SYDNEY BOYS HIGH SCHOOL

3 UNIT MATHEMATICS

Year 12 Assessment Task: April 1997

Time Allowed: 90 minutes (plus 5 minutes reading time)

Total Marks: 60

Examiner: Mr R Dowdell

INSTRUCTIONS:

- Attempt *all* questions.
- *All* questions are of equal value.
- All necessary working should be shown in every question. Full marks may not be awarded if work is careless or badly arranged.
- Standard integrals are provided on the back of this page. Approved calculators may be used.
- *Each* question attempted is to be returned in a *separate* Writing Booklet clearly marked Question 1, Question 2 etc. on the cover. Each booklet must show your name.
- If required, additional Writing Booklets may be obtained from the Examination Supervisor upon request.

Standard Integrals

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \ n \neq -1; \ x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \ a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a}\sin ax, \ a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a}\cos ax, \ a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a}\tan ax, \ a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a}\sec ax, \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a}, \ a > 0, \ -a < x < a$$

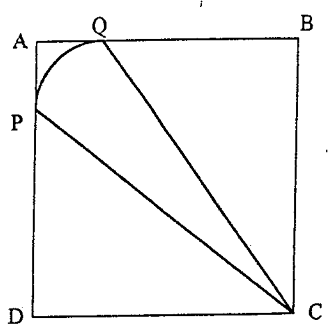
$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left\{x + \sqrt{x^2 - a^2}\right\}, \ |x| > |a|$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left\{x + \sqrt{x^2 + a^2}\right\}$$

NOTE: $\ln x = \log_e x$

Question 1: Marks

(a) In the figure shown, ABCD is a square of side $\sqrt{3}$ units. P and Q are points on AD and AB respectively such that C is the centre of the arc PQ and PD = QB= 1 unit. 4



Calculate the exact values of

(i) the length of the interval PC

(ii) the size of the angle PCD

(iii) the size of the angle PCQ

(iv) the length of the arc PQ

(v) the area of the region APQ.

(b) Find the largest possible domain of the function $f(x) = \ln\left(\frac{x-2}{x}\right)$ 2

(c) Given that $\log_8 2 = \log_x 5$, find x. 2

(d) Find the second derivative of e^{5x^2}. Discuss the concavity of $y = e^{5x^2}$. 2

Question 2: START A NEW BOOKLET Marks

(a) Find the derivative of $\ln\sqrt{\frac{x+1}{x-1}}$. 2

(b) Find the equation of the normal to the curve $y = \ln\sqrt{x}$ at the point where $x = 1$. 3

(c) Find the x coordinates for all the stationary points of $y = \cos\left(x^2\right)$. 2

(d) Show that $P(x) = x^2 + \cos \pi x$ is an even function. 1

(e) If $f(x) = \ln\left(\sin\frac{x}{2}\right)$, evaluate $f'\left(\frac{\pi}{2}\right)$ 2

Question 3: START A NEW BOOKLET — Marks

(a) If the area between the curve $y = \sqrt{\sin(\pi x)}$, the x axis and the lines $x = 0$ and $x = 1$ is rotated about the x axis, find the volume of the solid of revolution formed. — 2

(b) Given that $f(b) - f(a) = (b - a)f'(c)$ where $a < c < b$, find the exact value of c if $b = \ln 2$, $a = 0$, $f(x) = e^x$. — 2

(c) Evaluate $\lim\limits_{x \to 0} \dfrac{\sin 2x}{3x}$. — 1

(d) — 5

(i) Write down the period and amplitude of $y = \sin \pi x$

(ii) On the same set of axes, sketch $y = \sin \pi x$ and $2x + y = 2$.

(iii) Shade the region bounded by these curves and the y axis.

(iv) Find the exact area of the region in (iii).

