SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2005
HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK #1

Mathematics Extension 2

General Instructions

- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- Each question is to be returned in a separate bundle.
- All necessary working should be shown in every question.

Total Marks – 85

- Attempt questions 1 – 3
- All questions are not of equal value.

Examiner: *C. Kourtesis*

Question 1. (Start a new answer sheet.) (31 marks)

Marks

(a) Given that $w=\sqrt{3}+i$, express the following in the form $a+ib$ where a and b are real. **4**

(i) $-iw$

(ii) w^2

(ii) w^{-1}

(b) If $z=1-i$ find: **4**

(i) $|z|$ and $\arg z$

(ii) z^8 in exact form

(c) Consider the equation **3**

$$z^2+kz+(4-i)=0$$

Find the complex number k given that $2i$ is a root of the equation.

(d) If $z=x+iy$ prove that **3**

$$z+\frac{|z|^2}{z}=2\,\mathrm{Re}(z)$$

(e) Sketch the locus of z satisfying **4**

(i) $|z+2i|=2$

(ii) $\mathrm{Re}(z^2)=0$

(f) (i) Plot on the Argand diagram all complex numbers that are roots of $z^5=1$. **4**

(ii) Express z^5-1 as a product of real linear and quadratic factors.

(g) (i) By solving the equation $z^3+1=0$ find the three cube roots of -1. 9

(ii) Let ω be a cube root of -1, where ω is not real.
Show that $\omega^2+1=\omega$

(iii) Hence simplify $(1-\omega)^{12}$.

(iv) Find a quadratic equation with real coefficients whose roots are ω^2 and $-\omega$.

Question 2. (Start a new answer sheet.) (31 marks)

Marks

(a) Given that $\text{cis}\,\theta = \cos\theta + i\sin\theta$ find in exact form 3

$$\text{cis}\frac{\pi}{12}\,\text{cis}\frac{\pi}{6}$$

(b) The equation $x^3 + Ax + B = 0$ (A, B real) has three real roots α, β and γ. 9

(i) Evaluate $\alpha^{-1} + \beta^{-1} + \gamma^{-1}$ and $\alpha^2 + \beta^2 + \gamma^2$ in terms of A and B.

(ii) Prove that $A < 0$.

(iii) Find the cubic polynomial whose roots are α^2, β^2 and γ^2.

(c) It is given that $z = 1 + i$ is a zero of $P(z) = z^3 + pz^2 + qz + 6$ where p and q are real numbers. 4

(i) Explain why $\bar{z}$ is also a zero of $P(z)$. (State the theorem.)

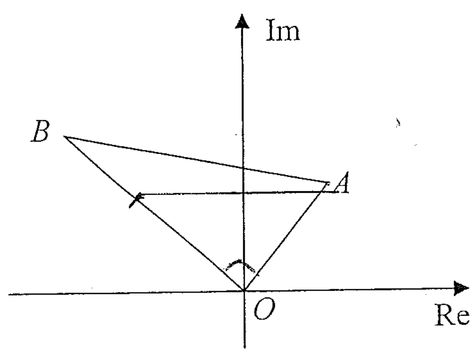
(ii) Find the values of p and q.

(d) Find the number of ways in which six women and six men can be arranged in three sets of four for tennis if: 5

(i) there are no restrictions.

(ii) each man has a woman as a partner.

(e) In the Argand diagram the points O, A and B are the vertices of a triangle with $\angle AOB = 90°$ and $\dfrac{OB}{OA} = 2$. 6



The vertices A and B correspond to the complex numbers z_1 and z_2 respectively.

