SYDNEY BOYS' HIGH SCHOOL

MOORE PARK, SURRY HILLS

S·H·S

VERITATE ET VIRTUTE

ASSESSMENT TASK - July 2001

MATHEMATICS

EXTENSION 1

Time allowed — One and a half hours
(Plus 5 minutes reading time)

Examiner: Mr A.M.Gainford

DIRECTIONS TO CANDIDATES

- *All* questions may be attempted.
- All necessary working should be shown in every question. Full marks may not be awarded for careless or badly arranged work.
- Approved calculators may be used.
- *Each question* is to be returned in a *separate* booklet, clearly marked Questions 1, Question 2, etc. Each booklet must also show your name and class.
- If required, additional booklets may be obtained from the Examination Supervisor upon request.

Question 1. (10 marks) **(Start a new booklet.)**

Marks

(a) Simplify $\dfrac{(2^3)^5\times 4^2}{8}$, giving your answer as a power of 2. **1**

(b) Find $\lim\limits_{x\to 0}\dfrac{\sin x}{5x}$. **1**

(c) Use the substitution $u=1-x^2$ to find $\displaystyle\int_0^1 6x\sqrt{1-x^2}\,dx$. **4**

(d) Write down the inverse function of $y=\dfrac{1}{x-2}$ as a function of x and state its domain. **2**

(e) State the exact value of: **2**

(i) $\sin^{-1}\dfrac{1}{\sqrt{2}}$

(ii) $\tan^{-1}\left(-\sqrt{3}\right)$

Question 2 (10 marks) **(Start a new booklet.)**

Marks

(a) Differentiate **5**

(i) $\cos 2x$.

(ii) $\sin^{-1}\left(\dfrac{x}{2}\right)$

(iii) $\ln(\sin x)$

(b) Find a primitive of: **5**

(i) $\dfrac{1}{\sqrt{9-x^2}}$

(ii) $\cos(2x-1)$

(iii) $\cos x\sin^2 x$ $+C$

Question 3. (10 marks) **(Start a new booklet.)**

Marks

(a) (i) State the domain and range of the function $f(x)=\cos^{-1}\left(\frac{x}{2}\right)$. 5

(ii) Sketch the graph of $y=f(x)$.

(iii) Find the gradient of the tangent to the curve at the point where it crosses the y-axis.

(b) Use the substitution $u=\log_e x$ to evaluate $\int_1^e \frac{\log_e x}{x}\,dx$. 3

(c) A body moves in a straight line so that at time t seconds its displacement from O, a fixed point on the line, is x metres, and its velocity is v m/sec. 2
If $v=\frac{1}{x}$, find the acceleration of the body when $x=\frac{1}{2}$ m.

Question 4 (10 marks) **(Start a new booklet.)**

Marks

(a) The velocity v of a particle moving in a straight line is given in terms of its displacement from a fixed point O by: 4

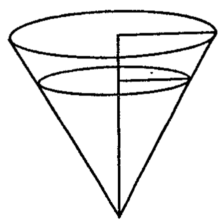
$$v=\sqrt{8x+1}$$

(i) Show that the acceleration is constant.

(ii) Given that the particle started at the origin, find the equation for x in terms of time t.

(b) A water tank consists of an inverted right circular cone, with vertex angle 90°. 6

The water drains through the apex at a rate (in m³/min) equal to one tenth of the depth (in metres) at the time.



(i) Write an equation expressing the flow rate of water in terms of h, the depth of water.

(ii) Find the rate of change of the depth of the water when the depth is 2 metres.

Question 5: (10 marks) **(Start a new booklet.)**

Marks

(a) The area bounded by the curve $y=\sec\left(\frac{\pi}{2}x\right)$, the x-axis, and the ordinates $x=-\frac{1}{2}$ and $x=\frac{1}{2}$ is rotated about the x-axis. 3

Find the volume of the solid of revolution so formed.

