SYDNEY BOYS HIGH SCHOOL

3 UNIT MATHEMATICS

Year 11 Yearly Examination

September 2000

Time Allowed: 90 minutes

Total Marks: 72

Examiner: Mr R Dowdell

INSTRUCTIONS:

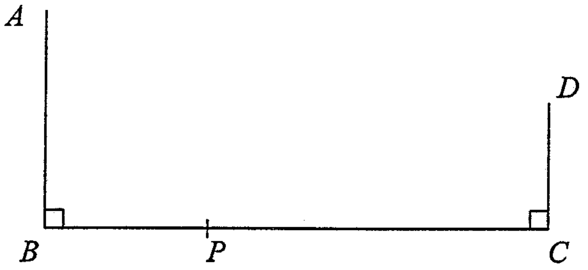
- Attempt *all* questions.
- *All* questions are of equal value.
- All necessary working should be shown in every question. Full marks may not be awarded if work is careless or badly arranged.
- Return your answers in 4 booklets. Each booklet must show your name.
- If required, additional Writing Booklets may be obtained from the Examination Supervisor upon request.

Question 1: (18 marks)

Marks

(a) Find the point $P(x, y)$ which divides the interval joining $X(-2, 7)$ and $Y(3, 17)$ internally in the ratio 3:2. 2

(b) 3



$AB = 9$ cm, $BC = 8$cm, $CD =7$ cm

If $AP = PD$, calculate the length of BP.

(c) If $x = 2 + \sin\alpha$ and $y = 4 + 3\cos\alpha$, find a relationship between x and y which does not involve α. 2

(d) For $P(x) = 2x^3 - 7x^2 - 7x + 30$, 4

(i) evaluate $P(3)$;

(ii) evaluate $P(-2)$;

(iii) find all the zeroes of $P(x)$.

(e) If α, β and γ are the roots of $7x^3 + 5x^2 - 11x + 2$, evaluate 7

(i) $\alpha + \beta + \gamma$;

(ii) $\alpha\beta + \alpha\gamma + \beta\gamma$;

(iii) $\alpha\beta\gamma$;

(iv) $\alpha^2 + \beta^2 + \gamma^2$;

(v) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$;

(vi) $(\alpha + 1)(\beta + 1)(\gamma + 1)$.

Question 2: (18 marks) START A NEW BOOKLET

Marks

(a) Find the acute angle (to the nearest degree) between the lines $y = 5x - 4$ and $y = -x + 3$. **2**

(b) If $\tan A$ and $\tan B$ are the roots of the equation $3x^2 - 5x - 1 = 0$, find the value of $\tan(A + B)$. **2**

(c) Find the general solution of the equation $\sin 2x = 2\cos^2 x$ **4**

(d) A monic cubic polynomial leaves a remainder of $x + 8$ when divided by $x^2 + 4$ and when divided by x leaves a remainder of -4. **4**

Find the polynomial in the form $\overset{1}{\cancel{a}}x^3 + bx^2 + cx + d$.

(e) Solve $\dfrac{x-3}{x^2-x} \geq -2$. **6**

Graph your solution on a number line.

$$\frac{x-3}{x^2-x} + 2 \geq 0$$

$$\frac{x-3+2(x^2-x)}{x^2-x}$$

$$\frac{x-3+2x^2-2x}{x^2-x} \geq 0$$

$$\frac{2x^2-x-3}{\cancel{x^2}-x} \geq \times (x^2-x)^2$$

$$(2x^2-x-3)(x^2-x) \geq 0$$

$$2x^4-2x^3-x^3+x^2-3x^2+3 \geq 0$$

$$2x^4-3x^3+2x^2+3 \geq 0$$

Question 3: (18 marks) START A NEW BOOKLET Marks

(a) Show that $\sin 8\theta \sin 2\theta \equiv \sin^2 5\theta - \sin^2 3\theta$. 4

(b) Solve the equation $x^2 + 2x - 4 + \frac{3}{x^2 + 2x} = 0$ 4

(c) If $\cos A = \frac{7}{9}$ and $\sin B = \frac{1}{3}$, $0 \le A \le \frac{\pi}{2}$ and $0 \le B \le \frac{\pi}{2}$, 4

(i) show, without a calculator, that $A = 2B$;

(ii) find the value of $\cos(A + B)$ in simplest surd form.