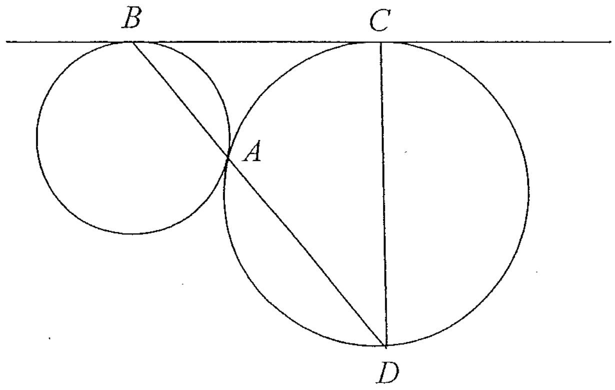
Show that:

(i) $2z_1 + iz_2 = 0$

(ii) the equation of the circle with AB as diameter and passing through O is given by

$$\left|z - z_1\left(\tfrac{1}{2} + i\right)\right| = \tfrac{\sqrt{5}}{2}\left|z_1\right|.$$

(f) **4**



The two circles touch at A and a common external tangent touches them at B and C. BA produced meets the larger circle at D.

Prove that CD is a diameter.

Question 3. (Start a new answer sheet.) (23 marks)

Marks

(a) In how many ways can three different trophies be awarded to five golfers if a golfer may receive at most two trophies? **3**

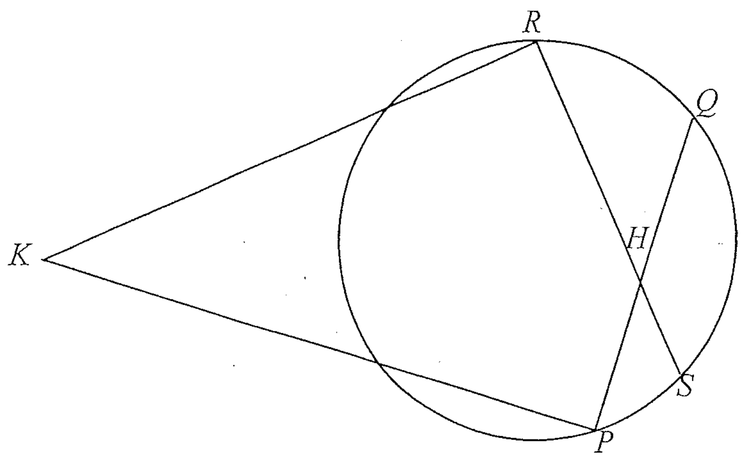
(b) Sketch the region in the Argand diagram consisting of all points z satisfying **4**

$$|\arg z| < \frac{\pi}{4} \text{ and } z + \overline{z} < 4 \text{ and } |z| > 2 .$$

(c) (i) Prove that $(1 + i\tan\theta)^n + (1 - i\tan\theta)^n = \dfrac{2\cos n\theta}{\cos^n\theta}$, where n is a positive integer. **6**

(ii) Hence or otherwise show that $(1+z)^4 + (1-z)^4 = 0$ has roots $\pm i\tan\frac{\pi}{8}$ and $\pm i\tan\frac{3\pi}{8}$

(d)



In the diagram above PQ and RS are two chords intersecting at H, and $\angle KPQ = \angle KRS = 90°$. **6**

(i) Copy the diagram onto your answer sheet, indicating the above information.

(ii) Prove that (α) $\angle PKH = \angle PQS$.

(β) KH produced is perpendicular to QS.

(e) If α is a real root of the equation $x^3 + ux + v = 0$ prove that the other two roots are real if $4u + 3\alpha^2 \leq 0$. **4**

End of paper.

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Mathematics Extension 2

Sample Solutions

Question	Marker
1	PSP
2	DH
3	PRB

Question 1

(a) $w = \sqrt{3} + i$

(i) $-iw = -i\left(\sqrt{3}+i\right) = 1 - i\sqrt{3}$

(ii) $w^2 = \left(\sqrt{3}+i\right)^2 = 2 + i2\sqrt{3}$

(iii) $w^{-1} = \dfrac{\overline{w}}{|w|^2} = \dfrac{\sqrt{3}-i}{4} = \dfrac{\sqrt{3}}{4} - i\left(\dfrac{1}{4}\right)$

