta}$

$= \frac{2\sin\theta}{\cos\theta\cos\theta}$

$= 2\tan\theta\sec\theta = LHS$.

2

(b) (ii) $\frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x}$

$= \frac{\sin 3x \cos x - \cos 3x \sin x}{\sin x \cos x}$

$\sin 2\theta = 2\sin\theta\cos\theta$

$= \frac{\sin(3x - x)}{\frac{1}{2}\sin 2x}$

$= \frac{2\cancel{\sin 2x}}{\cancel{\sin 2x}} = 2$ (2)

(c) $3x^3 - 6x^2 + 3x + 1 = 0$

$\alpha+\beta+\gamma = -\frac{b}{a} = -\frac{-6}{3} = 2$

$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a} = \frac{3}{3} = 1$

$\alpha\beta\gamma = -\frac{d}{a} = -\frac{1}{3}$

(i) $\alpha\beta\gamma = -\frac{1}{3}$ (1)

(ii) $\alpha\beta + \alpha\gamma + \beta\gamma = 1$ (1)

(iii) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \alpha\gamma + \beta\gamma}{\alpha\beta\gamma} = \frac{1}{-\frac{1}{3}} = -3$ (2)

(d) (i) $P(x) = x^3 + 6x^2 - x - 30$

$P(2) = 8 + 24 - 2 - 30 = 32 - 32 = 0$

$(x-2)$ is a factor. (1)

Could also have: $(x+3)$ or $(x+5)$.

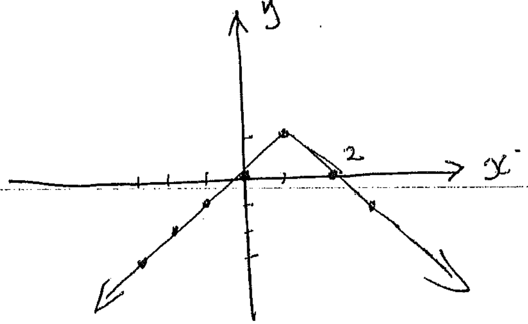
(ii)

$$\begin{array}{r|l} & x^2 + 8x + 15 \\ \hline x-2 & x^3 + 6x^2 - x - 30 \\ & x^3 - 2x^2 \\ \hline & 8x^2 - x - 30 \\ & 8x^2 - 16x \\ \hline & 15x - 30 \\ & 15x - 30 \\ \hline & 0 \end{array}$$

$P(x) = (x-2)(x^2 + 8x + 15)$

$= (x-2)(x+3)(x+5)$ (2)

(e) (i) $y = 1 - |x-1|$



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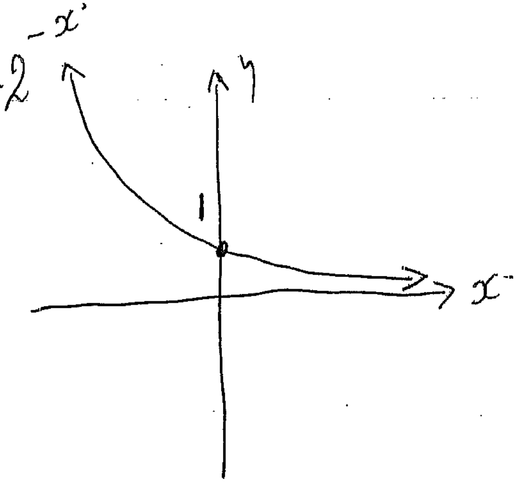
(ii) $y = 2^{-x}$



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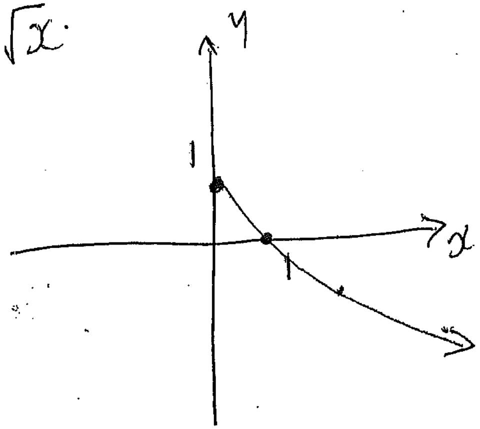
(iii) $y = 1 - \sqrt{x}$



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(f) $\sin 2x = \sin x$ radians

$2\sin x\cos x - \sin x = 0$

$\sin x(2\cos x - 1) = 0$

$\sin x = 0$

$2\cos x - 1 = 0 \Rightarrow \cos x = \frac{1}{2}$.

General solns are:

$\theta = \pi n + (-1)^n \times 0$ 2 n integer

and $\theta = 2\pi n \pm \frac{\pi}{3}$ 2 n integer.