(d) The elevation of a hill at a place *P* due east of it is 48°, and at a place *Q* due south of *P* the elevation is 30°. If the distance from *P* to *Q* is 500 metres, find the height of the hill (correct to 3 significant figures). 6

Question 4: (18 marks) START A NEW BOOKLET Marks

(a) If $f(x) = \dfrac{\sin(x-\frac{\pi}{4}) + \sin(x+\frac{\pi}{4})}{\cos(x-\frac{\pi}{4}) - \cos(x+\frac{\pi}{4})}$, 6

(i) comment on the value of $f(0)$;

(ii) simplify the expression for $f(x)$;

(iii) sketch $y = f(x)$ for $-2\pi \leq x \leq 2\pi$.

(b) (i) Simplify the square of $\dfrac{\sqrt{6}+\sqrt{2}}{4}$ and hence state the positive square root of $\dfrac{2+\sqrt{3}}{4}$. 6

(ii) Given that θ is acute and that $\cos\theta = \dfrac{\sqrt{6}-\sqrt{2}}{4}$, find the exact value of $\sin\theta$.

(iii) Hence, or otherwise, evaluate $\sin 2\theta$ and deduce the exact value(s) of θ, expressing your answer in radians.

(c) (i) Show that the distance from (p, q) to the line $y = x$ is given by 6

$$d = \frac{|p-q|}{\sqrt{2}}.$$

(ii) A point $P(x, y)$ moves such that its distance from the line $y = x$ is equal to its distance from the point $A(-2, 2)$.

(α) Show that the equation of the locus of P is $x^2 + 8x + y^2 - 8y + 2xy + 16 = 0$.

(β) What type of curve does this locus represent?

END OF PAPER

SBHS - SEPT 2000 PRELIMINARY

2000 Yrly Exam Bank

1. a). $X(\overset{x_1}{-2},\overset{y_1}{7})$ & $Y(\overset{x_2}{3},\overset{y_2}{17})$ $P(x,y)$ $\overset{m}{3}:\overset{n}{2}$

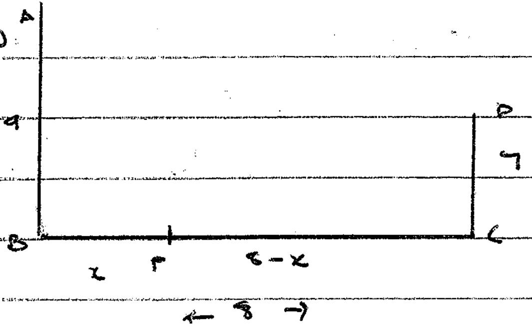
$x = \frac{mx_2 + nx_1}{m+n}$ $y = \frac{my_2 + ny_1}{m+n}$

$= \frac{9 + -4}{5}$ $= \frac{51 + 14}{5}$

$= 1$ $= 13$

$\therefore P(1, 13)$

b).



Let BP = x ,

$AP^2 = 9^2 + x^2$ $DP^2 = (8-x)^2 + 7^2$

$\therefore 9^2 + x^2 = (8-x)^2 + 7^2$

$81 + x^2 = 64 - 16x + x^2 \cancel{49}$

$= 32 - 16x = 0$

$x = 2$

BP = 2 , PC = 6.

c) $x = 2 + \sin a$ $y = 4 + 3\cos a$

$x - 2 = \sin a$ $\frac{y-4}{3} = \cancel{3}\cos a$

① $(x-2)^2 = \sin^2 a$ $\frac{(y-4)^2}{9} = \cancel{3}\cos^2 a$ ②.

① + ② $\sin^2 a + \cos^2 a = (x-2)^2 + \frac{(y-4)^2}{9}$

$9 = 9(x^2 - 4x + 4) + y^2 - 8y + 16$

$= 9x^2 - 36x + 36 + y^2 - 8y + 16$

$= 9x^2 - 36x + y^2 - 8y + 42 = 0.$

d) i) $P(x) = 2x^3 - 7x^2 - 7x + 30$

$P(3) = 54 - 63 - 21 + 30$

$= 0.$

ii) $P(-2) = -16 - 28 + 14 + 30$

$= 0.$

iii). zeros at $x = 3$ and $x = -2$.

