SYDNEY BOYS HIGH SCHOOL

MATHEMATICS COURSE

July 2001

Assessment Task # 3

Time Allowed: 2 hours (plus 5 minutes Reading Time)

Examiner: Ms B Opferkuch

INSTRUCTIONS:

- Attempt *all* questions.
- *All* questions are of equal value.
- All *necessary* working should be shown in every question. Full marks *may not* be awarded if work is careless or badly arranged.
- Standard integrals are provided on the back of this page. Approved calculators may be used.
- Return your answers in **5** sections: Question 1, Question 2, Question 3, Question 4 and Question 5. Each booklet **MUST** show your name.
- If required, additional Answer Booklets may be obtained from the Examination Supervisor upon request.

QUESTION 1. Use a *separate* Writing Booklet.

Marks
12

(a) Differentiate:

(i) $\dfrac{1}{\sqrt{x}}$

(ii) xe^{x}

(iii) $\cos^{3}x$

(b) Find $\int \sin 3x \; dx$.

(c) Find a primitive function of $x^{2}\sqrt{x}$.

(d) Evaluate the following integral: $\int_{1}^{e} \frac{4}{x} \; dx$.

(e) Find $\int (e^{-x}+e^{x})^{2} \; dx$

(f) Evaluate cos 2 to four significant figures.

(g) Convert 210^{0} to radians.

QUESTION 2 Use a *separate* Writing Booklet **Marks 12**

(a) During an experiment the following values for an unknown function $f(t)$ were recorded:

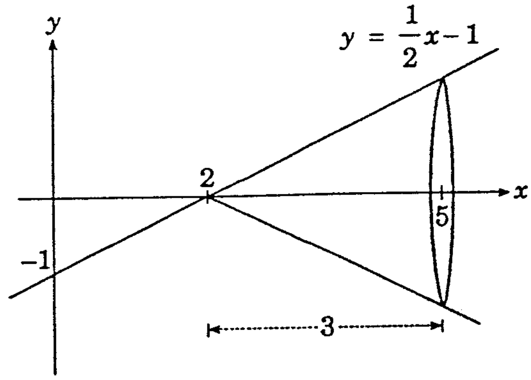
(i)

t	1.0	1.2	1.4	1.6	1.8	2.0
$f(t)$	0.528	0.728	0.876	0.985	0.683	0.563

(ii) Use these six values of the function and the trapezoidal rule to find the approximate value of

$$\int_1^2 f(t)\,dt.$$

(b) The line $y = \frac{1}{2}x - 1$ is rotated about the x-axis to form a cone with a height of 3 units.



Find the volume of this cone using calculus.

(c) Draw a sketch of $y = f(x)$ in the vicinity of $x = a$ for the following:
$a > 0,\ f(a) > 0,\ f'(a) > 0,\ f''(a) < 0$

QUESTION 3 Use a *separate* Writing Booklet

Marks
12

(a) Consider the curve given by $y = x^3 - 3x + 2$.

(i) Find the stationary points and determine their nature.

(ii) Find the point of inflexion.

(iii) Sketch the curve for the domain $-2 \leq x \leq 3$.

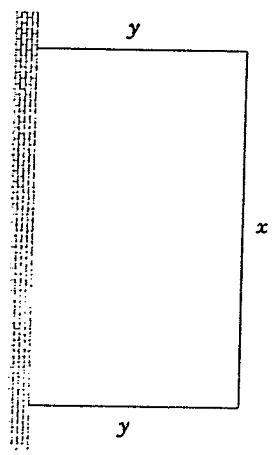
(iv) Find the equation of the tangent to the curve at the point R (-1,4).

(b) A gardener has 40 metres of wire fencing. She wants to make a rectangular enclosure using the wall of a shed as one side.

(i) If one side is x metres, express y in terms of x.

(ii) Find in terms of x an expression for the area of this enclosure.

(iii) Hence, find the largest area she can enclose.



(c.) Simplify $e^{2\ln x}$

QUESTION 4 Use a *separate* Writing Booklet **Marks**

12

(a) (i) On the same set of axes, on the graph paper provided, draw accurate graphs of $y = 2\sin x$ and $y = \frac{x}{2}$ for $0 \le x \le \pi$.