Question 4: START A NEW BOOKLET — Marks

(a) Evaluate $\displaystyle\int_{\pi}^{\frac{3\pi}{2}} \sin x \cos x \, dx$ — 2

(b) Differentiate $\ln(\ln x)$. — 2

Hence evaluate $\displaystyle\int_{e}^{e^2} \frac{dx}{x \ln x}$

(c) Differentiate the following, simplifying your answers: — 4

(i) $\dfrac{\sin 2x}{1 + \cos 2x}$

(ii) $x e^{x^2}$

(d) The curve $y = \ln x$ between the points where $x = 1$ and $x = 3$ is rotated about the y axis. Find the exact volume of the solid formed. — 2

Marks

Question 5: START A NEW BOOKLET

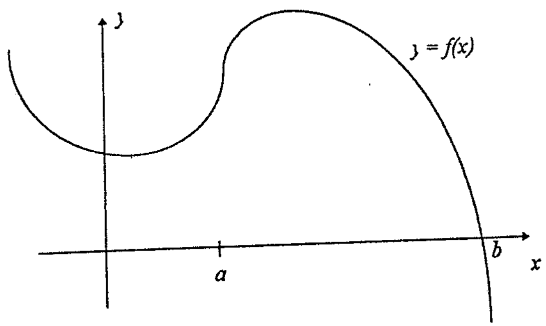
(a) Find the exact value of $f'\left(\frac{\pi}{4}\right)$ if 4

(i) $f(x) = x^2 \sec x$

(ii) $f(x) = \ln(\cot^3 x)$

(b) It is known that $\ln x + \sin x = 0$ has a root close to $x = 0.5$. Use one application of Newton's method to obtain a better approximation. 3

(c) 2



a is the first approximation of the root b of $y = f(x)$.

If Newton's method is used to obtain a second approximation to the root, will it be a better approximation to b?

Explain your answer, using an appropriate diagram if necessary.

(d) Differentiate $\ln\left(\frac{1}{x}\right)$ with respect to x, where $x > 0$. Write your answer in simplest form. 1

Marks

Question 6: START A NEW BOOKLET

(a) Consider $f(x) = \dfrac{\ln\sqrt{x}}{x}$. 6

(i) What is the domain of $f(x)$?

(ii) Find $f'(x)$ and hence find and classify all stationary points.

(iii) Find any points of inflexion.

(iv) Sketch the curve of $y = f(x)$, clearly showing its essential features.

(b) Show that $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$. 4

Hence show that $\sin^4\theta = a + b\cos 2\theta + c\cos 4\theta$, finding a, b and c.

Hence evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^4\theta \, d\theta$.

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V. Good work!

1) A) i.) In Δ PDC, using pythag. theorem,

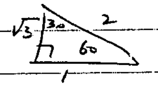
$PC^2 = 1+3$

$PC^2 = 4$

$PC = 2$ unit ($PC > 0 \therefore PC \neq -2$) ✓

b/c it is a length

ii.) $\tan \angle PCD = \frac{1}{\sqrt{3}}$



$\angle PCD = \frac{\pi}{6}$ ✓

iii.) $\Delta PDC \equiv \Delta CBQ$ (SAS)

$\therefore \angle PCD = \angle QCB = \frac{\pi}{6}$ (corresp. $\angle$ of congruent Δ are equal)

$\angle PCQ = 90° - 30° - 30°$ (right $\angle = 90°$)

$= 30°$

$\therefore \angle PCQ = 30°$ ✓

iv.) $l = r\theta$

$PQ = 2 \times \frac{\pi}{6} = \frac{\pi}{3}$ units ✓

v.) Area of APQ = Area AQCP − Area CPQ.

Area AQCP $= (\sqrt{3})^2 - 2\left(\frac{1}{2} \times 1 \times \sqrt{3}\right)$

$= 3 - \sqrt{3}\ u^2$

Area of CPQ $= \frac{\frac{\pi}{6}}{2\pi} \times \pi(2^2)$

$= \frac{\cancel{\pi}}{3\ \cancel{12\pi}} \times 4\pi = \frac{\pi}{3}\ u^2$

Area of APQ $= \left(3 - \sqrt{3} - \frac{\pi}{3}\right) u^2$ ✓

D) $P(x) = x^2 + \cos \pi x$

$P(-x) = (-x)^2 + \cos \pi(-x)$

$= x^2 + \cos(-\pi x)$

$= x^2 + \cos(\pi x)$ since $\cos(-\theta) = \cos\theta \rightarrow$ 4th quad.

$= P(x)$

$\therefore$ even f^n.