a) $7x^3 + 5x^2 - 11x + 2$

i) $\alpha + \beta + \gamma = \frac{-b}{a}$

$= \frac{-5}{7}$

ii) $\alpha\beta + \beta\gamma + \alpha\gamma = \frac{c}{a}$

$= \frac{-11}{7}$

iii) $\alpha\beta\gamma = \frac{-d}{a}$

$= \frac{-2}{7}$

iv) $\alpha^2 + \beta^2 + \gamma^2 =$

$= (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\beta)$

$= (-\frac{5}{7})^2 - 2(-\frac{11}{7})$

$= 3\frac{32}{49}.$

$(\alpha+\beta+\gamma)(\alpha+\beta+\gamma) = \alpha^2 - \alpha\beta + \alpha\gamma + \alpha\beta + \beta^2 + \beta\gamma + \alpha\gamma + \beta\gamma + \gamma^2$

$= \alpha^2 + \beta^2 + \gamma^2 + 2\alpha\gamma + 2\alpha\beta + 2\beta\gamma$

$= \alpha^2 + \beta\gamma^2 + 2(\alpha\gamma + \beta\gamma + \alpha\beta)$

v) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

$= \frac{\alpha+\beta+\gamma}{\alpha\beta\gamma}$

$= \frac{\frac{c}{a}}{\frac{-d}{a}} = \frac{-11}{7} \times \frac{7}{-2} = 11/2.$

vi). $(\alpha+1)(\beta+1)(\gamma+1)$

$= (\alpha\beta + \alpha + \beta + 1)(\gamma + 1)$

$= \alpha\beta\gamma + \alpha\gamma + \beta\gamma + \gamma + \alpha\beta + \alpha + \beta + 1.$

$= \alpha\beta\gamma + \alpha\gamma + \beta\gamma + \alpha\beta + \alpha + \beta + \gamma + 1$

$= \frac{-d}{a} + \frac{c}{a} - \frac{b}{a}$

$= \frac{-2}{7} - \frac{11}{7} \mp \frac{5}{7} = -\frac{18}{7}$

Question 2

a). $y = 5x - 4$ and $y = -x + 3$

$m_1 = 5 \qquad m_2 = -1$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\frac{5+1}{1-5}\right|$

$= 1$

$\theta = 45$

b). $3x^2 - 5x - 1 = 0$ roots at $\tan A$ and $\tan B$.

$\alpha + \beta = \frac{-b}{a} = 5/3 \qquad \alpha\beta = \tan A \tan B$

$\tan A + \tan B = 5/3. \qquad = \frac{c}{a}$

$= -1/3$

$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$= \frac{\frac{5}{3}}{1 - -1/3}$

$= 1\,1/4.$

c). $\sin 2x = 2\cos^2 x$

$2\cos^2 x$

$2\cos x \sin x =$

$\frac{2\cos x \sin x}{\sin x \sin x} = \frac{2\cos x \cos x}{\sin x \sin x}$

$2\cot x = 2\cot^2 x$

$2\cot x^2 - 2\cot x = 0.$

$2\cot^2 x(\cot x - 1) = 0.$

$2\cot x = 0$ <u>or</u> $\cot x = 1$

$\cot x = 0$ $\frac{1}{\tan x} = 1$

$\frac{1}{\tan x} = 0.$ $\tan x = 1$

$x = 45, 225$, etc.

$x = 90, 270$, etc.

$\theta = \underline{180n + 90}$ and $\underline{\overset{\theta}{180n} + 45.}$

$x \neq 0, 1$ $x < \frac{1}{2}$

d). $\frac{x-3}{x^2-x} \geq -2$

$(x-3)(x^2-x) \geq -2(x^2-x)^2$

$x^3 - x^2 - 3x^2 + 3x \geq -2(x^4 - 2x^3 + x^2)$

$x^3 - 4x^2 + 3x \geq -2(x^4 - 2x^3 + x^2).$

$x(x^2-4x+3) \geq -2x^2(x^2-2x+1)$

$x(x^2 - 4x + 3) + 2x^2(x^2 - 2x + 1) \geq 0.$

$x[x^2 - 4x + 3 + 2x(x^2 - 2x + 1)] \geq 0.$

$x \geq 0.$ $x^2 - 4x + 3 + 2x^3 - 4x^2 + 2x \geq 0.$

~~But since $x \geq 0$~~

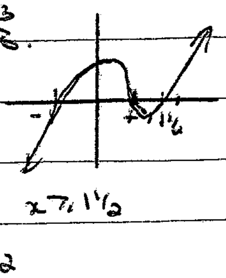
since $x \neq 0$ $2x^3 - 3x^2 - 2x + 3 \geq 0.$