(ii) From the graphs find an approximation to the positive root of the equation $2\sin x = \frac{x}{2}$.

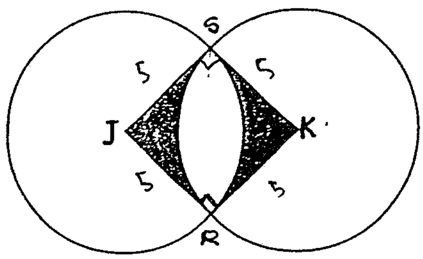
(b) Find the gradient of the normal to the curve $y = ln(2x+1)$ at the point where $x = 1$.

(c.) (i.) Sketch in your answer booklet $y = 2\cos 3x$, for $0 \le x \le \pi$.

(ii) Find the amplitude and period.

(iii) Find the area under the curve $y = 2\cos 3x$ between $x = \frac{\pi}{2}$ and $\frac{2\pi}{3}$.

(d) Two circles centres J and K respectively and radii 5cm intersect each other at S and R. Given JK = SR.

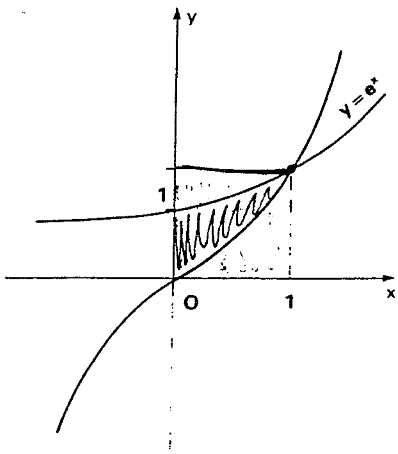


(i) Find the area of the sector SKR.

(ii) Hence find the area of the shaded region.

QUESTION 5 Use a *separate* Writing Booklet **Marks 12**

(a)



The diagram shows the graphs of $y = e^x$ and $y = xe^{x^2}$.

(i) Show that these graphs pass through the point $(1,e)$.

(ii) Find the area between the curves and the y-axis
Leave your answer in terms of e.

(b) Find the value of m where $\int_1^5 (x-1)dx = \int_2^m (x+1)dx$

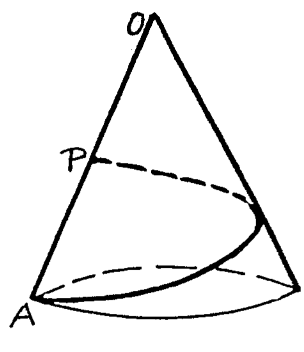
(c.) A thin sheet of smooth metal is in the shape of a sector of a circle with OA, OB as bounding radii each of length 10cm, and the angle AOB is 60^0.

(i) Find the length of the arc and the area of the major sector.

(ii) The sheet is now bent to form a right circular cone by welding the bounding radii OA, OB together (and intersecting a circular disc to close in the cone at the base)

(iii) Find the volume of this right circular cone.
Leave your answer in exact form.

(iv) On the surface of this cone a thin string is pulled tight starting with one end fixed at point A and passing once around the cone to the other end P which is at the midpoint of OA. (As illustrated in the diagram).
Find the exact length of this string.



Question 1

(a) (i) $\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) = \frac{d}{dx}\left(x^{-\frac{1}{2}}\right)$

$= -\frac{1}{2} x^{-3/2}$ 1

(ii) $\frac{d}{dx}\left(x e^{x}\right) = e^{x}.1 + x.e^{x}$

$= (x+1)e^{x}$ 1

(iii) $\frac{d}{dx}\left(\cos^3 x\right) = 3\cos^2 x \,.\, -\sin x$

$= -3\cos^2 x \sin x$ 1

(b) $\int \sin 3x \, dx = -\frac{1}{3}\cos 3x + c.$ 1

(c) $\int x^2\sqrt{x}\, dx = \int x^{5/2}\, dx$

$= \frac{2}{7}x^{7/2} + c.$

$= \frac{2}{7}x^3\sqrt{x} + c.$ 2

(d) $\int_1^e \frac{4}{x}\, dx = \left[4 \ln x\right]_1^e$

$= 4\ln e - 4\ln 1$ $\leftarrow 1\frac{1}{2}$

$= 4$ 2

(e) $\int (e^{-x} + e^{x})^2 dx = \int (e^{-2x} + 2 + e^{2x})\, dx$

$= -\frac{1}{2}e^{-2x} + 2x + \frac{1}{2}e^{2x} + c.$ 2

(f) $\cos 2 = -0.416146\ldots$

≈ -0.4161 1

(g) $210^\circ = 210 \times \frac{\pi}{180}{}^c$

$= \frac{7\pi}{6} \quad (\approx 3.665191429\ldots)$ 1

Solution to q(2).

