4

# SYDNEY BOYS' HIGH SCHOOL

AUGUST 1996 TRIAL HSC

# MATHEMATICS

## 3 UNIT (ADDITIONAL)
## AND
## 3/4 UNIT (COMMON)

*Time allowed - 2 hours*
*(Plus 5 minutes reading time)*

*Examiner: PS Parker*

**DIRECTIONS TO CANDIDATES**

- Attempt ALL questions.
- ALL questions are of equal value.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board-approved calculators may be used.
- Standard integrals are provided at the back of the examination paper.
- Each section is to be returned in a separate Writing Booklet clearly marked with the section and the questions on the cover. Start each question on a new page, clearly showing your name, class and teacher's name. Second and subsequent Writing Booklets are to be inserted in the first Writing Booklet for the section.
- If required, additional Writing Booklets may be obtained from the Examination Supervisor upon request.
- This is a trial paper and does not necessarily reflect the format or content of the HSC examination for this subject.

**Question 1 (Start a new page)** **Marks**

(a) The point $P(8, -2)$ divides the interval joining $Q(2, 7)$ and $R(6, 1)$ externally in the ratio $k:1$. What is the value of $k$? **2**

(b) A tank is emptied by a tap from which water flows so that, until the flow ceases, the rate after $t$ minutes is $R$ litres/minute where **4**

$$R = (t-3)^2$$

(i) What is the initial rate of flow?

(ii) How long does it take to empty the tank?

(iii) How long will it take (to the nearest second) for the flow to drop to 20 litres/minute?

(iv) How much water was in the tank initially?

(c) A circle has equation $x^2 + y^2 + 4x - 6y = 0$ **4**

(i) Find the centre and the radius of the circle.

(ii) The line $3x + 2y = 0$ meets this circle in two points, $A$ and $B$.

($\alpha$) Find the coordinates of $A$ and $B$.

($\beta$) Calculate the distance $AB$.

(d) Evaluate $\lim\limits_{x \to 0} \dfrac{\sin 7x}{6x}$ **2**

(e) $P(x) = 10x^4 - 33x^3 - 7x^2 + 45x + 9$ **3**

Given $P(-1) = P(3) = 0$. Find all the zeros of $P(x)$.

**Question 2 (Start a new page)** Marks

(a) A subcommittee of seven persons is chosen at random from 7 men and 5 women. Find the probability that the subcommittee **4**

(i) consists entirely of men.

(ii) included all the women.

(iii) includes a majority of women.

(b) Let $f(x) = 2x^3 + 2x - 1$ **4**

(i) Show that $f(x)$ has a root between $x = 0$ and $x = 1$.

(ii) By considering $f'(x)$, explain why this is the only root of $f(x)$.

(iii) Taking $x = 0$ as an initial approximation, use Newton's Method to find a closer approximation.

(c) Let $F(x) = 3\sin^{-1}(4x)$ $\qquad \sin^{-1}x \quad -1 \le x \le 1$ **3**

$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

(i) Write down the domain and range of $F(x)$.

(ii) Sketch $F(x)$.

(d) Find the indefinite integral $\displaystyle\int \frac{4x+9}{4+9x^2}\,dx$ **4**

**Question 3 (Start a new page)** **Marks**

(a) A preschool $P$, is due south of a digital phone tower and a surf club $S$ is due east of it. The house $H$ of one of the preschool children is between and on the line joining the preschool and the surf club. The angles of elevation to the top of the tower from points $P$, $S$ and $H$ are $25^0$, $32^0$ and $29^0$ respectively. The height of the phone tower is $h$ and $O$ is the base of the tower. **4**

(i) Draw a suitable diagram of the above information

(ii) Show that $OS = h\cot 32^0$.

(iii) Find similar results for the lengths of $OH$ and $OP$.

(iv) Use $\Delta POS$ to show that $\angle OSP = 53^0$.

(v) Use $\Delta OHS$ to show that $\angle OHS = 45^0$.

(vi) Show that the bearing of the house from the foot of the tower is $172^0$.

