SYDNEY BOYS HIGH SCHOOL

3 UNIT MATHEMATICS

Year 12 Assessment Task: June 2000

Time allowed: 90 minutes (plus 5 minutes reading time)

Total Marks: 72

Examiner: Mr C Kourtesis

INSTRUCTIONS:

- Attempt *all* questions.
- *All* questions are of equal value.
- All necessary working should be shown in every question. Full marks may not be awarded if work is careless or badly arranged.
- Standard integrals are provided on the back of this page. Approved calculators may be used.
- *Each* question attempted is to be returned in a *separate* Writing Booklet clearly marked Question 1, Question 2 etc. on the cover. Each booklet must show your name.
- If required, additional Writing Booklets may be obtained from the Examination Supervisor upon request.

Question 1 (12 marks)

(a) Differentiate the following with respect to x

(i) $y = e^{1-4x}$

(ii) $y = \log_e(\sin x)$

(iii) $y = \cos^{-1}(2x)$

(iv) $y = \cos^2 x$

(b) Find the indefinite integrals

(i) $\int \frac{dx}{1+2x}$

(ii) $\int \frac{dx}{4+9x^2}$

(c) Find $\int_0^1 (e^{5x} - 1)dx$ in terms of e.

Question 2 (12 marks)

(a) Find $\lim_{x \to 0} \frac{\sin 5x}{2x}$

(b) Given the curve

$f(x) = (\log_e x)^2$

Find:

(i) $f''(x)$

(ii) the values of x for which the curve is concave up.

(c) If $f(x) = \cos x - x^2$

(i) Taking $x = 1$ as the first approximation to $f(x) = 0$, use one application of Newton's Method to find a better approximation.

(ii) What other solution does $f(x) = 0$ have? (Give reasons)

Question 3 (12 marks)

(a) Use the substitution $u = x^2 + 4$ to find the indefinite integral

$$\int \frac{x}{\sqrt{x^2 + 4}} \, dx$$

(b) Given that

$$f(x) = 4 + 3x - x^2$$

(i) Find the largest possible domain consisting of positive real numbers such that $f(x)$ is one to one.

(ii) Find the inverse function $f^{-1}(x)$ stating clearly the domain and range.

(c) Evaluate

$$\int_0^{\frac{\pi}{2}} \sin x \cos x dx$$

(d) If $\sin\left[2\tan^{-1}\frac{3}{7}\right] = \frac{a}{b}$ where a and b are integers, find the values of a and b.

Question 4 (12 marks)

(a) (i) Find the equation of the tangent to

$y = \log_e x$ at the point (a,b)

(ii) Find the values of a and b given that the tangent to $y = \log_e x$ passes through the origin

(b) Consider the function

$f(x) = xe^{-x}$

(i) Determine the coordinates of any stationary points and determine their nature.

(ii) Find the coordinates of any point of inflexion.

(iii) Find $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$

(iv) Sketch the graph of $y = f(x)$

Question 5 (12 marks)

(a) If $y = e^{\sin^{-1}(4x)}$ find $\frac{dy}{dx}$

(b) (i) Show that $f(x) = \sin(\sin^{-1} x)$ is an odd function

(ii) Sketch the graph of $f(x) = \sin(\sin^{-1} x)$

(c) Using the substitution $u = e^x - 1$ evaluate

$$\int_1^2 \frac{e^{2x}}{e^x - 1} dx$$

(d) Find the volume of the solid of revolution formed when the area bounded by the curve $y = \cos 2x$, the x axis and the ordinates at $x = 0$ and $x = \frac{\pi}{2}$ is rotated about the x axis.

Question 6 (12 marks)

(a) (i) Show that $\frac{d}{dx}(x\tan^{-1}x) = \tan^{-1}x + \frac{x}{1+x^2}$

(ii) Hence find the area enclosed by the curve $y = \tan^{-1}x$, the x axis and the lines $x=0$ and $x = 4$

(b) A particle moves in a straight line and its displacement, x cm from a fixed origin point after t seconds is determined by the function:

$$x = \sin t - \sin t \cos t - 2t$$

(i) Find the initial displacement and velocity of the particle.