(a)

$$\frac{h}{2}\left[(y_1+y_6)+2(y_2+\cdots+y_5)\right] \quad 2$$

$$h = \frac{2-1}{5} = 0.2 \quad 2$$

$$0.1\left[1.091+2(3.272)\right]$$

$$= 0.7635$$

4

(b)

$$V = \pi\int_2^5 y^2\,dx$$

$$= \pi\int_2^5\left(\frac{x^2}{4}-x+1\right)dx \quad 1$$

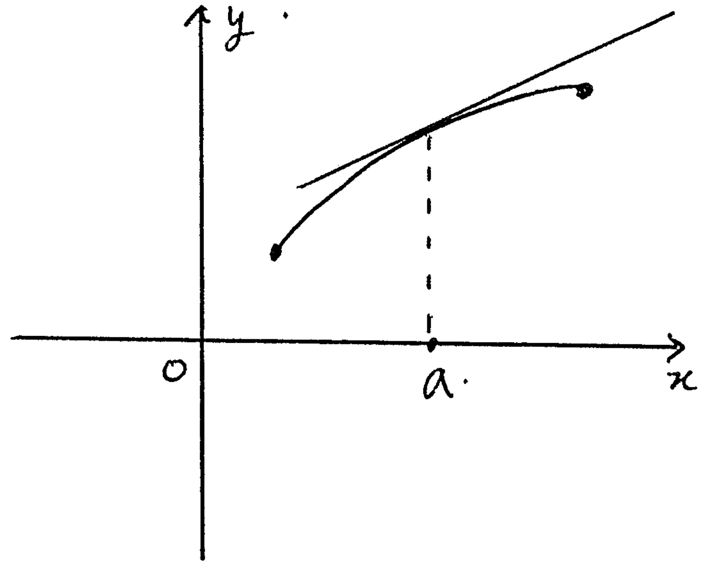
$$= \pi\left[\frac{x^3}{12}-\frac{x^2}{2}+x\right]_2^5 \quad 2$$

$$= \pi\left[\left(10\tfrac{5}{12}-\tfrac{25}{2}+5\right)-\left(\tfrac{2}{3}-2+2\right)\right] \quad 1$$

$$= \frac{9\pi}{4} \qquad (7.069)$$

4

(c)



$f(a) > 0$ 1

$f'(a) > 0$ 1

$f''(a) < 0$ 2

4

3)

(a) i) $y = x^3 - 3x + 2$ $\quad 1-3+2 \quad -8+6+2$

$-1+3+2 \quad 27-9+2$

20

$\frac{dy}{dx} = 3x^2 - 3, = 0 \quad x^2 = \pm 1$

$\frac{d^2y}{dx^2} = 6x$ (1) (1)

at $x = 1$ $\frac{d^2y}{dx^2}$ +ve MIN $(1,0)$ is a MIN.

$x = -1$ " -ve MAX $(-1,4)$ is a MAX.

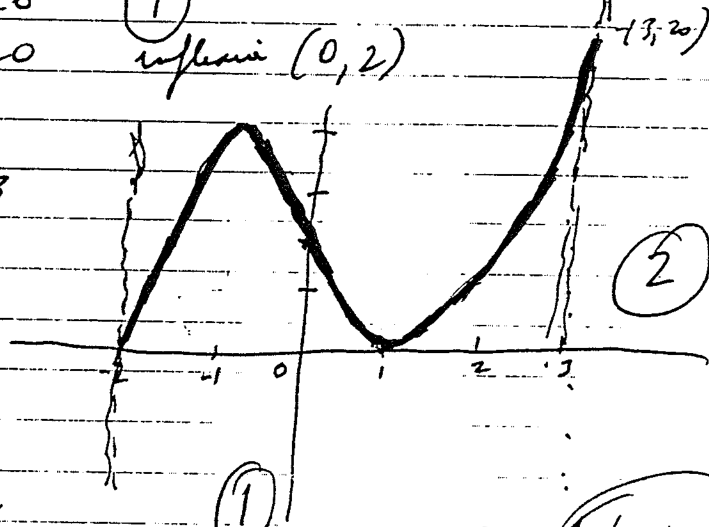
(1) (1)