(b) $z = 1 - i$

(i) $|z| = \sqrt{2}, \arg(z) = -\dfrac{\pi}{4}$

(ii) $z^8 = \left(\sqrt{2}\operatorname{cis}\left(-\dfrac{\pi}{4}\right)\right)^8 = 16\operatorname{cis}\left(-\dfrac{8\pi}{4}\right) = 16\operatorname{cis}(-2\pi) = 16$

(c) $p(z) = z^2 + kz + (4 - i)$

$p(2i) = 0 \Rightarrow (2i)^2 + k(2i) + 4 - i = 0$

$\therefore -4 + 2ki + 4 - i = 0$

$\therefore 2ki = i$

$\therefore k = \dfrac{1}{2}$

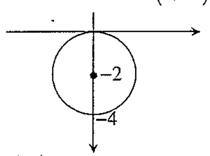
(d) $z = x + iy$

$\because \dfrac{1}{z} = \dfrac{\overline{z}}{|z|^2}$

$\therefore z + \dfrac{|z|^2}{z} = z + \overline{z} = 2\operatorname{Re} z$

(e) (i) $x^2 + (y+2)^2 = 4$

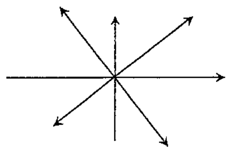
A circle with centre $(0,-2)$ ie $-2i$ and radius 2



(ii) $\operatorname{Re}(z^2) = x^2 - y^2 = 0$

$\therefore x^2 = y^2$

$\therefore y = \pm x$



(f) (i) $z^5 = 1\times \text{cis}(0)$

$= \text{cis}(0+2k\pi), k\in\mathbb{Z}$

$= \text{cis}(2k\pi)$

$z = [\text{cis}(2k\pi)]^{1/5}$

$= \text{cis}\left(\frac{2k\pi}{5}\right)$ (deMoivre's Theorem)

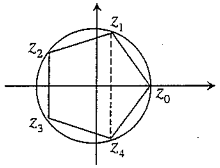
$k=0$: $z_0 = \text{cis}(0) = 1$

$k=1$: $z_1 = \text{cis}\left(\frac{2\pi}{5}\right)$

$k=2$: $z_2 = \text{cis}\left(\frac{4\pi}{5}\right)$ $|z_k| = 1$

$k=-1$: $z_3 = \text{cis}\left(-\frac{2\pi}{5}\right)$

$k=-2$: $z_4 = \text{cis}\left(-\frac{4\pi}{5}\right)$



The 5 roots must form a regular pentagon inscribed in a unit circle.
As well:
z_1 and z_4 are conjugates
z_2 and z_3 are conjugates

(ii) $(z-\alpha)(z-\bar{\alpha}) = z^2 - 2\text{Re}(\alpha)z + |\alpha|^2$

$z^5 - 1 = (z-z_0)(z-z_1)(z-z_2)(z-z_3)(z-z_4)$

$= (z-1)(z-z_1)(z-\bar{z}_1)(z-z_2)(z-\bar{z}_2)$

$= (z-1)\left(z^2 - (2\text{Re}\,z_1)z + |z_1|^2\right)\left(z^2 - (2\text{Re}\,z_2)z + |z_2|^2\right)z$

$= (z-1)\left(z^2 - 2z\cos\frac{2\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{5} + 1\right)$

(g) (i) $z^3 = -1$

$= 1\times\text{cis}(\pi)$

$= \text{cis}(\pi + 2k\pi), k\in\mathbb{Z}$

$= \text{cis}(2k+1)\pi$

$z = [\text{cis}(2k+1)\pi]^{1/3}$

$= \text{cis}(2k+1)\frac{\pi}{3}$ (deMoivre's Theorem)

$k=0$: $z = \text{cis}\frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

$k=1$: $z = \text{cis}\frac{3\pi}{3} = -1$

$k=-1$: $\text{cis}\left(-\frac{\pi}{3}\right) = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