(ii) Show that the particle never comes to rest and always moves in one particular direction, stating what this direction is.

(iii) Show that the particle initially has zero acceleration and find the first occasion after this when zero acceleration occurs again.

SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2000
HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK # 3

Mathematics Extension 1

Sample Solutions

1 (a) (i) $\frac{d}{dx}\left(e^{1-4x}\right) = -4e^{1-4x}$

(ii) $\frac{d}{dx}\left(\ln|\sin x|\right) = \frac{\cos x}{\sin x} = \cot x$

(iii) $\frac{d}{dx}\left(\cos^{-1}2x\right) = \frac{-2}{\sqrt{1-4x^2}}$

(iv) $\frac{d}{dx}\left(\cos^2 x\right) = -2\sin x\cos x$

(b) (i) $\int \frac{dx}{1+2x} = \frac{1}{2}\int \frac{2\,dx}{1+2x}$

$= \frac{1}{2}\ln|1+2x| + c$

(ii) $\int \frac{dx}{4+9x^2}$

$= \frac{1}{9}\int \frac{dx}{\frac{4}{9}+x^2}$

$= \frac{1}{9} \times \frac{3}{2} \times \tan^{-1}\left(\frac{3x}{2}\right) + c$

$= \frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right) + c$

(c) $\int_0^1 (e^{5x}-1)\,dx$

$= \left[\frac{1}{5}e^{5x} - x\right]_0^1$

$= \left(\frac{1}{5}e^5 - 1\right) - \left(\frac{1}{5} - 0\right)$

$= \frac{1}{5}(e^5 - 6)$

(d) $\sin^{-1}\left(\frac{1}{2}\right) + \cos^{-1}\left(-\frac{1}{2}\right)$

$= \frac{\pi}{6} + \frac{2\pi}{3}$

$= \frac{5\pi}{6}$

OR

$\sin^{-1}\left(\frac{1}{2}\right) + \pi - \cos^{-1}\left(\frac{1}{2}\right)$

$= \pi + \frac{\pi}{6} - \frac{\pi}{3}$

$= \frac{5\pi}{6}$

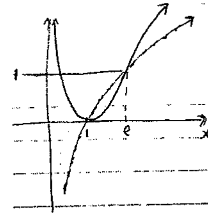
2 (a) $\lim_{x\to 0}\frac{\sin 5x}{2x} = \lim_{x\to 0}\frac{5x}{2x} = \frac{5}{2}$ $\left(\lim_{x\to 0}\sin x = \lim_{x\to 0} x\right)$

ie $\sin x \doteqdot x$, x small

(b) $f(x) = (\ln x)^2$

(i) $f'(x) = \frac{2\ln x}{x}$ ⇒ $f''(x) = \frac{x \times \frac{2}{x} - 2\ln x}{x^2} = \frac{2(1-\ln x)}{x^2}$

2 (b) (ii) concave up $f''(x) > 0$



ie. $\frac{2(1-\ln x)}{x^2} > 0$ $(x \neq 0)$

$2(1-\ln x) > 0$

$\therefore \ln x < 1$

$\therefore x < e$

$\therefore \boxed{0 < x < e}$ $f(x): x > 0$

(c) $f(x) = \cos x - x^2$

$f'(x) = -\sin x - 2x$

(remember RADIANS)

(i) $x_0 = 1$

$\alpha = x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{\cos 1 - 1}{-\sin 1 - 2}$

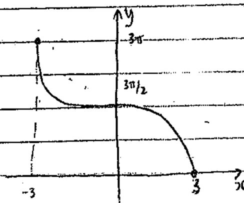
$\doteqdot 0.84$ (2dp)

(ii) $f(x)$ is even, so $x = -\alpha$ is also an approximation to a root.