ii) Inflexion $6x = 0$ (1)

$x = 0$ inflexion $(0,2)$

iii) Sketch $-2 \le x \le 3$

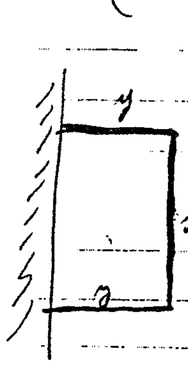
$x=-2$	0
$x=3$	20



(2)

iii) $(-1,4)$ $\underline{y = 4}$ (1)

I like eye-hand

b) 

i) $x + 2y = 40$ (1)

$\boxed{y = \frac{40-x}{2}}$

ii) Area $= xy = \frac{40x - x^2}{2}$ (1)

iii) $\frac{dA}{dx} = 20 - x = 0 \quad \underline{\underline{x = 20}} \quad \frac{d^2A}{dx^2} = \underline{-ve}$

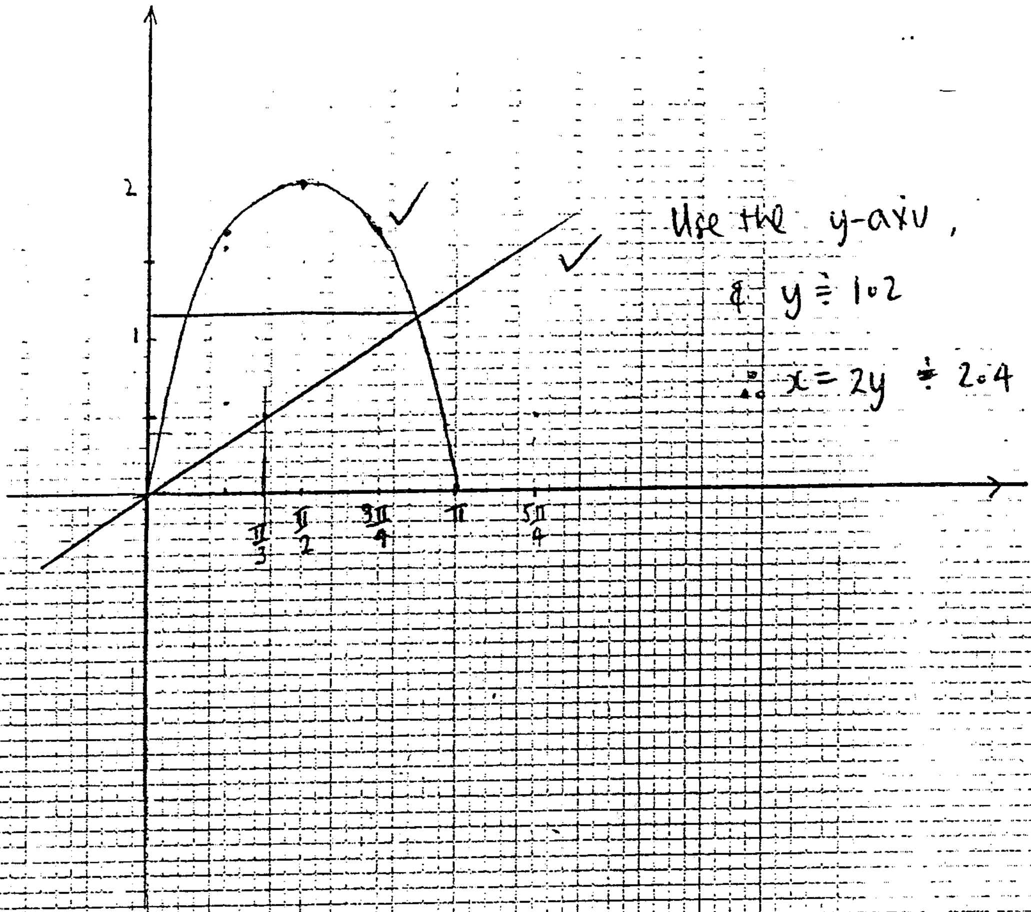
largest area $= 20 \times 10$

$= 200m^2$. (1)

c) $e^{2\ln x} = x^2$. (1)