(b) A country's population with a constant annual growth rate, $k$, and a constant immigration rate of $I$ persons per year entering the country is governed by the equation: **7**

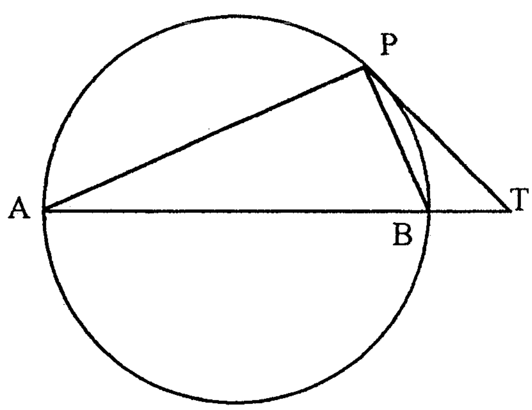
$$\frac{dP}{dt} = kP + I$$

(i) Show that a solution of this equation is $P(t) = P_0 e^{kt} + \frac{I}{k}(e^{kt} - 1)$,

where $P_0$ is a constant

(ii) The US population was 222 million people in 1980. Allowing for immigration at the rate of half a million people per year for the next 20 years, assuming a natural growth rate of 1% annually, what will be the population in 2000?

(c) AB is a diameter of the circle. PT is the tangent and $\angle APT = 108^0$ **4**




Calculate $\angle ATP$ giving reasons.

**Question 4 (Start a new page)** **Marks**

(a) $P(x, y)$ is a variable point on the line $x = 2$ **4**

(i) Sketch a diagram of this situation

(ii) Show that $\theta = \tan^{-1}\left(\frac{y}{2}\right)$, where $\theta$ is the angle between OP and the positive direction of the $x$ axis. Hence find $\frac{d\theta}{dy}$.

(b) (i) Show that $\int_0^{\frac{\pi}{4}} \cos^2\theta \, d\theta = \frac{\pi + 2}{8}$ **8**

(ii) Hence using the substitution $x = 2\sin\theta$, or otherwise, evaluate $\int_0^{\sqrt{2}} \sqrt{4 - x^2}\, dx$ .

(c) If $\alpha, \beta$ and $\gamma$ are the roots of $8x^3 - 6x + 1 = 0$ then evaluate $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$ **3**

**Question 5 (Start a new page)** **Marks**

(a) $f(x) = g(x) - \ln\{g(x)+1\}$ **5**

(i) Prove that $f'(x) = \dfrac{g(x).g'(x)}{g(x)+1}$

(ii) Hence evaluate $\displaystyle\int \frac{\sin 2x \cos 2x}{\sin 2x + 1}\,dx$

(b) By using the substitution $u^2 = x+1$, find the volume of the solid formed by rotating the area bounded by the curve $y = \dfrac{x-1}{\sqrt{x+1}}$, the $x$ axis and the lines $x = 3$ and $x = 8$, about the x axis **6**
Leave your answer as an exact value.

(c) $T(2t, t^2)$ is a variable point on the parabola $x^2 = 4y$ whose vertex is $O$. $N$ is the foot of the ordinate from $T$ and the perpendicular from $N$ to $OT$ meets $OT$ at $P$. Prove that the locus of $P$ is a circle and state its centre and radius. **4**

**Marks**

## Question 6 (Start a new page)

(a) In an acute angled triangle $ABC$, angle B > angle C. The line $BD$ is drawn so that $\angle DBC = \angle ACB$ and $BD = AC$. If this line cuts $AC$ in $O$ and $AD$ and $DC$ are joined, prove that: **5**

(i) $AO = OD$

(ii) $\Delta ADB \equiv \Delta DAC$

(iii) $AD \parallel BC$

(b) A particle, $P$, is moving in a straight line, with its motion given by $\ddot{x} = -9x$ where $x$ is the displacement of $P$ from $O$. **7**
Initially $P$ is 4 m on the right side of $O$ and is moving towards $O$ with velocity 12 m/s.

(i) Show that $\ddot{x} = \dfrac{d}{dx}(\tfrac{1}{2}v^2)$

(ii) Show that its speed at position $x$ is $3\sqrt{32 - x^2}$ m/s

(iii) Verify that $x = 4\sqrt{2}\cos(\tfrac{\pi}{4} + 3t)$ and hence find its velocity, $v$, as a function of $t$.