$x > 0.$ $P(1) = 0$, $P(-1) = 0.$

$$\begin{array}{r} 2x^2 - x - 3 \\ (x-1)\overline{)2x^3 - 3x^2 - 2x + 3} \\ \underline{2x^3 - 2x^2} \\ -x^2 - 2x \\ \underline{-x^2 + x} \\ -3x + 3 \\ \underline{-3x + 3} \\ 0. \end{array}$$

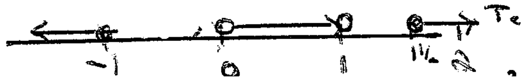
$P(x) = (x-1)(2x^2 - x - 3)$

$(x-1)(x+1)(2x-3) \geq 0$



$-1 \leq x < 1$ and $x \geq 1\frac{1}{2}$

$\therefore$ $\underline{\underline{\overset{x \leq -1}{}\ 0 < x < 1,\ x \geq 1\frac{1}{2}}}$ Test $x = 2$ $= -\frac{1}{2} \geq -2$



Test $x = 0$ = No Sol.

Test $x = \frac{1}{2}$ =

Question 2.

d). $P(x) = (x^2+4)Q(x) + x + 8$

$= xT(x) - 4$

$P(0) = -4$

$\therefore d = -4$ $\quad a = 1$ (since monic)

$$x^2+4 \overline{\smash{)}\, x^3 + bx^2 + cx - 4} \quad \text{quotient } x + b$$

$x^2 + 4x$

$bx^2 + x(c-4)$

$bx^2 + 4b$

$x(c-4) - 4b - 4$

$c - 4 = 1$

$c = 5$.

$-(4b - 4) = 8$.

$-4b = +4$.

$b = -1$

$\therefore \underline{p(x) = x^3 - x^2 + 5x - 4.}$

2e) Alternatively

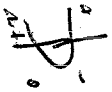
$\frac{x-3}{x^2-x} \geq -2 \qquad x \neq 0, 1$

$\frac{x - 3 + 2(x^2-x)}{x^2-x} \geq 0$

$\frac{x - 3 + 2x^2 - 2x}{x^2-x} \geq 0$

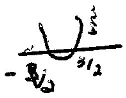
$\frac{2x^2 - x - 3}{x^2-x} \geq 0$

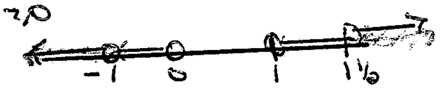
$\therefore 2x^2 - x - 3 \geq 0 \cap x^2 - x > 0$.



$(2x - 3)(2x+2) \geq 0 \qquad x(x-1) \geq 0$

$(2x-3)(x+1) \geq 0$



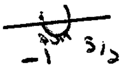


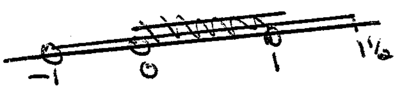
$\therefore x \leq -1 \qquad x \geq 1\frac{1}{2}$

AND $2x^2 - x - 3 \leq 0 \cap x^2 - x < 0$.

$x(x-1) < 0$.

$(2x-3)(x+1) \leq 0$





$\therefore 0 \leq x \leq 1$

$0 < x < 1$.

$RHS = (\sin 5\theta + \sin 3\theta)(\sin 5\theta - \sin 3\theta)$

3.a). $\downarrow \quad \downarrow$

$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

$= (2\sin 4\theta \cos\theta)(2\cos 4\theta \sin\theta)$

$= (2\sin 4\theta \cos 4\theta)(2\cos\theta \sin\theta)$

$= \sin 8\theta \sin 2\theta$

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$\sin A + \sin B = \sin\alpha\cos\beta + \cancel{\cos\alpha\sin\beta} + \sin\alpha\cos\beta - \cancel{\cos\alpha\sin\beta}$

$= 2\sin\alpha\cos\beta$

$= 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$A = \alpha + \beta$

$B = \alpha - \beta$

$A + B = 2\alpha$

$\alpha = \frac{A+B}{2}$

$A = \frac{A+B}{2} + \beta$

$2A = A + B + 2\beta$

$2\beta = A - B$

$\beta = \frac{A-B}{2}$

$\sin A - \sin B$

$= \cancel{\sin\alpha\cos\beta} + \cos\alpha\sin\beta - \cancel{\sin\alpha\cos\beta} + \cos\alpha\sin\beta$

$= 2\cos\alpha\sin\beta$

$= 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$.

b)

$x^2 + 2x - 4 + \frac{3}{x^2+2x} = 0$.