80

a)



Use the y-axis,

& $y \doteqdot 1.2$

$\therefore x = 2y \doteqdot 2.4$

SOLUTIONS q4 (continued)

b) $y = \ln(2x+1)$

$y' = \frac{2}{2x+1}$ ✓

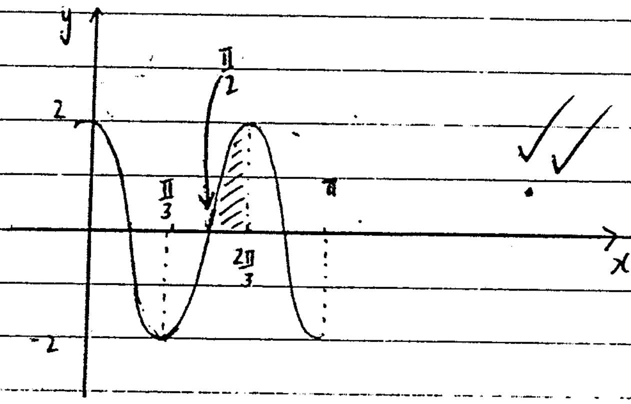
at $x=1$ $y' = \frac{2}{3}$

$\therefore$ gradient of normal $= -\frac{3}{2}$ ✓

c) $y = 2\cos 3x$.

(ii) amplitude $= 2$

period $= \frac{2\pi}{3}$ ✓ (i)



✓✓

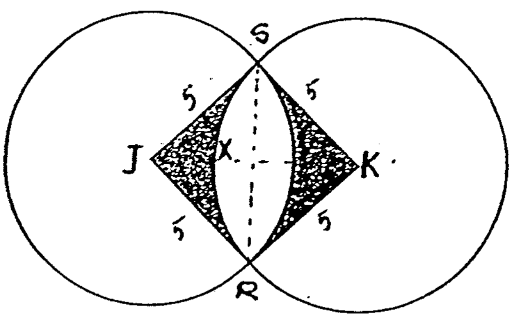
(iii) $A = \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} 2\cos 3x \, dx = 2\int_{\frac{\pi}{2}}^{2\pi/3} \cos 3x \, dx$

$= \frac{2}{3}\left[\sin 3x\right]_{\frac{\pi}{2}}^{\frac{2\pi}{3}}$

$= \frac{2}{3}\left(\sin 2\pi - \sin \frac{3\pi}{2}\right)$ ✓

$= \frac{2}{3}$

d)



Since JK = SR

then JSKR is a square.

(i) $SKR = \frac{1}{2} \times 5^2 \times \frac{\pi}{2} = \frac{25\pi}{4}$ ✓ $(\doteqdot 19.63)$

(ii) $\Delta SKR = \frac{25}{2}$

$\therefore$ segment $SXR = \frac{25\pi}{4} - \frac{25}{2} = \frac{25\pi - 50}{4}$ ✓

SOLUTIONS q4 (continued)

d) (iii) unshaded area $= 2 \times \left(\frac{25\pi - 50}{4}\right) = \frac{25\pi - 50}{2}$

$\therefore$ shaded area $= 25 - \frac{25\pi - 50}{2}$

$= \frac{100 - 25\pi}{2}$ ✓ $(\doteqdot 10.73)$

[alternative & shorter]

or $\boxed{\frac{1}{2} \times \text{shaded area} = 2 \times \Delta SKR - \text{sector } SRK}$