(iv) Find the greatest (α) speed of $P$

(β) acceleration of $P$

(δ) displacement of $P$ from $O$

(v) Find the period of the motion

(c) (i) Simplify the following expression $\dfrac{\sin(x - \frac{\pi}{6}) + \sin(x + \frac{\pi}{6})}{\cos(x - \frac{\pi}{6}) - \cos(x + \frac{\pi}{6})}$ **3**

(ii) If $f(x) = \dfrac{\sin(x - \frac{\pi}{6}) + \sin(x + \frac{\pi}{6})}{\cos(x - \frac{\pi}{6}) - \cos(x + \frac{\pi}{6})}$, for what values of $x$ is $f(x)$ independent of $x$. Hence sketch the function.

**Question 7 (Start a new page)** **Marks**

(a) Solve $2\cos(x - \frac{7\pi}{18}) + 1 = 0$ for $0 \le x \le 2\pi$ **3**

(b) (i) Prove that $1^2 + 2^2 + 3^2 + \ldots + n^2 = \sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$, by the process of mathematical induction. **6**

(ii) Draw the graph of $y = x^2$ and construct $n$ trapezia between the curve and the $x$ axis from $x = 0$ to $x = 1$. The width of each trapezium is $\frac{1}{n}$ units.

(α) If $S$ denotes the sum of the areas of these trapezia, show that

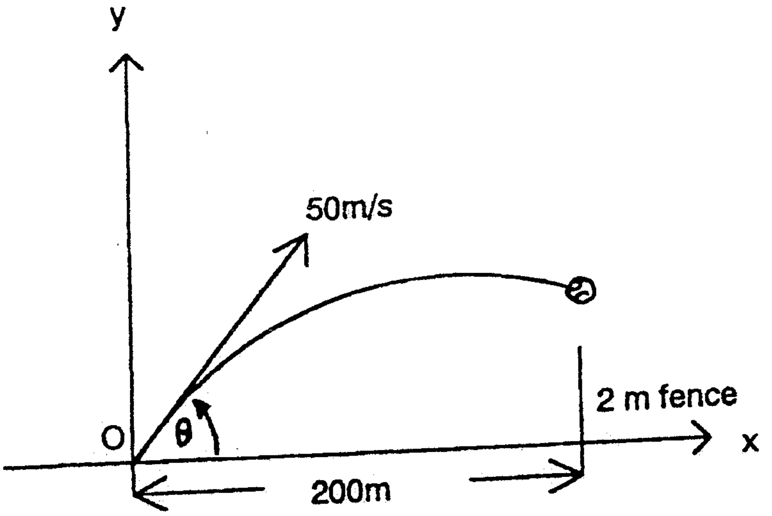
$$S = \frac{1}{2} \cdot \frac{1}{n}\left\{(0+1) + \frac{2}{n^2}\left(1^2 + 2^2 + 3^2 + \ldots + (n-1)^2\right)\right\}$$

(β) Using the result from part (i) above, show that

$$S = \frac{1}{2n}\left\{1 + \frac{1}{3n}(n+1)(2n+1) - 2\right\}$$

(δ) If A denotes the exact area under the curve show that $A = \lim_{n \to \infty} S$ and hence evaluate A.

(c) A method to score a home run in a baseball game is to hit the ball over the boundary fence on the full. 6