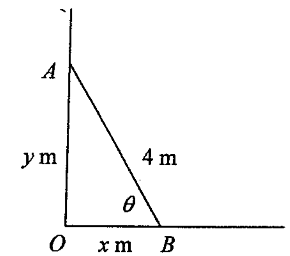
(b) Find the exact value of $\int_0^{\frac{\pi}{4}}\cos^2 x\,dx$. 4

(c) By writing $\frac{x^2}{1+x^2}$ as $\frac{(1+x^2)-1}{1+x^2}$, evaluate $\int_0^1\frac{x^2}{1+x^2}\,dx$. 3

Question 6: (10 marks) **(Start a new booklet.)**

Marks

(a) A ladder AB, 4 m long, Stands on horizontal ground against a vertical wall. 5



The base B is then drawn away from the wall at a constant rate of $0\cdot 3$ m/sec.

(i) Prove that $\frac{dx}{d\theta}=-4\sin\theta$.

(ii) Find the rate at which θ is changing when $x=2\cdot 5$.

(iii) Find the rate at which A is moving when $x=2\cdot 5$.

(b) Consider the curves $y=\cos x$ and $y=\sin x$. 5

(i) Sketch the graphs of the two curves on the same axes, in the domain $-\pi\le x\le\pi$.

(ii) Verify by substitution that the curves intersect at $x=-\frac{3\pi}{4}$ and $x=\frac{\pi}{4}$.

(iii) Determine the area of the region bounded by the curves between the points of intersection.

Question 7: (10 marks) **(Start a new booklet.)**

Marks

(a) (i) Show that $x = 0\cdot 8$ is an approximate solution of the equation $e^x + x = 3$. **4**

(ii) Use one application of Newton's Method to find a better solution.

(b) (i) Prove that $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$. **6**

(ii) Hence or otherwise find the exact values of x and y which satisfy the simultaneous equations:

$$\sin^{-1} x - \cos^{-1} y = \frac{\pi}{12}\ ;\ \sin^{-1} y + \cos^{-1} x = \frac{5\pi}{12}.$$

This is the end of the paper.

Stephanie Sin

3U

Sydney Boys Assessment July 2001 — 1/6/02

Question One ~

A.) $\frac{2^{15} \times (2^2)^2}{2^3} = \frac{2^{15} \times 2^4}{2^3} = \frac{2^{19}}{2^3} = 2^{16}$ ✓

B.) $\lim_{x\to 0} \frac{\sin x}{5x} = \lim_{x \to 0} \frac{\sin x}{x} \times \frac{1}{5}$

$= \frac{1}{5}$ ✓

c) $\int_0^1 6x\sqrt{1-x^2}\, dx$ — Let $u = 1-x^2$

$\frac{du}{dx} = -2x$ ✓

$du = -2x\,dx$

$= \int_1^0 -3\sqrt{u}\, du$ ✓

$= -3\left(\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right)_1^0$ ✓

$= -3\left(-1 \times \frac{2}{3}\right) = -\cancel{3} \times -\frac{2}{\cancel{3}} = 2$ ✓

D) $y = \frac{1}{x-2}$

$x = \frac{1}{y-2}$ — $y-2 = \frac{1}{x}$; $y = \frac{1}{x} + 2$ ✓

~~D: x ≠ 0~~ — $\therefore f^{-1}(x) = \frac{1}{x} + 2$ ✓

~~D: x ≠ 0~~ ✓

E) i.) $\sin^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4}$, ~~$\frac{3\pi}{4}$~~ ✓

ii.) $\tan^{-1}(-\sqrt{3}) = -\tan^{-1}\sqrt{3} = \boxed{-\frac{\pi}{3}}$, ~~$\frac{2\pi}{3}, \frac{5\pi}{3}$~~ ✓

Question Two ~

A.) i.) $y = \cos 2x$

$y' = -2\sin 2x$ ✓

ii.) $y = \sin^{-1}\left(\frac{x}{2}\right)$

$$y' = \frac{1}{\sqrt{1-\frac{x^2}{4}}} \times \frac{1}{2} \checkmark$$

$$= \frac{1}{\sqrt{\frac{4-x^2}{4}}} \times \frac{1}{2} = \frac{\cancel{2}}{\sqrt{4-x^2}} \times \frac{1}{\cancel{2}} = \frac{1}{\sqrt{4-x^2}} \checkmark$$