Let $u = x^2 + 2x$

$u - 4 + \frac{3}{u} = 0$.

$u^2 - 4u + 3 = 0$.

$(u-1)(u-3) = 0$.

$u = 1$

$x^2 + 2x = 1$

$x^2 + 2x - 1 = 0$

$x = \frac{-2 \pm \sqrt{4+4}}{2}$.

$u = 3$

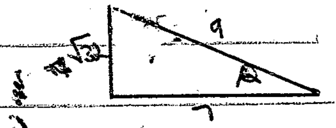
$x^2 + 2x = 3$

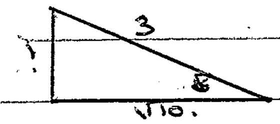
$x^2 - 2x - 3 = 0$.

$(x-3)(x+1) = 0$.

$x = 3, x = -1$

$\cos A = \frac{7}{9}$ $\sin B = {}^1/_3$





i) $\cos A = \cos 2B$

c) ii $= 1 - 2\sin^2 B$

LHS $= \frac{7}{9}$ RHS $= 1 - 2(\frac{1}{3})^2$

$= 1 - \frac{2}{9}$ ∴ A = 2B.

$= \frac{7}{9}$

ii). $\cos(A+B) \overset{LHS}{=} \cos A \cos B - \sin A \sin B$

$= \frac{7}{9} \times \frac{3}{\sqrt{3}} - \frac{\sqrt{32}}{9} \times \frac{B}{3}$

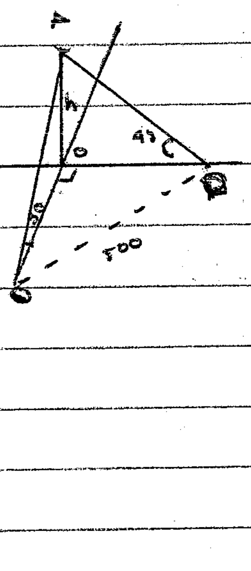
$= \frac{21}{9\sqrt{3}} - \frac{\sqrt{32}}{9 \times \sqrt{3}}$

$= \frac{21 - 2\sqrt{8}}{9\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{21\sqrt{3} - 2\sqrt{24}}{27}$

$= \frac{21\sqrt{3} - 4\sqrt{6}}{27}$

d)



~~hr~~ $\tan 48 = \frac{h}{OP}$

$h = $ ~~tan~~ $OP \tan 48$

$OP = h \cot 48$

~~h~~ $\tan 30 = \frac{h}{OQ}$

$h = OQ \tan 30$.

$OQ = h \cot 30$

In Δ OPQ

$OP^2 + OQ^2 = OP^2$ (Pyth)

$h^2 \cot^2 48 + h^2 \cot^2 30 = 500^2$

$h^2(\cot^2 48 + \cot^2 30) = 500^2$

$h^2 = \frac{500^2}{\cot^2 48 + \cot^2 30}$

$h = \sqrt{\frac{500^2}{\cot^2 48 + \cot^2 30}}$

$\doteqdot 256.13$

Question

$$f(x) = \frac{\sin(x-\frac{\pi}{4}) + \sin(x+\frac{\pi}{4})}{\cos(x-\frac{\pi}{4}) - \cos(x+\frac{\pi}{4})}$$

i) $f(0) = \dfrac{\sin\frac{-\pi}{4} + \sin\frac{\pi}{4}}{\cos -\frac{\pi}{4} - \cos\frac{\pi}{4}}$

$=$ Undefined

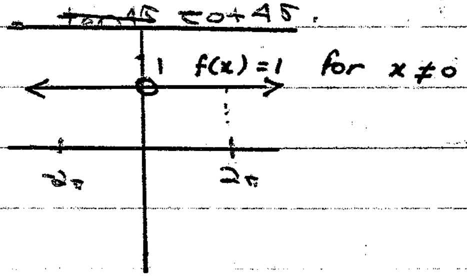
ii) $f(x) = \dfrac{\sin x\cos 45 - \cancel{\sin 45\cos x} + \sin x\cos 45 + \cancel{\sin 45\cos x}}{\cancel{\cos x\cos 45} + \sin 45\sin x - [\cancel{\cos x\cos 45} - \sin 45\sin x]}$