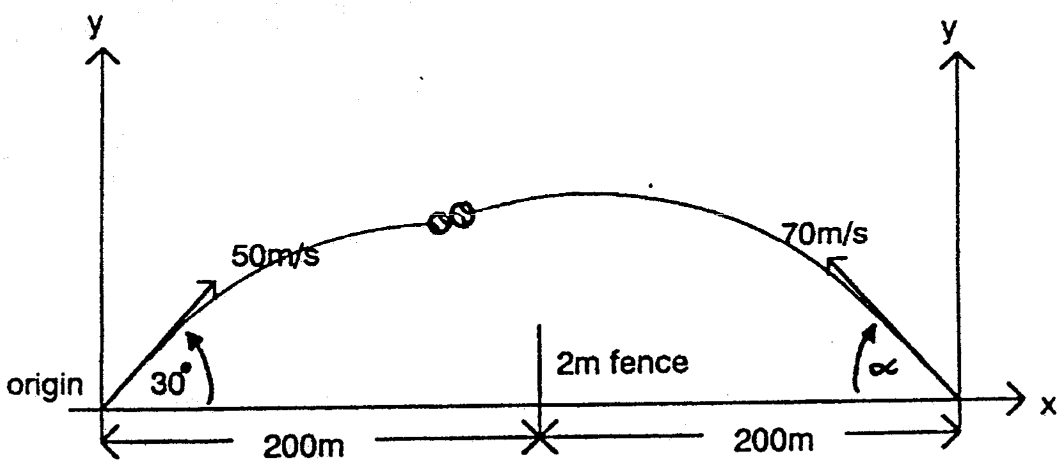



A ball is hit at 50 metres per second. The fence 200 metres away is 2 metres high. You may neglect air resistance and acceleration due to gravity can be taken as 10 metres per second per second and you may assume the following equations of motion:
$x = 50t\cos\theta$ and $y = 50t\sin\theta - 5t^2$

(i) Show that if ball just clears the 2 metre boundary fence then $80\tan^2\theta - 200\tan\theta + 82 = 0$, where $\theta$ is the angle of projection.

(ii) In what range of values must $\theta$ lie to score a home run by this method?

(iii) In an adjacent field another ball is hit at the same instant at 70 metres per second and the balls collide. Assuming that $\theta = 30^\circ$, find the angle of projection, $\alpha$, of the second ball and the time and position where the balls collide.




**END OF THE PAPER**

Stephanie Lim

Sydney Boys 3/4 U 1996 HSC trial.

102/105

Excellent work

Question 1

A) $8 = \frac{k(6)-2}{k-1}$ ; $-2 = \frac{k(1)-7}{k-1}$

$8k - 8 = 6k - 2$

$2k - 8 = -2$ ✓

$2k = 6$

$\underline{k = 3}$ ✓

B) $R = (t-3)^2$

i.) when $t=0$, $R = (-3)^2 = \underline{9\ L/min}$ ✓

ii.) ~~Find $t$ when $R=0$.~~

~~$0 = (t-3)^2$ ; $t = 3$ ∴ it takes 3 min~~

$\frac{dV}{dt} = (t-3)^2$

$\int_V^0 dV = \int_0^t (t-3)^2\,dt$

$[V]_V^0 = \left[\frac{(t-3)^3}{3}\right]_0^t$

$\therefore -V = \frac{(t-3)^3}{3} - (-9)$

$\therefore V = 9 - \frac{(t-3)^3}{3}$

When $V = 0$

$\frac{(t-3)^3}{3} = 9$

$t - 3 = \sqrt[3]{27}$

$\underline{t = 6s}$.

iii.) $20 = (t-3)^2$

$t - 3 = \sqrt{20}$ $(R>0)$

$t = 2\sqrt{5} + 3$

$= 7 min\ 28 sec$ ✓

(iv) When $t = 0$

$V = 9 - \frac{(-3)^3}{3}$

$= 9 + 9$

$= \underline{18\ Litres}.$

c) $x^2+y^2+4x-6y=0.$

i.) complete the square.

$(x^2+4x+4)+(y^2-6y+9)-4-9=0$

$(x+2)^2+(y-3)^2=13.$

$\therefore$ eqt of a circle with centre $(-2,3)$, radius $=\sqrt{13}$

ii.) $2y+3x=0$

A) $2y=-3x$

$y=\frac{-3x}{2}$ (sub into eqt. of circle)

$(x+2)^2+\left(\frac{-3x}{2}-3\right)^2-13=0$

$x^2+4x+\cancel{4}+\frac{9x^2}{4}+\cancel{9}+9x-\cancel{13}=0$

$4x^2+52x+9x^2=0$

$13x^2+52x=0$

$x^2+4x=0$ ; $x(x+4)=0$

$x=0$ or $x=-4$

$y=0$ $y=6$

$\therefore$ $A(0,0)$, $B(-4,6)$

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B) $AB=\sqrt{6^2+(-4)^2}=\sqrt{36+16}=\sqrt{52}$