iii.) $y = \ln(\sin x)$

$$y' = \frac{1}{\sin x} \times \cos x = \cot x. \checkmark$$

B) i.) $\int \frac{1}{\sqrt{9-x^2}}\,dx = \sin^{-1}\frac{x}{3} + c \checkmark$

ii.) $\int \cos(2x-1)\,dx = \frac{1}{2}\sin(2x-1) + c \checkmark$

iii.) $\int \cos x \sin^2 x\,dx.$

let $u = \sin x$

$\frac{du}{dx} = \cos x \checkmark$; $du = \cos x\,dx \checkmark$

$$= \int u^2\,du = \frac{u^3}{3} + c$$

$$= \frac{\sin^3 x}{3} + c \checkmark$$

Question Three ~

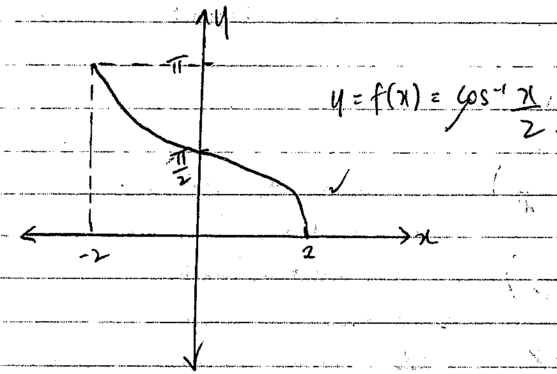
A.) i.) $f(x) = \cos^{-1}\frac{x}{2}$

$$-1 \le \frac{x}{2} \le 1$$

D: $-2 \le x \le 2$ ✓

R: $0 \le y \le \pi$ ✓

ii.)



iii.) $y = \cos^{-1}\frac{x}{2}$

$$y' = \frac{-1}{\sqrt{1-\frac{x^2}{4}}} \times \frac{1}{2} = \frac{-1}{\sqrt{\frac{4-x^2}{4}}} \times \frac{1}{2} = \frac{-\cancel{2}}{\sqrt{4-x^2}} \times \frac{1}{\cancel{2}}$$

$$= \frac{-1}{\sqrt{4-x^2}} \checkmark$$

At $x = 0$, $y' = \frac{-1}{\sqrt{4}} = \frac{-1}{2} \checkmark$

$\therefore$ where it crosses the y-axis, gradient $= \frac{-1}{2}$ ✓

B) $\int_1^e \frac{\log_e x}{x}\,dx.$ Let $u = \log_e x$

$\frac{du}{dx} = \frac{1}{x}.$

$= \int_0^1 u\,du$ ✓ $du = \frac{dx}{x}.$ ✓

$= \left(\frac{u^2}{2}\right)_0^1 = \frac{1}{2}$ ✓

C) $v = \frac{1}{x}.$

$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$= \frac{d}{dx}\left(\frac{1}{2}\cdot\frac{1}{x^2}\right)$

$= \frac{d}{dx}\left(\frac{1}{2x^2}\right)$

$= \frac{1}{\cancel{2}} \times \frac{-\cancel{2}}{x^3} = \frac{-1}{x^3}$ ✓

$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$= \frac{d}{dx}\left(\frac{1}{2}\cdot\frac{1}{x^2}\right)$

$= \frac{d}{dx}\left(\frac{1}{2x^2}\right)$

$= \frac{1}{\cancel{2}} \times \frac{-\cancel{2}}{x^3} = \frac{-1}{x^3}$

when $x = \frac{1}{2}$, $a = \frac{-1}{\frac{1}{8}} = -8m/\sec^2$ ✓

Question Four ~

A) $v = \sqrt{8x+1}$

i) $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$= \frac{d}{dx}\left(\frac{1}{2}\cdot 8x+1\right)$ ✓

$= \frac{d}{dx}\left(\frac{8x+1}{2}\right) = \frac{d}{dx}\left(4x + \frac{1}{2}\right)$

$= 4$ ✓ ∴ acceleration is a constant

$\therefore 4t + c$ $\quad \frac{dv}{dx} = \frac{8}{2\sqrt{8x+1}} = \frac{4}{\sqrt{8x+1}}$

ii.) At $t=0$, $x=0$.

$v = \frac{dx}{dt} = \sqrt{8x+1}$

$\frac{dx}{dv}\times\frac{dv}{dt} = \frac{\sqrt{8x+1}\times 4}{4} = \sqrt{8x+1}$

$\frac{dt}{dx} = \frac{1}{\sqrt{8x+1}}$ ✓

$\frac{dx}{dt} = \sqrt{8x+1}$

$t = \int \frac{1}{\sqrt{8x+1}}\,dx$

$\frac{dt}{dx} = \frac{1}{\sqrt{8x+1}}$

$t = \int (8x+1)^{-\frac{1}{2}}\,dx$

$t = 2\sqrt{8x+1} + c$ ✓

when $t=0$, $x=0$.