(ii) $z^3 + 1 = (z+1)(z^2 - z + 1)$

$\omega^3 = -1, \omega \neq -1$

$\therefore \omega^3 + 1 = (\omega+1)(\omega^2 - \omega + 1) = 0$

$\therefore \omega^2 - \omega + 1 = 0$ $(\because \omega \neq -1)$

$\therefore \omega^2 + 1 = \omega$

(iii) $(1-\omega)^{12} = (-\omega^2)^{12}$ (from (ii))

$= (\omega^3)^8$

$= (-1)^8$

$= 1$

(iv) $(z-\omega^2)(z+\omega) = 0$

$z^2 + (\omega - \omega^2)z - \omega^3 = 0$

$\therefore z^2 + (1)z - (-1) = 0$ (from (ii))

$\therefore z^2 + z + 1 = 0$

OR more simply since $z^3 + 1 = (z+1)(z^2 - z + 1)$
and the three roots of -1 are so that $z^2 - z + 1 = 0$ must have roots $\omega, -\omega^2$.
So let $y = -z$ and $y^2 + y + 1 = 0$ MUST have roots $-\omega, \omega^2$.

Question 2

(a) Method 1: [3]

$$\text{cis}\frac{\pi}{12}\,\text{cis}\frac{\pi}{6} = \text{cis}\left(\frac{\pi}{12}+\frac{\pi}{6}\right), \text{ by de Moivre's theorem}$$
$$= \text{cis}\frac{\pi}{4},$$
$$= \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}.$$

Method 2:

$$\text{cis}\frac{\pi}{12}\,\text{cis}\frac{\pi}{6} = \left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right),$$
$$= \cos\frac{\pi}{12}\cos\frac{\pi}{6} - \sin\frac{\pi}{12}\sin\frac{\pi}{6} + i\left(\sin\frac{\pi}{12}\cos\frac{\pi}{6} + \cos\frac{\pi}{12}\sin\frac{\pi}{6}\right),$$
$$= \cos\frac{\pi}{4} + i\sin\frac{\pi}{4},$$
$$= \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}.$$

(b) i. [3]

$$\alpha+\beta+\gamma = 0,$$
$$\alpha\beta+\alpha\gamma+\beta\gamma = A,$$
$$\alpha\beta\gamma = -B.$$

Now, $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}$,
$$= -\frac{A}{B}.$$

Also, $(\alpha+\beta+\gamma)^2 = \alpha^2+\beta^2+\gamma^2+2(\alpha\beta+\alpha\gamma+\beta\gamma)$.

$\therefore\ \alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$,
$$= 0-2A,$$
$$= -2A.$$

ii. Method 1: [2]

$A = -\frac{1}{2}(\alpha^2+\beta^2+\gamma^2)$.

But $\alpha^2+\beta^2+\gamma^2 > 0$ if $\alpha\neq\beta\neq\gamma$.

$\therefore\ A<0$.

Method 2:

$P'(x) = 3x^2 + A$.

If $A>0$ then $P(x)$ is monotonic increasing so there can be only one real root. But there are 3 real roots so $A<0$.

iii. Put $X = x^2$. [4]

$\therefore\ x = \sqrt{X}$.

$$X\sqrt{X} + A\sqrt{X} + B = 0,$$
$$\sqrt{X}(X+A) = -B,$$
$$X(X^2+2XA+A^2) = B^2.$$

New equation is $x^3 + 2Ax^2 + A^2x - B^2 = 0$.