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E) $f(x) = \ln\left(\sin \frac{x}{2}\right)$

$f'(x) = \frac{1}{\sin\frac{x}{2}} \times \frac{1}{2}\cos\frac{x}{2}$

$= \frac{\cos\frac{x}{2}}{2\sin\frac{x}{2}} = \frac{1}{2}\cot\frac{x}{2}$ ✓

when $x = \frac{\pi}{2}$,

$f'\left(\frac{\pi}{2}\right) = \frac{1}{2}\cot\frac{\pi}{4}$

$= \frac{1}{2} \times \frac{1}{\tan\frac{\pi}{4}} = \frac{1}{2} \times 1$

$= \frac{1}{2}$ ✓

QUESTION 3

A) $V = \pi \int_0^1 \sin(\pi x)\,dx$

$= \pi \left(-\frac{1}{\pi}\cos(\pi x) \right)_0^1$ ✓

$= \pi \left(-\frac{1}{\pi}\cos\pi + \frac{1}{\pi}\cos 0 \right)$

$= \pi \left(\frac{1}{\pi} + \frac{1}{\pi} \right) = \pi \left(\frac{2}{\pi} \right) = 2u^3$ ✓

B) $f(b) - f(a) = (b-a) f'(c)$

$b = \ln 2$

$a = 0$

$f(x) = e^x \; ; f'(x) = e^x$

$e^b - e^a = (b-a)e^c$ ✓

$e^{\ln 2} - e^0 = (\ln 2) e^c$

$2 - 1 = (\ln 2) e^c$

$1 = (\ln 2) e^c$

$e^c = \frac{1}{\ln 2}$

$c = \ln\left(\frac{1}{\ln 2}\right)$ ✓

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c) $\lim_{x\to 0} \frac{\sin 2x}{3x}$

$= \lim_{x\to 0} \frac{\sin 2x}{2x} \times \frac{2}{3}$

$= 1 \times \frac{2}{3} = \frac{2}{3}$ ✓

D) i.) $y = \sin(\pi x)$

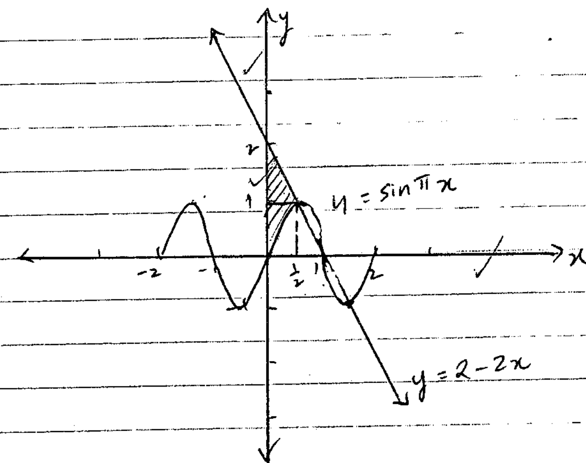
Period $= \frac{2\pi}{n} = \frac{2\pi}{\pi} = 2$

Amplitude $= 1$ ✓

ii.) $y = \sin(\pi x)$

$y = 2 - 2x$

iii.)



iv.) $y = \sin(\pi x)$

$$\text{Area} = \left(1\times\frac{1}{2}\right) - \int_0^{\frac{1}{2}} \sin\pi x \, dx + \frac{1}{2}\times\frac{1}{2}\times 1$$

$$= \frac{1}{2} - \left(-\frac{1}{\pi}\cos\pi x\right)_0^{\frac{1}{2}} + \frac{1}{4}$$

$$= \frac{3}{4} - \left(-\frac{1}{\pi}\cos\frac{\pi}{2} + \frac{1}{\pi}\cos 0\right)$$

$$= \frac{3}{4} - \left(\frac{1}{\pi}\right) = \left(\frac{3}{4} - \frac{1}{\pi}\right) u^2$$

QUESTION 4

A) $\int_{\pi}^{\frac{3\pi}{2}} \sin x \cos x \, dx$

$$= \frac{1}{2}\int_{\pi}^{\frac{3\pi}{2}} \sin 2x \, dx$$

$$= \frac{1}{2}\left(-\frac{1}{2}\cos 2x\right)_{\pi}^{\frac{3\pi}{2}}$$

$$= \frac{-1}{4}\left(\cos 2\left(\frac{3\pi}{2}\right) - \cos 2\pi\right)$$

$$= \frac{-1}{4}(-1-1) = -\frac{1}{4}\times -2 = \frac{1}{2}$$

B) $\frac{d}{dx}\ln(\ln x) = \frac{1}{\ln x}\times\frac{1}{x}$

$$= \frac{1}{x\ln x}$$

$$\int_e^{e^2} \frac{dx}{x\ln x} = \left[\ln(\ln x)\right]_e^{e^2}$$

$$= \ln(\ln e^2) - \ln(\ln e)$$

$$= \ln(2) - \ln(1)$$

$$= \ln 2$$

c) i.) $\frac{d}{dx}\frac{\sin 2x}{1+\cos 2x} = \frac{d}{dx}\frac{2\sin x\cos x}{1+(1-2\sin^2 x)}$