$=2\sqrt{13}$ units

D) $\lim_{x\to 0}\frac{\sin 7x}{6x}=\lim_{x\to 0}\frac{\sin 7x}{7x}\times\frac{7}{6}$

$=1\times\frac{7}{6}=\boxed{\frac{7}{6}}$

E) $P(x)=10x^4-33x^3-7x^2+45x+9$

Given that $x=-1$ and $x=3$ are zeros of $P(x)$. $\therefore$ $(x+1)$ and $(x-3)$ are factors of $P(x)$.

$(x+1)(x-3)=x^2-2x-3$

$$\begin{array}{r l}
 & 10x^2-13x-3 \\
x^2-2x-3 \,\big) & 10x^4-33x^3-7x^2+45x+9 \\
 & 10x^4-20x^3-30x^2 \\
\hline
 & -13x^3+23x^2+45x \\
 & -13x^3+26x^2+39x \\
\hline
 & -3x^2+6x+9 \\
 & -3x^2+6x+9 \\
\hline
 & 0\;\;0\;\;0
\end{array}$$

$\therefore P(x)=(x+1)(x-3)(10x^2-13x-3)$

Zeros of $P(x)$ are $\boxed{x=-1}$, $\boxed{x=3}$, $\therefore x = \dfrac{13 \pm \sqrt{169-4(10)(-3)}}{20}$

$= \dfrac{13 \pm 17}{20}$

$\boxed{x = \dfrac{3}{2}}$ or $\boxed{x = -\dfrac{1}{5}}$ ✓

Question 2

A) i.) P(entirely all men) $= \dfrac{{}^7C_7 \times {}^5C_0}{{}^{12}C_7} = \dfrac{1}{792}$ ✓

ii.) P(included all women) $= \dfrac{{}^7C_2 \times {}^5C_5}{{}^{12}C_7} = \dfrac{7}{264}$ ✓

iii) P(include majority women) $= P(4W, 3M) + P(5W, 2M)$

$= \dfrac{{}^5C_4 \times {}^7C_3}{{}^{12}C_7} + \dfrac{{}^5C_5 \times {}^7C_2}{{}^{12}C_7} = \dfrac{49}{198}$ ✓

B) $f(x) = 2x^3 + 2x - 1$

i.) $f(0) = -1 < 0$

$f(1) = 3 > 0$ $f(x)$ is a continuous function

$\therefore f(x)$ has a root $0 < x < 1$.

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ii.) $f'(x) = 6x^2 + 2 > 0$ for all real $x$.

$\therefore$ the curve is constantly increasing with NO turning pt.

$\therefore$ the graph / curve cuts the $x$-axis only ONCE.

$\therefore$ there is only one root of $f(x)$.

iii.) $x = 0$

$x_1 = x - \dfrac{f(x)}{f'(x)}$ ✓

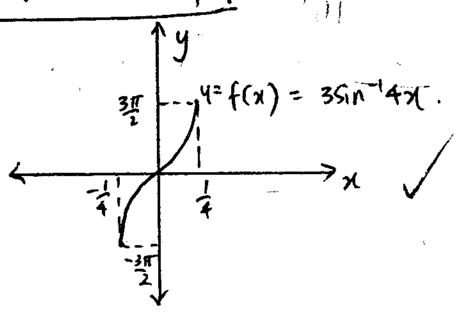
$= 0 - \dfrac{f(0)}{f'(0)} = 0 - \left(\dfrac{-1}{2}\right) = \dfrac{1}{2}$ ✓

c) $f(x) = 3\sin^{-1} 4x$

i.) D: $-1 \le 4x \le 1$

$\boxed{D: -\dfrac{1}{4} \le x \le \dfrac{1}{4}}$ ✓ $\boxed{R: -\dfrac{3\pi}{2} \le f(x) \le \dfrac{3\pi}{2}}$ ✓

ii.)