$0 = 2\sqrt{1} + c$; $c = -2$

$\therefore t = 2\sqrt{8x+1} - 2$

$2\sqrt{8x+1} = t+2$

$\sqrt{8x+1} = \frac{t+2}{2}$

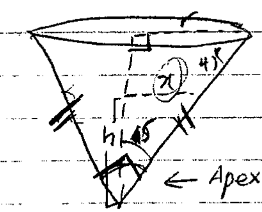
$8x+1 = \left(\frac{t+2}{2}\right)^2$ ✓

$8x = \frac{(t+2)^2}{4} - 1$

$x = \frac{(t+2)^2 - 4}{32}$ ✓

$x = \frac{t^2+4t}{32}$ ✓

B)



$\frac{dV}{dt} = \frac{1}{10}h$

i.) $\frac{dV}{dt} = \frac{h}{10}$ ✓

$r = h$ since isos Δ, base ∠ equal. two sids equal

ii.) Find $\frac{dh}{dt}$

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi h^3$ ✓

$\frac{dV}{dh} = \cancel{\frac{1}{3}\pi r^2}\ \pi h^2$

$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$ ✓

$\cancel{= \frac{3}{\pi r^2} \times \frac{h}{10} = \frac{3h}{10\pi r^2}}$

~~when~~ $\cancel{h = 2,\ \frac{dh}{dt} = \frac{6}{10\pi r^2} = \frac{3}{5\pi r^2}\ m/min}$

how do I get rid of r-value?

$\frac{dh}{dt} = \frac{1}{\pi h^2} \times \frac{h}{10}$ ✓

$= \frac{1}{10\pi h}$

when $h = 2$, $\frac{dh}{dt} = \frac{1}{20\pi}$ ✓

∴ when $h=2$, rate of change of depth of water $= \frac{1}{20\pi}\ m/min$ ✓

Question 5

A) $V = \pi \int_{-\frac{1}{2}}^{\frac{1}{2}} \sec^2\left(\frac{\pi x}{2}\right) dx.$

$\int \sec^2 x = \tan x + c$

$= \pi \left(\tan \frac{\pi}{2} x \times \frac{1}{\frac{\pi}{2}}\right)_{-\frac{1}{2}}^{\frac{1}{2}}$ ✓

$= \pi\left(\frac{2}{\pi} \tan \frac{\pi x}{2}\right)_{-\frac{1}{2}}^{\frac{1}{2}} = \pi\left(\frac{2}{\pi}\tan\frac{\pi}{4} + \frac{2}{\pi}\tan\frac{\pi}{4}\right)$

$= 4 \tan\frac{\pi}{4} = 4\ u^3$ ✓

B) $\int_0^{\frac{\pi}{4}} \cos^2 x\ dx.$

$\cos 2x = 2\cos^2 x - 1$

$\cos^2 x = \frac{\cos 2x + 1}{2}$ ✓

$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \cos 2x + 1\ dx$

$= \frac{1}{2}\left(\frac{1}{2}\sin 2x + x\right)_0^{\frac{\pi}{4}}$ ✓

$= \frac{1}{2}\left(\frac{1}{2}\sin\frac{\pi}{2} + \frac{\pi}{4}\right) = \frac{1}{2}\left(\frac{1}{2} + \frac{\pi}{4}\right) = \frac{1}{4}\left(1 + \frac{\pi}{2}\right)$ ✓

C) $\int_0^1 \frac{x^2}{1+x^2} dx = \int_0^1 \frac{(1+x^2)-1}{1+x^2} dx$

$= \int_0^1 1 - \frac{1}{1+x^2}\ dx$ ✓

$= (x - \tan^{-1} x)_0^1$ ✓

$= 1 - \tan^{-1} 1 = 1 - \frac{\pi}{4}$ ✓

Question six

i.) R.T.P. $\frac{dx}{d\theta} = -4\sin\theta$

$\cos\theta = \frac{x}{4}$

$x = 4\cos\theta$

$\frac{dx}{d\theta} = 4\times -\sin\theta = -4\sin\theta$

$\therefore \frac{dx}{d\theta} = -4\sin\theta$.