$= \dfrac{\cancel{2}\sin\cancel{x}\cos 45}{\cancel{2}\sin 45\cancel{\sin x}}$

$= \dfrac{\cos 45}{\sin 45}$

$=$ ~~tan 45~~ $\cot 45$.

iii) ~~[illegible]~~



b). i) $\left(\dfrac{\sqrt{6}+\sqrt{2}}{4}\right)^2 = \dfrac{6+2\sqrt{12}+2}{16}$

$= \dfrac{8+2\sqrt{12}}{16}$

$= \dfrac{8+4\sqrt{3}}{16}$

$= \dfrac{2+\sqrt{3}}{4}$

Positive root of $\dfrac{2+\sqrt{3}}{4}$

$= \dfrac{\sqrt{6}+\sqrt{2}}{4}$.

$\sqrt{\dfrac{2+\sqrt{3}}{2}}$

~~$= (a+b)^2 = 2+\sqrt{3}$.~~

~~$a^2+2ab+b^2 = 2+\sqrt{3}$~~

~~$a=1 \therefore 1+2b+b^2 = 2+\sqrt{3}$~~

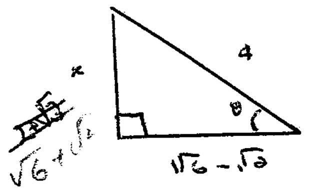
~~$2b = \sqrt{3}$~~

~~$b = 1$~~

~~$(1+\frac{\sqrt{3}}{2})^2$~~

~~$(\frac{2+\sqrt{3}}{2})^2$~~

~~$4+$~~



$4^2 = (\sqrt6-\sqrt2)^2 + x^2$

$= 6 - 2\sqrt{12} + 2 + x^2$

$= 8 - 2\sqrt{12} + x^2$

$= x^2 - 8 - 2\sqrt{12} = 0.$

$(a+\sqrt b)^2 = 8 + 2\sqrt{12}$

$\Rightarrow a^2 + 2a\sqrt b + b = 8 + 2\sqrt{12}.$ $4\sqrt3$

$a^2 + b = 8 \quad 2a\sqrt b = 2\sqrt{12}.$

$x^2 = 8 + 2\sqrt{12}$

$x = \sqrt{8 + 2\sqrt{12}}$

$\frac{x^2}{16} = \frac{8+2\sqrt{12}}{16} = \frac{2+\sqrt3}{4}$

$\frac{x}{16} = \frac{(\sqrt6+\sqrt2)}{16}.$

$x = \sqrt6 + \sqrt2$

$\therefore \sin\theta = \frac{O}{H}$

$= \frac{\sqrt6+\sqrt2}{4}.$

iii). $\sin 2\theta$

$= 2\sin\theta\cos\theta$

$= 2\left(\frac{\sqrt6+\sqrt2}{4}\right)\left(\frac{\sqrt6-\sqrt2}{4}\right)$

$\sin 2\theta = \frac{1}{2}. \quad (0 \le x < \frac{\pi}{2})$

$\therefore 2\theta = 30, 150$

$\theta = 15, 75$

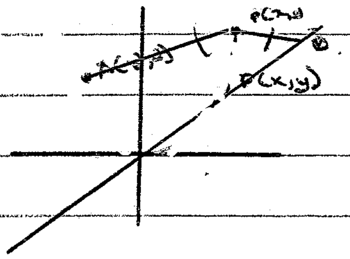
$= \frac{\pi}{12}, \frac{5\pi}{12}.$

c). i). $d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$ $y = x$ $x - y = 0.$

$= \frac{|p(1) + q(1)|}{\sqrt{1+1}}$ $a = 1$ $b = 1$

$= \frac{|p+q|}{\sqrt2}$

ii).



$AP = PQ$

$\therefore PQ = \frac{|p-q|}{\sqrt2}$ (proved above)

$\therefore \frac{x-y}{\sqrt2} = \sqrt{(x+2)^2 + (y-2)^2}$

$x - y = \sqrt2 \times \sqrt{(x+2)^2 + (y-2)^2}$

$(x-y)^2 = 2[(x+2)^2 + (y-2)^2]$

$x^2 - 2xy + y^2 = 2[x^2 + 4x + 4 + y^2 - 4y + 4$

$x^2 - 2xy + y^2 = 2x^2 + 8x + 16 + 2y^2 - 8y$

$\therefore x^2 + 8x + y^2 + 16 + 2xy = 0.$

iii). A slanted parabola.