$$= \frac{d}{dx}\frac{2\sin x\cos x}{2-2\sin^2 x}$$

$$= \frac{d}{dx}\frac{\sin x\cos x}{1-\sin^2 x} = \frac{d}{dx}\frac{\sin x\cos x}{\cos^2 x}$$

$$= \frac{d}{dx}\frac{\sin x}{\cos x} = \frac{d}{dx}\tan x$$

$$= \sec^2 x$$

ii.) $\frac{d}{dx} x e^{x^2}$

$$= e^{x^2} + x(2xe^{x^2})$$

$$= e^{x^2}(1+2x^2)$$

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D) $y = \ln x$

when $x=1$, $y=0$

when $x=3$, $y=\ln 3$.

$x = e^y$

$$V = \pi\int_0^{\ln 3} e^{2y} \, dy$$

$$= \pi\left(\frac{1}{2}e^{2y}\right)_0^{\ln 3}$$

$$= \pi\left(\frac{1}{2}e^{2\ln 3} - \frac{1}{2}\right)$$

$$= \frac{\pi}{2}\left(e^{\ln 9} - 1\right) = \frac{\pi}{2}(9-1) = (4\pi)u^3$$

QUESTIONS

A) $f'(\frac{\pi}{4})$

i.) $f(x) = x^2 \sec x = \frac{x^2}{\cos x}$

$$f'(x) = \frac{\cos x(2x) - x^2(-\sin x)}{\cos^2 x}$$

$$= \frac{2x\cos x + x^2 \sin x}{\cos^2 x}$$ ✓

$$f'(\tfrac{\pi}{4}) = \frac{2(\frac{\pi}{4})\cos(\frac{\pi}{4}) + \frac{\pi^2}{16}\sin\frac{\pi}{4}}{\cos^2\frac{\pi}{4}}$$

$$= \frac{\frac{\pi}{2}\cdot\frac{1}{\sqrt{2}} + \frac{\pi^2}{16}\cdot\frac{1}{\sqrt{2}}}{\frac{1}{2}}$$

$$= \left(\frac{\pi}{2\sqrt{2}} + \frac{\pi^2}{16\sqrt{2}}\right) \times 2$$

$$= \frac{8\pi + \pi^2}{16\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \times 2$$ ✓

$$= \frac{\sqrt{2}(8\pi + \pi^2)}{16} = \frac{\sqrt{2}\pi(8+\pi)}{16}$$

ii.) $f(x) = \ln(\cot^3 x)$

$$f'(x) = \frac{1}{\cot^3 x} \times 3\cancel{\cot^2 x} \times -\mathrm{cosec}^2 x$$

$$= \frac{-3\,\mathrm{cosec}^2 x}{\cot x} = \frac{-3}{\sin^2 x} \times \frac{\cancel{\sin x}}{\cos x}$$

$$= \frac{-3}{\sin x \cos x}$$ ✓

$$f'(\tfrac{\pi}{4}) = \frac{-3}{\sin\frac{\pi}{4}\cos\frac{\pi}{4}} = \frac{-3}{\frac{1}{2}} = \boxed{-6}$$ ✓

B) Let $P(x) = \ln x + \sin x = 0$

$$P'(x) = \frac{1}{x} + \cos x$$ ✓

$x = 0.5$

$$x_1 = x - \frac{P(x)}{P'(x)}$$ ✓

$$x_1 = 0.5 - \frac{P(0.5)}{P'(0.5)} = 0.574 \text{ (3dp)}$$ ✓

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c) ~~The~~ If Newton's method is used to obtain a second approx. to the root, it will be worse than b as the tangent drawn to the curve at $x=a$ cuts the x-axis further away from the root. ✓ The tangent at $x = x_1$ ~~...~~ has a gradient of zero, $\frac{1}{P'(x)}$ is undefined.