~~ii.) $\frac{d\theta}{dt} = \frac{-1}{4\sin\theta}$~~

~~$\sin\theta = \frac{y}{4}$~~

~~$= \frac{\sqrt{16-x^2}}{4}$~~

~~$x^2+y^2=16$ (pythag theo)~~

~~$y^2 = 16-x^2$~~

~~$y = \sqrt{16-x^2}$~~

~~$= \frac{-1}{4\cdot\frac{\sqrt{16-x^2}}{4}}$~~

~~$= \frac{-1}{\sqrt{16-x^2}}$~~

~~when $x = 2.5$, $\frac{d\theta}{dt} = \frac{-1}{\sqrt{16-6.25}}$~~

~~$= \frac{-1}{\sqrt{9.75}}$~~

ii.) Must find $\frac{d\theta}{dt}$.

$\frac{d\theta}{dt} = \frac{d\theta}{dx} \times \frac{dx}{dt}$

$= \frac{-1}{4\sin\theta} \times \frac{3}{10} = \frac{-3}{40\sin\theta}$

$\sin\theta = \frac{y}{4}$

$y = \sqrt{16-x^2}$

$\sin\theta = \frac{\sqrt{16-x^2}}{4}$

$= \frac{-3}{10\sqrt{16-x^2}}$

when $x = 2.5$, $\frac{d\theta}{dt} = \frac{-3}{10\sqrt{16-6.25}} = \frac{-3}{10\sqrt{9.75}}$ °/sec

want to find $\frac{dy}{dt}$

iii.) Find $\frac{dy}{dt}$.

$\sin\theta = \frac{y}{4}$; $y = 4\sin\theta$

$\frac{dy}{d\theta} = 4\cos\theta$

$\cos\theta = \frac{x}{4}$

$\frac{dy}{dt} = \frac{dy}{d\theta} \times \frac{d\theta}{dt}$

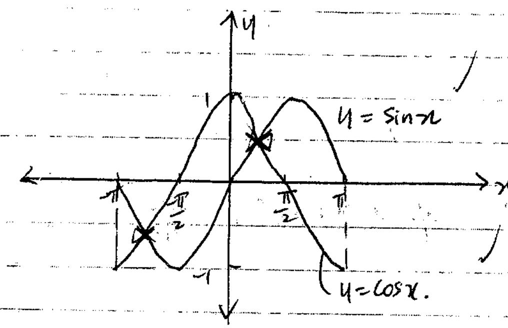
when $x = 2.5$, $\cos\theta = \frac{2.5}{4}$

$= 0.625$

$= 4\cos\theta \times \frac{-3}{10\sqrt{9.75}}$

$= 4(0.625) \times \frac{-3}{10\sqrt{9.75}} = \frac{-7.5}{10\sqrt{9.75}} = \frac{-3}{4\sqrt{9.75}}$ m/s

B) i.)



ii.) The curves intersect at the pt given by $\cos x = \sin x$.

Sub $x = -\frac{3\pi}{4}$, $\cos -\frac{3\pi}{4} = \frac{-1}{\sqrt{2}}$, $\sin -\frac{3\pi}{4} = \frac{-1}{\sqrt{2}} = \cos -\frac{3\pi}{4}$.

Sub $x = \frac{\pi}{4}$, $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \cos\frac{\pi}{4}$.

$\therefore$ they intersect at $x = -\frac{3\pi}{4}$ and $\frac{\pi}{4}$

iii.) Area $= \int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} \cos x - \sin x \, dx$

$= (\sin x + \cos x)_{-\frac{3\pi}{4}}^{\frac{\pi}{4}}$

$= \left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right) - \left(\sin\left(-\frac{3\pi}{4}\right) + \cos -\frac{3\pi}{4}\right)$

$= \left(\frac{1}{\sqrt2} + \frac{1}{\sqrt2}\right) - \left(+\frac{1}{\sqrt2} - \frac{1}{\sqrt2}\right) = \frac{2}{\sqrt2}$ $= \frac{4}{\sqrt2} = \frac{4\sqrt2}{2}$

$= 2\sqrt2\, u^2$

Question Seven

A.) i.) $e^x + x - 3 = 0$.

Let $e^x + x - 3$ be $f(x)$.