(d) $y = 3\cos^{-1}\left(\frac{x}{2}\right)$

D: $-1 \le \frac{x}{2} \le 1 \Rightarrow -2 \le x \le 2$

R: $0 \le \frac{y}{3} \le \pi \Rightarrow 0 \le y \le 3\pi$



(NO HPOI at $x=0$)

3 (a) $\int \frac{x}{\sqrt{x^2+4}}\,dx$ | $u = x^2+4$, $du = 2x\,dx$

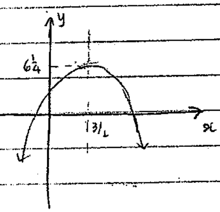
$= \frac{1}{2}\int \frac{du}{\sqrt{u}}$

$= \frac{1}{2} \times 2\sqrt{u} + C$

$= \sqrt{x^2+4} + C$

(b) $f(x) = 4 + 3x - x^2$

(i) $= \frac{25}{4} - (x - \frac{3}{2})^2$



$\therefore x \geq 3/2$ f is 1:1

(ii) $f: x \geq \frac{3}{2},\ y \leq 6\frac{1}{4}$

$f^{-1}: x \leq 6\frac{1}{4},\ y \geq 3/2$

$f: y = 4 + 3x - x^2$

$f^{-1}: x = 4 + 3y - y^2$

$\therefore y^2 - 3y + x - 4 = 0$

$\therefore y = \frac{3 \pm \sqrt{9 - 4(x-4)}}{2}$

$= \frac{3}{2} \pm \frac{\sqrt{25-4x}}{2}$

$\therefore y = \frac{3 + \sqrt{25-4x}}{2} \quad (y \geq \frac{3}{2})$

(c) $\int_0^{\pi/2} \sin x \cos x\, dx$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin 2x\, dx$

$= -\frac{1}{4} \cos 2x \Big]_0^{\pi/2}$

$= \frac{-1}{4}[-1-1]$

$= \frac{1}{2}$

OR

$= \int_0^{\pi/2} \sin x\, d(\sin x)$

$= \frac{1}{2}\sin^2 x \Big]_0^{\pi/2}$

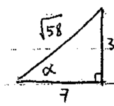
$= \frac{1}{2}[1-0]$

$= \frac{1}{2}$

3 (d) $\sin\left[2\tan^{-1}\frac{3}{7}\right]$

let $\alpha = \tan^{-1}\frac{3}{7}$

$\therefore \tan\alpha = \frac{3}{7}$



$= 2\sin\alpha\cos\alpha$

$= 2 \cdot \frac{3}{\sqrt{58}} \cdot \frac{7}{\sqrt{58}}$

$= \frac{42}{58}$

$= \frac{21}{29}$

$= \frac{a}{b}$

(4) (a) (i) $y = \ln x$

$y' = \frac{1}{x}$

$\therefore m = \frac{1}{a}$

$y - b = \frac{1}{a}(x - a)$

$\therefore ay - ab = x - a$

$\therefore x - ay + ab - a = 0$

$[b = \ln a]$

(ii) $ab - a = 0$

$a(b-1) = 0$

$a = 0,\ b = 1$

BUT $a \neq 0 \because \ln a \Rightarrow a > 0$

$\therefore b = 1 \Rightarrow x = e$

$\boxed{\therefore a = e,\ b = 1}$

(b) $y = xe^{-x}$

(i) $y' = e^{-x} - xe^{-x} = e^{-x}(1-x)$

$y' = 0 \Rightarrow x = 1$

$y = e^{-1}$

$y'' = -e^{-x} - (e^{-x}(1-x))$

$= e^{-x}(x-2)$

$x = 1 \quad y'' = e^{-1}(-1) < 0$

$\therefore$ max

(ii) P.O.I. $y'' = 0$

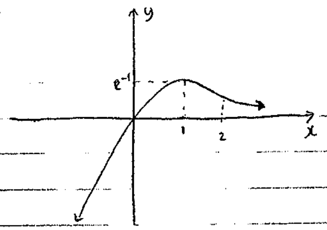
$\therefore x = 2,\ y = 2e^{-2}$

x	1	2	3
y''	$-e^{-1}$	0	e^{-1}

$\therefore (2, 2e^{-2})$ is a P.O.I.