$= 25 - \frac{25\pi}{4}$

$= \frac{100 - 25\pi}{4}$

$\therefore$ shaded area $= 2 \times \left(\frac{100 - 25\pi}{4}\right)$

$= \frac{100 - 25\pi}{2}$

$$\frac{m^2}{2} + m - 4 = 0$$

$$m^2 + 2m - 8 = 16$$

$$m^2 + 2m - 8 - 16 = 0$$

$$m^2 + 2m - 24 = 0$$

$$(m+6)(m-4) = 0$$

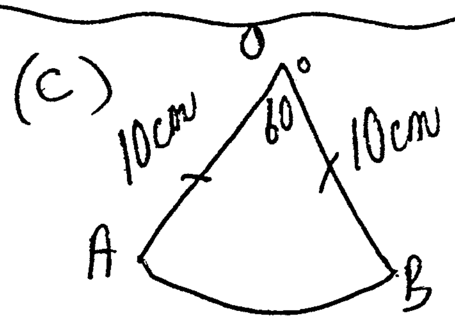
$$m = -6,\ m = 4$$

So accept both.

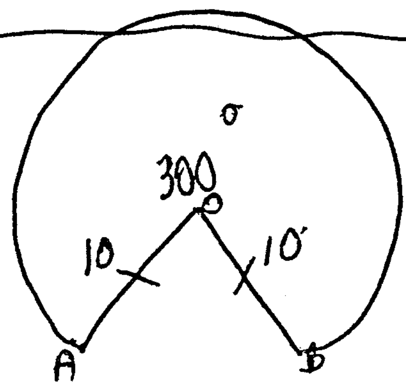
4

$$2\pi - \frac{\pi}{3} = \frac{5\pi}{3}.$$

(c)



OR



(i) $L = r\theta$

$$= 10 \times \frac{\pi}{3}$$

1 $$= \frac{10\pi}{3}\text{ cm or } \doteq 10.47\text{ cm}$$

$$A = \frac{1}{2}r^2\theta$$

$$= \frac{1}{2} \times 100 \times \frac{\pi}{3} = \frac{50\pi}{3}\text{ cm}^2$$

1 or $\doteq 52.36\text{ cm}^2$.

$L = r\theta$

$$= 10 \times \frac{5\pi}{3}$$

$$= \frac{50\pi}{3}\text{ cm or } \doteq 52.36\text{ cm}.$$

$$A = \frac{1}{2}r^2\theta$$

$$= \frac{1}{2} \times 100 \times \frac{5\pi}{3}$$

$$= \frac{250\pi}{3}\text{ cm}^2$$

or $\doteq 261.80\text{ cm}^2$.

5 (a) (i) $e^{x} = xe^{x^2}$

$e^{x} - xe^{x^2} = 0$

When $x=1$, $e^{1} - 1e^{1^2} = e - e = 0$

When $x=1$, $y = e^{1} = e$.

So $y=e^{x}$ and $y = xe^{x^2}$ pass through $(1, e)$ 1

12

(ii) $\int_0^1 e^{x}\,dx - \int_0^1 xe^{x^2}\,dx$

$= e^{x}\Big]_0^1 - \frac{1}{2}e^{x^2}\Big]_0^1$

$= (e^1 - e^0) - (\frac{1}{2}e^1 - \frac{1}{2}e^0)$

$= e - 1 - \frac{1}{2}e + \frac{1}{2}$

$= \frac{1}{2}e - \frac{1}{2} = \frac{1}{2}(e-1)\ u^2$ 2

(b) $\int_1^5 (x-1)\,dx = \int_2^m (x+1)\,dx$

LHS $\frac{x^2}{2} - x\Big]_1^5$

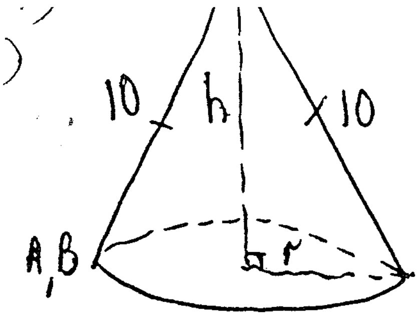
$= \left(\frac{25}{2} - 5\right) - \left(\frac{1}{2} - 1\right)$