Question 3

a) 4

$$\left(x+\frac{1}{x}\right)^2 - 6\left(x+\frac{1}{x}\right) + 8 = 0$$

let $u = x + \frac{1}{x}$

$\therefore u^2 - 6u + 8 = 0$

$(u-4)(u-2) = 0$ (1)

$u = 4$ or $u = 2$ (1)

$x + \frac{1}{x} = 4$ or $x + \frac{1}{x} = 2$ (1)

$x^2 - 4x + 1 = 0$ or $x^2 - 2x + 1 = 0$

$x = 2 \pm \sqrt{3}$ or $x = 1$ (1)

b) 2 $y = \frac{1}{3}x - 2$ and $y = -2x - 5$

where $m_1 = \frac{1}{3}$ ($\frac{1}{2}$) $m_2 = -2$ ($\frac{1}{2}$)

Acute angle $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$$= \left|\frac{\frac{1}{3} - -2}{1 + \frac{1}{3}(-2)}\right|$$

$\tan\theta = 7$ ($\frac{1}{2}$)

$\theta = 81°52'$ ($\frac{1}{2}$)

d) $P(2ap, ap^2)$. $S(0, a)$

(i) Coords of M are

$$\left(\frac{2ap+0}{2}, \frac{ap^2+a}{2}\right)$$

ie $M\left(ap, \frac{a}{2}(p^2+1)\right)$ (1)

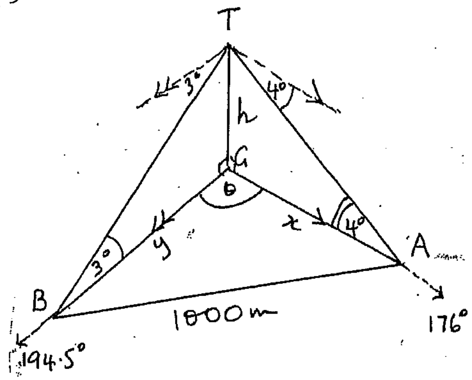
(ii) let $x = ap$, $y = \frac{a}{2}(p^2+1)$ $\frac{1}{2}$

Since $p = \frac{x}{a} \Rightarrow y = \frac{a}{2}\left(\frac{x^2}{a^2}+1\right)$

or $x^2 = 2a\left(y - \frac{a}{2}\right)$ $\frac{1}{2}$

Focal length is $\frac{a}{2}$ units $\frac{1}{2}$

(c) 5



(i) $AGB = \theta = 18.5°$ (1)

(ii) In ΔAGT, $\tan 4° = \frac{h}{x}$ ($\frac{1}{2}$)

In ΔBGT, $\tan 3° = \frac{h}{y}$ ($\frac{1}{2}$)

In ΔABG by COSINE RULE

$1000^2 = x^2 + y^2 - 2xy\cos 18.5°$ (1)

(iii) $x = h\cot 4°$, $y = h\cot 3°$

$1000^2 = h^2\cot^2 4° + h^2\cot^2 3° - 2h^2\cot 4°\cot 3°\cos 18.5°$

$$h^2 = \frac{1000^2}{\left[\cot^2 4° + \cot^2 3° - 2\cot 4°\cot 3°\cos 18.5°\right]}$$ (1)

$\therefore h \doteqdot 140$m. (1)

(e) $x^2 + 3x - 4 = (x+4)(x-1)$ $\therefore$ If a factor then $(x+4)$ and $(x-1)$ are factors

3 $\therefore$ $P(-4) = -64 + 16 - 4a + b = 0$

$P(1) = 1 + 1 + a + b = 0$

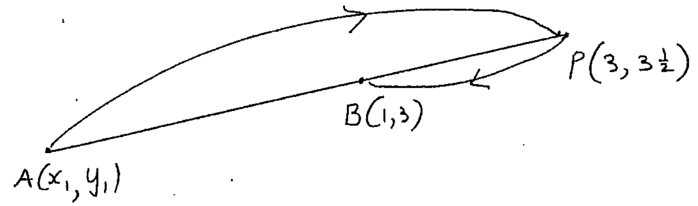
$\Rightarrow -4a + b = 48$ — (1)

$a + b = -2$ — (2)

$a = -10$

$b = 8$

(f) 2



let $AP : PB = -3 : 1$ (external division)

$$\left[\frac{1(x_1) + -3(1)}{-3+1}, \frac{1(y_1) + -3(3)}{-3+1}\right] \equiv \left[3, 3\tfrac{1}{2}\right]$$ (1)

$\frac{x_1 - 3}{-2} = 3 \Rightarrow x_1 = -3$ and $\frac{y_1 - 9}{-2} = 3\frac{1}{2} \Rightarrow y_1 = 2$

$\therefore$ Coords of $A\ (-3, 2)$ (1)

(g) 4 $y = x^3 - x$ — (C)

(i) grad. tangent $\frac{dy}{dx} = 3x^2 - 1$ and when $x = a$ grad $= 3a^2 - 1$ (1)

$\therefore$ Eq^n tangent is $y - (a^3 - a) = (3a^2-1)(x-a)$

or $y = x\left[3a^2 - 1\right] - 2a^3$ — (D) (1)

(ii) Solving (C) and (D) $\Rightarrow x^3 - x = x\left[3a^2-1\right] - 2a^3$

ie $x^3 - 3a^2x + 2a^3 = 0$ (*)

$x = a$ is a double root of this equation (since tangent at $x = a$)

$\therefore$ sum of roots of (*) is (2)

$$a + a + \gamma = -\frac{(-3a^2)}{1} = 3a^2$$

$\gamma = 3a^2 - 2a$

$\therefore$ This is the x coord. of other pt where tangent meets curve again.