(c) i. If $a+ib$ is a complex zero of the polynomial $P(x)$ of degree $n\geq 1$ with real coefficients, then $a-ib$ is also a zero of $P(x)$. [1]

ii. Let the roots be α, $1+i$, $1-i$, then [3]

$$z^3+pz^2+qz+6 = (z-\alpha)(z-1-i)(z-1+i),$$
$$= (z-\alpha)(z^2-2z+2),$$
$$= z^3-(\alpha+2)z^2+(2\alpha+2)z-2\alpha.$$

Equating coefficients gives $\alpha = -3$.
$$p = -(-3+2),$$
$$= 1.$$
$$q = -6+2,$$
$$= -4.$$

(d) i. There are ${}^{12}C_4$ ways of getting the first group and ${}^{8}C_4$ ways of getting the second group leaving the third group. As the group order does not matter, we have $\frac{{}^{12}C_4\times{}^{8}C_4}{3!} = 5775$. [2]

ii. There are ${}^{6}C_2\times{}^{6}C_2$ ways of getting the first and ${}^{4}C_2\times{}^{4}C_2$ ways of getting the second group, leaving the third group. As before, the group order does not matter, so we have $\frac{\left({}^{6}C_2\times{}^{4}C_2\right)^2}{3!} = 1350$. [3]

Note that we are not asked to arrange the people within the groups, only to form the groups.

(e) i. Method 1: [2]

$$z_2 = 2iz_1 \text{ (Twice the length and rotated anti-clockwise by } 90^\circ\text{)},$$
$$iz_2 = -2z_1,$$
$$\therefore\ 2z_1 + 2iz_2 = 0.$$

Method 2:

$$\text{Let } z_1 = a+ib,$$
$$z_2 = 2i(a+ib),$$
$$= 2ai-2b.$$
$$\therefore\ 2z_1 = 2a+2bi,$$
$$iz_2 = -2a-2bi.$$

So $2z_1 + iz_2 = 0$.

ii. Method 1: [4]

$$\text{Centre} = \frac{z_1+z_2}{2},$$
$$= \frac{z_1}{2} - \frac{2z_1}{2i}\times\frac{i}{i},$$
$$= z_1\left(\tfrac{1}{2}+i\right).$$

Radius $= \frac{1}{2}|z_1 - z_2|$,

$= \frac{1}{2}|z_1 - 2z_1 i|$,

$= \frac{1}{2}|z_1|\,|1 - 2i|$,

$= \frac{1}{2}|z_1|\sqrt{1^2 + 2^2}$,

$= \frac{\sqrt{5}}{2}|z_1|$.

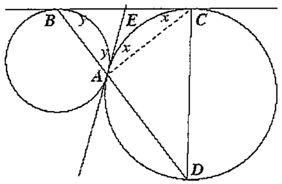
$\therefore \quad \left|z - z_1\left(\frac{1}{2} + i\right)\right| = \frac{\sqrt{5}}{2}|z_1|$.

Method 2:

$$\text{Centre} = \frac{a-2b}{2} + \frac{i}{2}(b+2a),$$
$$= \frac{a+ib}{2} + \frac{2ai-2b}{2},$$
$$= \frac{z_1}{2} + \frac{z_2}{2},$$
$$= \frac{z_1}{2} - \frac{2z_1}{2i} \times \frac{i}{i},$$
$$= z_1\left(\tfrac{1}{2} + i\right).$$
$$\text{Radius}^2 = \left(\frac{a-2b}{2}\right)^2 + \left(\frac{b+2a}{2}\right)^2,$$
$$= \frac{a^2 - 4ab + 4b^2 + b^2 + 4ab + 4a^2}{4},$$
$$= \frac{5a^2 + 5b^2}{4}.$$
$$\text{Radius} = \frac{\sqrt{5}}{2}\sqrt{a^2 + b^2},$$
$$= \frac{\sqrt{5}}{2}|z_1|$$

$\therefore \quad \left|z - z_1\left(\frac{1}{2} + i\right)\right| = \frac{\sqrt{5}}{2}|z_1|$.