✓

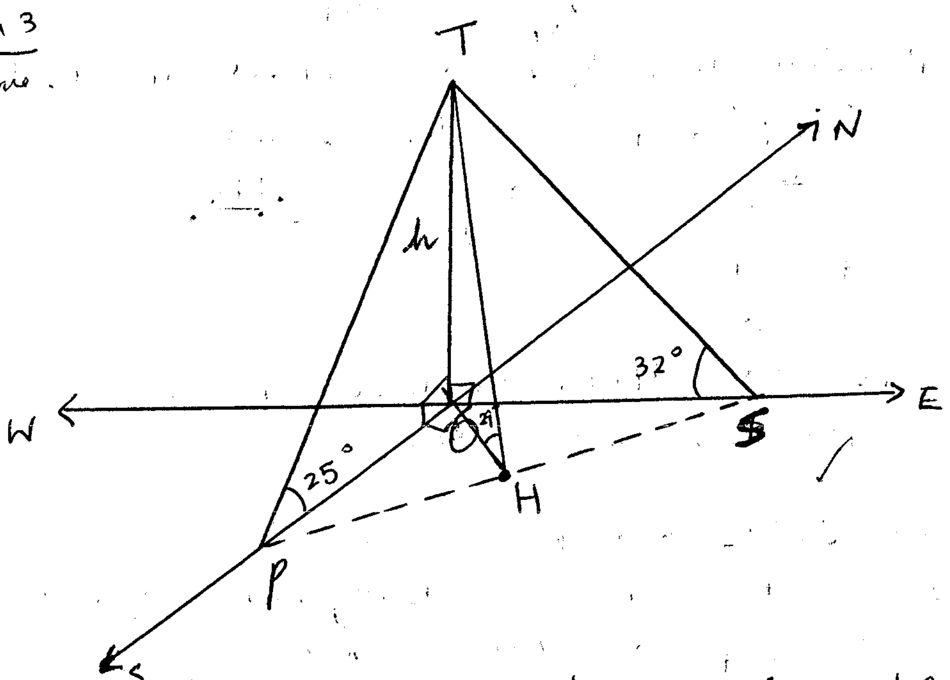
D) $\displaystyle\int \frac{4x+9}{4+9x^2}\,dx = \int \frac{4x}{4+9x^2} + \frac{9}{4+9x^2}\,dx$ ✓

$= \frac{4}{18}\ln(4+9x^2) + 9\int \frac{1}{9(\frac{4}{9}+x^2)}\,dx$ ✓ ✓

$= \frac{4}{18}\ln(4+9x^2) + \frac{3}{2}\tan^{-1}\frac{3x}{2} + C$ ✓

Question 3

i.) done.




ii.) In $\Delta TOS$, $\tan 32 = \frac{h}{OS}$ ; $OS = \frac{h}{\tan 32}$ $\therefore$ $OS = h\cot 32$ ✓

iii.) In $\Delta TOH$, $\tan 29 = \frac{h}{OH}$ ; $OH = h\cot 29$ ✓

In $\Delta TOP$, $\tan 25 = \frac{h}{OP}$ ; $OP = h\cot 25$ ✓

iv.) In $\Delta POS$, $\tan\angle OSP = \frac{OP}{OS}$

$= \frac{\cancel{h}\cot 25}{\cancel{h}\cot 32}$ ✓

$\tan\angle OSP = \frac{1}{\tan 25} \times \tan 32$

$\angle OSP = 53°16'$ ✓

$= 53°$ (nearest degree)

v) In $\Delta OHS$, $\frac{\sin 53^\circ}{\cancel{h}\cot 29} = \frac{\sin \angle OHS}{\cancel{h}\cot 32}$.

$\sin \angle OHS = \sin 53 \cdot \frac{1}{\tan 32} \times \tan 29$.