4 (b) (iii) $x \to \infty$, $y \to 0$

$x \to -\infty$, $y \to -\infty$



5 (a) $y = e^{\sin^{-1}(4x)}$

$\frac{dy}{dx} = \frac{4}{\sqrt{1-16x^2}} e^{\sin^{-1}(4x)}$

(b) (i) $f(x) = \sin(\sin^{-1}x)$

$f(-x) = \sin(\sin^{-1}(-x))$

$= \sin(-\sin^{-1}x)$ [$\sin^{-1}x$ is odd]

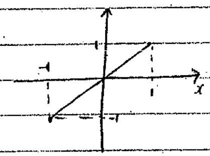
$= -\sin(\sin^{-1}x)$ [$\sin x$ is odd]

$= -f(x)$

$\therefore f(x)$ is odd

(ii) $f(x)$: $-1 \le x \le 1$

$f(x) = x$



(c) $\int_1^2 \frac{e^{2x}}{e^x - 1} dx$

$u = e^x - 1$

$\therefore du = e^x dx$

$x = 1$, $u = e - 1$

$x = 2$, $u = e^2 - 1$

$= \int_1^2 \frac{e^x \cdot e^x}{e^x - 1} dx$

$= \int_{e-1}^{e^2-1} \frac{(u+1)}{u} du = \int_{e-1}^{e^2-1} \left(1 + \frac{1}{u}\right) du$

$= u + \ln u \Big]_{e-1}^{e^2-1}$

$= e^2 - 1 - (e-1) + \ln\left(\frac{e^2-1}{e-1}\right) = e^2 - e + \ln(e+1)$

(5) (d) $V = \pi \int_0^{\pi/2} \cos^2 2x \, dx$

$= \frac{\pi}{2} \int_0^{\pi/2} 2\cos^2 2x \, dx$

$= \frac{\pi}{2} \int_0^{\pi/2} (1 + \cos 4x) \, dx$

$= \frac{\pi}{2} \left[x + \frac{1}{4}\sin 4x\right]_0^{\pi/2}$

$= \frac{\pi}{2} \left[\frac{\pi}{2} + \frac{1}{4} \times 0\right]$

$= \frac{\pi^2}{4}$ c.u.

(6) (a) (i) $y = x\tan^{-1}x$

$y' = \tan^{-1}x + x \times \frac{1}{1+x^2}$

$= \tan^{-1}x + \frac{x}{1+x^2}$

$\therefore \int \left(\tan^{-1}x + \frac{x}{1+x^2}\right) dx = x\tan^{-1}x$

(ii) $\int_0^4 \tan^{-1}x \, dx$

$= x\tan^{-1}x \Big]_0^4 - \int_0^4 \frac{x}{1+x^2} dx$

$= 4\tan^{-1}4 - \frac{1}{2}\int_0^4 \frac{2x}{1+x^2} dx$

$= 4\tan^{-1}4 - \frac{1}{2}\ln(1+x^2)\Big]_0^4$

$= 4\tan^{-1}4 - \frac{1}{2}\ln 17$

(b) $x = \sin t - \sin t\cos t - 2t$

$= \sin t - \frac{1}{2}\sin 2t - 2t$

(i) $t = 0$: $x = 0$

$v = \dot{x} = \cos t - \cos 2t - 2$

$t = 0$: $v = 1 - 1 - 2$

$= -2$

(ii) $v = \cos t - \cos 2t - 2$

$= \cos t - (2\cos^2 t - 1) - 2$

$= \cos t - 2\cos^2 t - 1$

$v = 0 \Rightarrow 2\cos^2 t - \cos t + 1 = 0$

$\Delta = 1 - 4 \times 2 \times 1$

$= -7 < 0$

$\therefore v \ne 0$

$\therefore$ initially $v = -2$, so it is always moving to the left.

6 (b) (iii)

$v = \cos t - \cos 2t - 1$

$a = -\sin t + 2\sin 2t$

$t=0$: $a = -0 + 2\times 0 = 0$

$a = -\sin t + 4\sin t\cos t$

$= \sin t\,(4\cos t - 1)$

$\therefore a = 0 \Rightarrow \sin t = 0$ or $\cos t = \frac{1}{4}$

$t = 0, \pi, 2\pi$ $\quad t = \cos^{-1}(\frac{1}{4}), 2\pi - \cos^{-1}(\frac{1}{4}), \ldots$

$\therefore$ next time is $\cos^{-1}(\frac{1}{4})$ secs.

$\doteqdot 1.32$ (2dp)