$= 8$

RHS $\frac{x^2}{2} + x\Big]_2^m$

$\left(\frac{m^2}{2} + m\right) - (2+2)$

i)



ircumference $= 2\pi r = \dfrac{10\pi}{3}$

$$r = \frac{10\pi}{2\pi \times 3} = \frac{5}{3}\text{ cm.}$$

OR $\doteqdot 1.\dot{6}$ cm

so $\left(\dfrac{5}{3}\right)^2 + h^2 = 100$

$$h^2 = 100 - \frac{25}{9}$$

$$= \frac{875}{9} \Rightarrow$$

$$h = \sqrt{\frac{875}{9}} = \frac{5\sqrt{35}}{3}\text{ cm}$$ OR $\doteqdot 9.87$ cm

so $V = \frac{1}{3}\pi r^2 h$

$$= \frac{1}{3} \times \pi \times \frac{25}{9} \times \frac{5\sqrt{35}}{3}$$

$$= \frac{125\sqrt{35}\,\pi}{81}\text{ cm}^3$$

OR $\doteqdot 8.42\pi\text{ cm}^3$

OR circumference $= 2\pi r = \dfrac{50\pi}{3}$

$$r = \frac{50\pi}{2\pi \times 3}$$

$$= \frac{25}{3}\text{ cm}$$

OR 8.33 cm

So $\left(\dfrac{25}{3}\right)^2 + h^2 = 100$

$$h^2 = 100 - \frac{625}{9}$$

$$= \frac{275}{9}$$

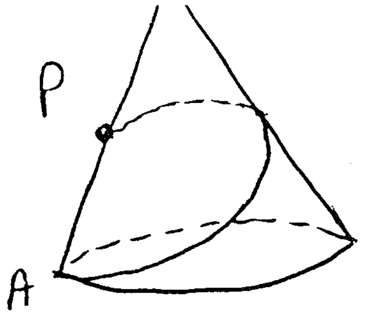
$$h = \frac{\sqrt{275}}{3} = \frac{5\sqrt{11}}{3}\text{ cm}$$

Thus

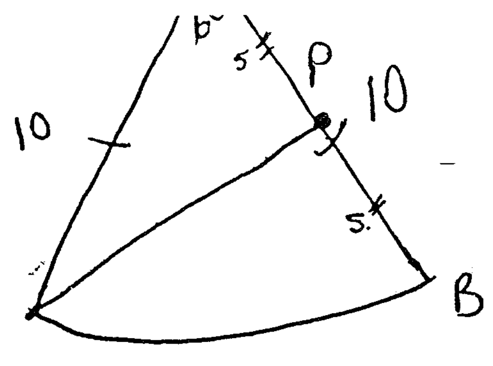
$$V = \frac{1}{3} \times \pi \times \frac{625}{9} \times \frac{5\sqrt{11}}{3}$$

$$= \frac{3125\sqrt{11}\,\pi}{81}\text{ cm}^3.$$

(iv)



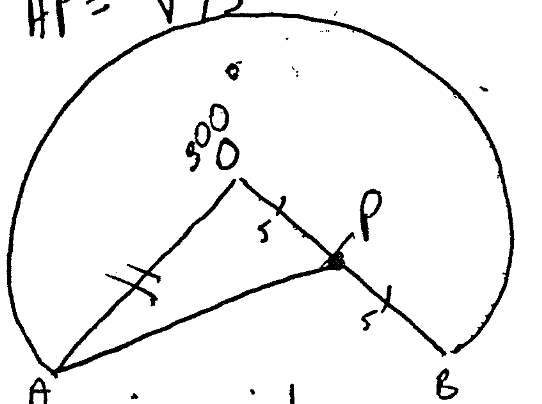
$\Rightarrow$



$$(AP)^2 = 10^2 + 5^2 - 2 \times 10 \times 5 \times \cos 60^\circ$$
$$= 100 + 25 - 100 \times \tfrac{1}{2}$$
$$= 75$$
$$AP = \sqrt{75} = 5\sqrt{3} \text{ cm} \quad (8.66\text{cm})$$

1

OR



By cosine rule

$$AP^2 = 10^2 + 5^2 - 2 \times 10 \times 5 \times \cos 300$$
$$= 125 - 100 \times \tfrac{1}{2}$$
$$= 75$$
$$AP = 5\sqrt{3} \text{ cm}$$

No solution since AP goes through open space.