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$\angle OHS = 45^\circ 7'$

$= \underline{45^\circ}$ (nearest degree)

vi.) **NOW** in $\Delta OSH$, $\angle SOH = 180^\circ - 45^\circ - 53^\circ$ ($\angle$ sum $\Delta = 180^\circ$)

$= \underline{82^\circ}$

Also the $\angle$ btw North and East $= 90^\circ$

$\therefore$ bearing of H from O is $90^\circ + \cancel{82}^\circ = \underline{172^\circ}$

B) i.) $P(t) = P_0 e^{kt} + \frac{\ell}{k}(e^{kt} - 1)$ ; $P_0 e^{kt} = P - \frac{\ell}{k}(e^{kt} - 1)$ — (1)

$\frac{dP}{dt} = kP_0 e^{kt} + \frac{\ell}{\cancel{k}}(\cancel{k}e^{kt})$ (sub in (1))

$= k\left(P - \frac{\ell}{k}(e^{kt} - 1)\right) + \ell e^{kt}$

$= kP - \ell(e^{kt} - 1) + \ell e^{kt} = kP - \ell e^{kt} + \ell e^{kt} + \ell$

$= \underline{kP + \ell}$

ii.) when $t = 0$, $P = 222,000,000$

$222,000,000 = P_0 + \frac{\ell}{k}(0)$ ; $\underline{P_0 = 222,000,000}$

Given: $\ell = 500,000$

$t = 20$

$k = 0.01 + 1 = \underline{1.01}$

$P(20) = 222,000,000 e^{20(1.01)} + \frac{500,000}{1.01}(e^{20(1.01)} - 1)$

$= \underline{1.318465854 \times 10^{17} \text{ people}}$

c)

Given: $\angle APT = 108^\circ$

$\angle APB = 90^\circ$ ($\angle$ in semicircle $= 90^\circ$)

$\therefore \angle BPT = 108^\circ - 90^\circ = \underline{18^\circ}$

$\angle BPT = \angle PAB = 18^\circ$ ($\angle$ made by tgt and chord equals $\angle$ in alt segment)

~~$\angle PBT = 18^\circ + 90^\circ$ (exterior $\angle$ equals sum of interior opp $\angle$)~~

~~$= 108^\circ$~~

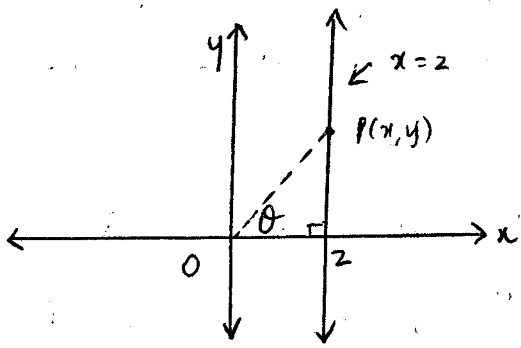
~~In $\Delta BPT$, $\angle$~~

In $\Delta APT$, $\angle ATP = 180° - 108° - 18°$ ($\angle$ sum $\Delta = 180°$)

$= \underline{\underline{54°}}$ ✓

Question 4

A) i.)




✓

ii.) $\tan\theta = \dfrac{y}{2}$ (from diagram)

$\therefore \theta = \tan^{-1}\left(\dfrac{y}{2}\right)$ ✓

$$\frac{d\theta}{dy} = \frac{1}{1+\frac{y^2}{4}} \times \frac{1}{2} = \frac{\cancel{4}\,2}{4+y^2} \times \frac{1}{\cancel{2}} = \underline{\frac{2}{4+y^2}} \checkmark$$

R.T.P.

B) i.) $\displaystyle\int_0^{\frac{\pi}{4}} \cos^2\theta \, d\theta = \frac{\pi+2}{8}$

LHS $\displaystyle\int_0^{\frac{\pi}{4}} \cos^2\theta \, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}} 1+\cos 2\theta \, d\theta$ ✓

$$= \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)_0^{\frac{\pi}{4}} \checkmark$$

$$= \frac{1}{2}\left(\frac{\pi}{4} + \frac{1}{2}\sin\frac{\pi}{2}\right) \checkmark$$

$$= \frac{\pi}{8} + \frac{1}{4} = \frac{\pi+2}{8} = \underline{\underline{RHS}}$$

ii.) $\displaystyle\int_0^{\sqrt{2}} \sqrt{4-x^2}\, dx$ $\quad x = 2\sin\theta$

$\dfrac{dx}{d\theta} = 2\cos\theta$ ✓

$$= 2\int_0^{\frac{\pi}{4}} \sqrt{4-4\sin^2\theta}\,.\cos\theta\, d\theta \qquad dx = 2\cos\theta\, d\theta \checkmark$$