$f(0.6) < 0$

$f(1) > 0$, $f(x)$ is continuous

$\therefore$ a root lie btw $x = 0.6$ and $x = 1$.

~~Halve the interval.~~ $\therefore x = 0.8$ is an approximate solution of $e^x + x = 3$

~~$f(0.8) > 0$~~

OR simply sub $x = 0.8$ into eqn

$e^{0.8} + 0.8 = 3.03 \approx 3$

ii.) $x_1 = 0.8$

$f(x_1) = e^{0.8} - 2.2$

$f'(x_1) = e^{0.8} + 1$

$f(x) = e^x + x - 3$

$f'(x) = e^x + 1$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= 0.792$ (3dp)

B) i.) R.T.P. $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

Let $\sin^{-1} x = \alpha$

$x = \sin\alpha$

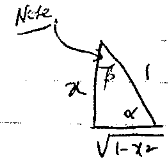
Note

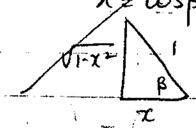
$\cos\beta = \frac{x}{1}$

$\therefore \alpha + \beta = \frac{\pi}{2}$

~~Let $\cos^{-1} x = \beta$~~

~~$x = \cos\beta$~~





LHS $\sin^{-1}x + \cos^{-1}x = \alpha + \beta$.

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$= x.x + \sqrt{1-x^2}.\sqrt{1-x^2}$

$= x^2 + 1 - x^2 = 1$

$\sin(\alpha+\beta) = 1$

$\alpha + \beta = \sin^{-1} 1$

$= \frac{\pi}{2}$ = RHS

ii.) ~~$\sin^{-1}x - \cos^{-1}y = \frac{\pi}{12}$~~

$\sin^{-1}x - \cos^{-1}y = \frac{\pi}{12}$

~~$\alpha - \beta = \frac{\pi}{12}$~~

Let $\sin^{-1} x = \alpha$

and $\cos^{-1} y = \beta$

~~$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$~~

~~$= x.x - (\sqrt{1-x^2}.\sqrt{1-x^2})$~~

~~$= x^2 - 1 + x^2 = 2x^2 - 1$~~

~~$\sin(\alpha-\beta) = 2x^2 - 1$~~

~~$\alpha - \beta = \sin^{-1}(2x^2-1)$~~

~~$-1 \le 2x^2 - 1 \le 1$~~

~~$0 \le 2x^2 \le 2$~~

~~$0 \le x^2 \le 1$~~

~~$0 \le x \le 1$~~

~~$\sin^{-1}(2x^2-1) = \frac{\pi}{12}$~~

~~$\sin^{-1} x = \frac{7\pi}{24}$~~

~~$2x^2 - 1 = \sin\frac{\pi}{12}$~~

$x^2 = \frac{\sin\frac{\pi}{12} + 1}{2}$

$\sin^{-1}y + \cos^{-1}x = \frac{5\pi}{12}$

PROVEN $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

$\sin^{-1}x - \cos^{-1}y = \frac{\pi}{12}$

$\alpha - \beta = \frac{\pi}{12}$ — (1) ✓

$\sin^{-1}y + \cos^{-1}x = \frac{5\pi}{12}$

Let

$\cos^{-1}y = \beta$

and $\sin^{-1}x = \alpha$

$\frac{\pi}{2} - \beta + \frac{\pi}{2} - \alpha = \frac{5\pi}{12}$

$\pi - \frac{5\pi}{12} = \alpha + \beta$ ✓

$\frac{7\pi}{12} = \alpha + \beta$ — (2). ✓

$\alpha + \beta = \frac{7\pi}{12}$

$\alpha - \beta = \frac{\pi}{12}$.

$2\alpha = \frac{2\pi}{3}$

$\alpha = \frac{\pi}{3}$ ✓

$\frac{\pi}{3} - \beta = \frac{\pi}{12}$

$\beta = \frac{\pi}{3} - \frac{\pi}{12} = \frac{\pi}{4}$. ✓

$\cos^{-1}y = \frac{\pi}{4}$

$y = \cos\frac{\pi}{4}$

$y = \frac{1}{\sqrt{2}}$ ✓

$\sin^{-1}x = \alpha$

$x = \sin\frac{\pi}{3}$

$= \frac{\sqrt{3}}{2}$ ✓