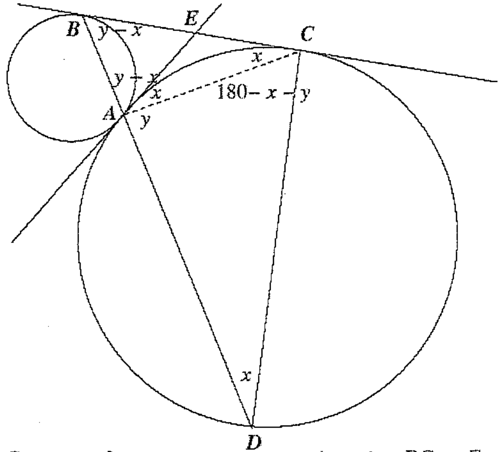
(f) Method 1:



Construct the common tangent at A cutting BC at E.
Join AC.
Let $A\widehat{C}E = x$, $E\widehat{B}A = y$.
$EC = EA = EB$ (equal tangents from external point),

$E\widehat{C}A = E\widehat{A}C = x$ (equal angles of isosceles $\triangle$),
$E\widehat{B}A = B\widehat{A}E = y$ (equal angles of isosceles $\triangle$),
$2x + 2y = 180°$ (angle sum of $\triangle ABC$),
$x + y = 90° = B\widehat{A}E$,
$\therefore \quad C\widehat{A}D = 90°$ (supplementary to $B\widehat{A}E$),
$\therefore \quad CD$ is a diameter (angle in a semi-circle is a right angle).

Method 2:



Construct the common tangent at A cutting BC at E.
Join AC.
Let $A\widehat{D}C = x$, $C\widehat{A}D = y$.
$A\widehat{C}D = 180° - x - y$ (angle sum of $\triangle$),
$E\widehat{C}A = x$ (angle in alternate segment),
$D\widehat{B}C = y - x$ (angle sum of $\triangle$).
$EC = EA = EB$ (equal tangents from external point),
$E\widehat{C}A = E\widehat{A}C = x$ (equal angles of isosceles $\triangle$),
$E\widehat{B}A = B\widehat{A}E = y - x$ (equal angles of isosceles $\triangle$),
$B\widehat{A}D = 2y = 180°$ (supplementary angles),
$\therefore \quad y = 90°$
$B\widehat{C}D = 180° - y = 90°$.
$\therefore \quad CD$ is a diameter (radius $\perp$ tangent at the point of tangency).

Question 3

(a) **Method 1:**

Case 1: 3 different golfers receive prizes

$\binom{5}{3}$ picks the golfers and then the prizes can be awarded in 3! ways

ie $\binom{5}{3} \times 3! = 60$ ways.

Case 2: 1 golfer receives two prizes

Pick the golfer to receive the prize in $\binom{5}{1}$ ways and his prizes in $\binom{3}{2}$ ways.

Then the remaining prize can go to one of the 4 others

ie $\binom{5}{1} \times \binom{3}{2} \times \binom{4}{1} = 60$ ways

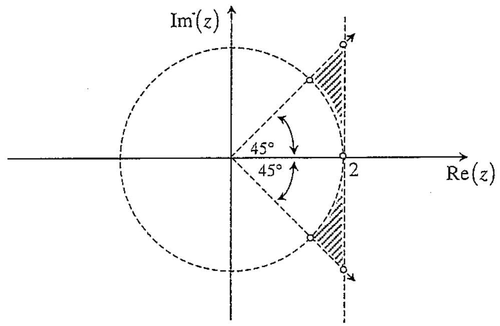
Total = 60 + 60 = 120

Method 2:

There are $5^3 = 125$ ways of dividing up the prizes with no restrictions.
There are 5 ways in which a golfer can get all the prizes.
So there are $125 - 5 = 120$ ways in dividing up the prizes so that a golfer gets no more than 2 prizes.