D) $$\frac{d}{dx}\ln\left(\frac{1}{x}\right) = \frac{1}{\frac{1}{x}} \times -\frac{1}{x^2}$$

$$= \frac{d}{dx}(\ln x^{-1}) = \frac{d}{dx}(-\ln x) = x \times -\frac{1}{x^2} = -\frac{1}{x}$$ ✓

$$= -\frac{1}{x}.$$

QUESTION 6

A) $f(x) = \frac{\ln\sqrt{x}}{x} = \frac{\frac{1}{2}\ln x}{x}$

i.) $x \neq 0$

AND $\sqrt{x} > 0$ $\quad\therefore$ Domain is $x>0$ ✓

$x > 0$

ii.) $f'(x) = \frac{x \times \frac{1}{\sqrt{x}} \times \frac{1}{2\sqrt{x}} - \ln\sqrt{x}}{x^2}$

$= \frac{\frac{x}{2x} - \ln\sqrt{x}}{x^2} = \frac{\frac{1}{2} - \ln\sqrt{x}}{x^2}$ ✓

$= \frac{1 - 2\ln\sqrt{x}}{2x^2} = \frac{1 - \ln x}{2x^2}$

At stat. pts, $f'(x) = 0$

$\frac{1 - 2\ln\sqrt{x}}{2x^2} = 0$

$1 - 2\ln\sqrt{x} = 0$

$2\ln\sqrt{x} = 1; \quad \ln\sqrt{x} = \frac{1}{2}$

$\sqrt{x} = e^{\frac{1}{2}}$

$x = e, \quad y = \frac{\ln\sqrt{e}}{e} = \frac{1}{2e}$

$\therefore$ stat. pt $= \left(e, \frac{1}{2e}\right)$ ✓

iii.) $f''(x) = \frac{2x^2\left(\frac{-2}{\sqrt{x}} \times \frac{1}{2\sqrt{x}}\right) - (1 - 2\ln\sqrt{x})4x}{4x^4}$

$= \frac{2x^2\left(\frac{-2}{2x}\right) - 4x(1 - 2\ln\sqrt{x})}{4x^4}$

$= \frac{-2x - 4x + 8x\ln\sqrt{x}}{4x^4}$

$= \frac{8x\ln\sqrt{x} - 6x}{4x^4}$

$= \frac{2x(4\ln\sqrt{x} - 3)}{4x^4}$

$= \frac{4\ln\sqrt{x} - 3}{2x^3}$ ✓

Note: if you had simplified logs first:

$f'(x) = \frac{1 - \ln x}{2x^2}$

$f''(x) = \frac{(2x^2)(-\frac{1}{x}) - (1 - \ln x)(4x)}{(2x^2)^2}$

$= \frac{-2x(1 - 2 + 2\ln x)}{4x^4}$

$= \frac{-3 + 2\ln x}{2x^3}$

At poss. pts of inflexion, $f''(x) = 0$

$\frac{4\ln\sqrt{x} - 3}{2x^3} = 0$

$4\ln\sqrt{x} - 3 = 0$

$4\ln\sqrt{x} = 3; \quad \ln\sqrt{x} = \frac{3}{4}$

$\sqrt{x} = e^{\frac{3}{4}}$

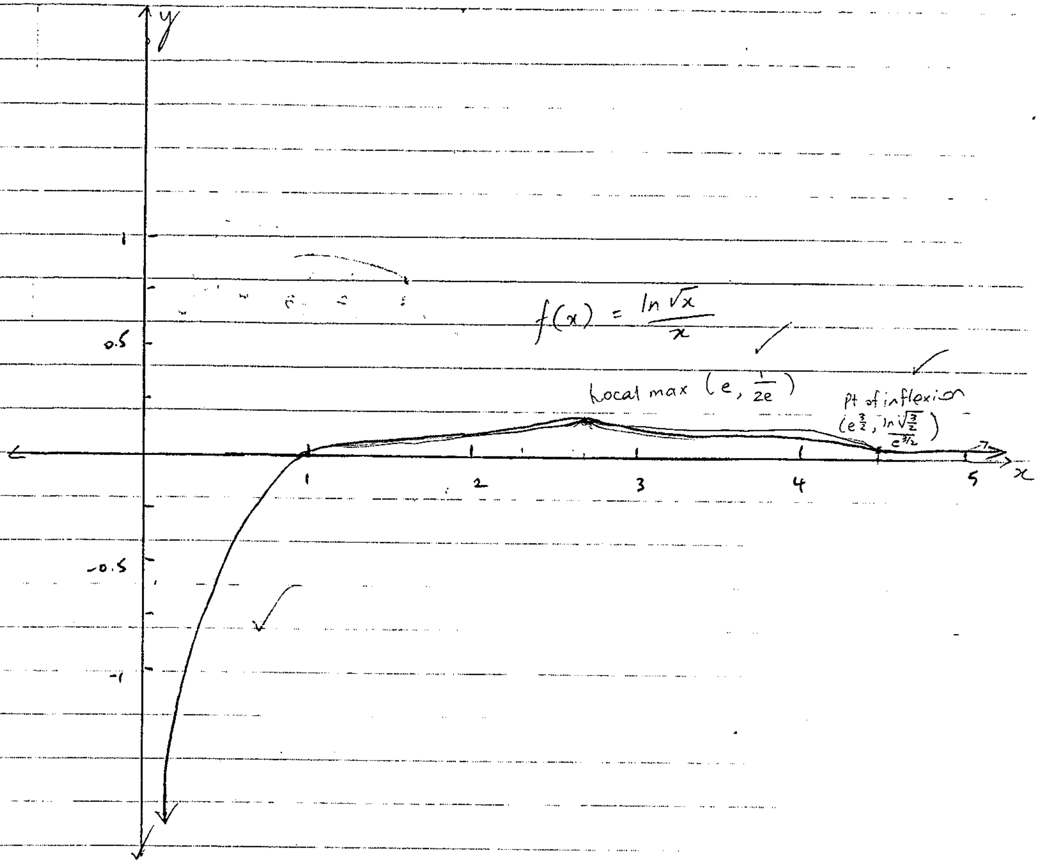
$x = e^{\frac{3}{2}}, \quad f(x) = \frac{\ln\sqrt{\frac{3}{2}}}{e^{\frac{3}{2}}}$ ✓

Check for change in concavity.

x	$e^{\frac{3}{2}-}$	$e^{\frac{3}{2}}$	$e^{\frac{3}{2}+}$
$f''(x)$	<0	0	>0

$\therefore$ since there is change in concavity, ✓

$\left(e^{\frac{3}{2}}, \frac{\ln\sqrt{\frac{3}{4}}}{e^{\frac{3}{2}}}\right)$ is a pt of inflexion



B). Show that $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= 1 - 2\sin^2\theta$

$-2\sin^2\theta = \cos 2\theta - 1$

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$\therefore \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$\sin^4\theta = a + b\cos 2\theta + c\cos 4\theta$

$\left[\frac{1}{2}(1-\cos 2\theta)\right]^2 = a + b\cos 2\theta + c\cos 4\theta$

$\frac{1}{4}(1-2\cos 2\theta + \cos^2 2\theta) = a + b\cos 2\theta + c\cos(2\theta+2\theta)$

$= a + b\cos 2\theta + c(2\cos^2 2\theta - 1)$

$\frac{1}{4}(1-2\cos 2\theta + \cos^2 2\theta) = a + b\cos 2\theta + 2c\cos^2 2\theta - c$

~~1 − 2cos2θ + cos~~ ²2 Equating coefficients.

$\frac{1}{4} = a - c$

$\frac{1}{4} \times -2 = b$; $b = -\frac{1}{2}$

$\frac{1}{4} = 2c$; $c = \frac{1}{8}$

$\frac{1}{4} = a - \frac{1}{8}$; $a = \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$

$\therefore a = \frac{3}{8},\ b = -\frac{1}{2},\ c = \frac{1}{8}$

$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^4\theta \, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{3}{8} - \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta \, d\theta$

$= \left(\frac{3}{8}\theta - \frac{1}{2} \times \frac{1}{2}\sin 2\theta + \frac{1}{8} \times \frac{1}{4}\sin 4\theta\right)_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$= \left(\frac{3}{8}\theta - \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta\right)_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$$= \left(\frac{3}{8}\theta - \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta \right)_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \left[\frac{3}{8}\left(\frac{\pi}{2}\right) - \cancel{\frac{1}{4}\sin\pi} + \cancel{\frac{1}{32}\sin 2\pi} \right] - \left[\frac{3}{8}\left(\frac{-\pi}{2}\right) - \frac{1}{4}\sin(-\pi) + \frac{1}{32}\sin(-2\pi) \right]$$

$$= \frac{3\pi}{16} + \frac{3\pi}{16} = \frac{6\pi}{16} = \frac{3\pi}{8}$$

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