(b) $|\arg z| < \frac{\pi}{4} \Rightarrow -\frac{\pi}{4} < \arg z < \frac{\pi}{4}$

$z + \bar{z} < 4 \Rightarrow x < 2$



(c) (i) $\text{LHS} = (1 + i\tan\theta)^n + (1 - i\tan\theta)^n$

$= \left(1 + i\frac{\sin\theta}{\cos\theta}\right)^n + \left(1 - i\frac{\sin\theta}{\cos\theta}\right)^n$

$= \left(\frac{\cos\theta + i\sin\theta}{\cos\theta}\right)^n + \left(\frac{\cos\theta - i\sin\theta}{\cos\theta}\right)^n$

$= \frac{[\text{cis}\theta]^n + [\text{cis}(-\theta)]^n}{\cos^n\theta}$

$= \frac{\text{cis}n\theta + \text{cis}(-n\theta)}{\cos^n\theta}$ (deMoivre's Theorem)

$= \frac{2\cos n\theta}{\cos^n\theta}$ $(z + \bar{z} = 2\text{Re}\,z)$

= RHS

(ii) $(1+z)^4 + (1-z)^4 = \frac{2\cos 4\theta}{\cos^4\theta}$ where $z = i\tan\theta$ from (i)

$(1+z)^4 + (1-z)^4 = 0 \Leftrightarrow \frac{2\cos 4\theta}{\cos^4\theta} = 0$

$\therefore \cos 4\theta = 0$

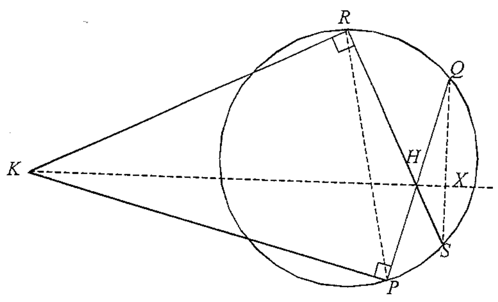
$\therefore 4\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$

$\therefore \theta = \pm\frac{\pi}{8}, \pm\frac{3\pi}{8}$

$\because z = i\tan\theta \Rightarrow z = i\tan\left(\pm\frac{\pi}{8}\right), i\tan\left(\pm\frac{3\pi}{8}\right)$

$\therefore z = \pm i\tan\left(\frac{\pi}{8}\right), \pm i\tan\left(\frac{3\pi}{8}\right)$ $[\because \tan(-x) = -\tan(x)]$

(d) (i)



Join QS and produce KH to intersect with QS at X.
Join RP

(ii) (α)

$PKRH$ is a cyclic quadrilateral	(opposite angles are supplementary)
$\angle PKH = \angle PRH$	(angles in the same segment)
$\angle PRH = \angle PQS$	(angles in the same segment)
$\therefore \angle PKH = \angle PQS$	

(β)

$\angle PHK + \angle PKH = 90°$	$(\because \angle KPH = 90°)$
$\angle QHX = \angle PHK$	(vertically opposite angles)
$\therefore \angle QHX + \angle PQS = 90°$	$(\because \angle PKH = \angle PQS)$
$\therefore \angle QXH = 90°$	(angle sum of Δ)
$\therefore KH$ (produced) $\perp QS$	

(e) If α is a real root of the equation $x^3 + ux + v = 0$ then $\alpha^3 + u\alpha + v = 0$

Now $x^3 + ux + v = (x-\alpha)(x^2 + Ax + B)$

$$x^2 + \alpha x + (u + \alpha^2)$$

$$(x-\alpha)\overline{)x^3 + 0x^2 + ux + v}$$

$$x^2 - \alpha x^2$$

$$(x-\alpha)\overline{)0 + \alpha x^2 + ux}$$

$$\alpha x^2 - \alpha^2 x$$

$$(x-\alpha)\overline{)0 + (u+\alpha^2)x + v}$$

$$(u+\alpha^2)x - (u+\alpha^2)\alpha$$

$$0$$

$\because v + (u+\alpha^2)\alpha = 0$

$\therefore x^3 + ux + v = (x-\alpha)\left[x^2 + \alpha x + (u+\alpha^2)\right]$

With $x^2 + \alpha x + (u+\alpha^2) = 0$ to have real roots then

$\Delta = \alpha^2 - 4(u+\alpha^2) = -3\alpha^2 - 4u \geq 0$

$\therefore 3\alpha^2 + 4u \leq 0$