$$= 2\int_0^{\frac{\pi}{4}} 2\sqrt{1-\sin^2\theta}\,.\cos\theta\, d\theta \checkmark$$

$$= 4\int_0^{\frac{\pi}{4}} \sqrt{\cos^2\theta}\,.\cos\theta \, d\theta$$

$$= 4\int_0^{\frac{\pi}{4}} \cos^2\theta \, d\theta = 4\left(\frac{\pi+2}{8}\right) \text{ (from part (i.))}$$

$$= \boxed{\frac{\pi+2}{2}}$$

c) Let $P(x) = 8x^3 - 6x + 1 = 0$
∴ roots be $\alpha, \beta$ and $\gamma$.

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$$\alpha+\beta+\gamma = \frac{-b}{a} = 0.$$

$$\alpha\beta+\alpha\gamma+\beta\gamma = \frac{c}{a} = -\frac{6}{8} = -\frac{3}{4}$$

$$\alpha\beta\gamma = \frac{-d}{a} = -\frac{1}{8}$$

$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2} = \frac{\beta^2\gamma^2+\alpha^2\gamma^2+\alpha^2\beta^2}{\alpha^2\beta^2\gamma^2}$$

NOW $(\alpha\beta+\alpha\gamma+\beta\gamma)^2 = \alpha^2\beta^2+\alpha^2\beta\gamma+\alpha\beta^2\gamma+\alpha^2\beta\gamma+\alpha^2\gamma^2+\alpha\beta\gamma^2+\alpha\beta^2\gamma+\alpha\beta\gamma^2+\beta^2\gamma^2$

$$\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2 = (\alpha\beta+\alpha\gamma+\beta\gamma)^2-\alpha^2\beta\gamma-\alpha\beta^2\gamma-\alpha^2\beta\gamma-\alpha\beta\gamma^2-\alpha\beta^2\gamma-\alpha\beta\gamma^2$$

$$= (\alpha\beta+\alpha\gamma+\beta\gamma)^2 - 2(\alpha^2\beta\gamma+\alpha\beta^2\gamma+\alpha\beta\gamma^2)$$

$$= (\alpha\beta+\alpha\gamma+\beta\gamma)^2 - 2\alpha\beta\gamma(\alpha+\beta+\gamma)$$

$$= \left(\frac{-3}{4}\right)^2 - 2\left(-\frac{1}{8}\right)(0)$$

$$= \frac{9}{16}$$

$$\therefore \frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2} = \frac{\frac{9}{16}}{\frac{1}{64}} = \frac{9}{16} \times 64 = \boxed{36}$$

Question 5

A) $f(x) = g(x) - \ln\{g(x)+1\}$

i.) R.T.P. $f'(x) = \dfrac{g(x) \cdot g'(x)}{g(x)+1}$

LHS $f'(x) = g'(x) - \dfrac{1}{g(x)+1} \cdot g'(x)$ ✓

$= g'(x) - \dfrac{g'(x)}{g(x)+1}$

$= \dfrac{g'(x)[g(x)+1] - g'(x)}{g(x)+1}$ ✓ $= \dfrac{g'(x) \cdot g(x) + \cancel{g'(x)} - \cancel{g'(x)}}{g(x)+1}$ ✓

$= \dfrac{g'(x) \cdot g(x)}{g(x)+1}$ = RHS

ii.) $\displaystyle\int \frac{\sin 2x \cos 2x}{\sin 2x + 1}\, dx$

✓

$= \frac{1}{2}\left[\sin 2x - \ln(\sin 2x + 1)\right] + C$

B) Volume $= \pi \displaystyle\int_3^8 \frac{(x-1)^2}{x+1}\, dx$

$u^2 = x+1$
$x = u^2 - 1$
$\dfrac{dx}{du} = 2u$
$dx = 2u\,du$ ✓

$= \pi \displaystyle\int_2^3 \frac{(u^2-1-1)^2 \cdot 2u\, du}{u^2}$

$= 2\pi \displaystyle\int_2^3 \frac{u^4 - 4u^2 + 4}{u}\, du$

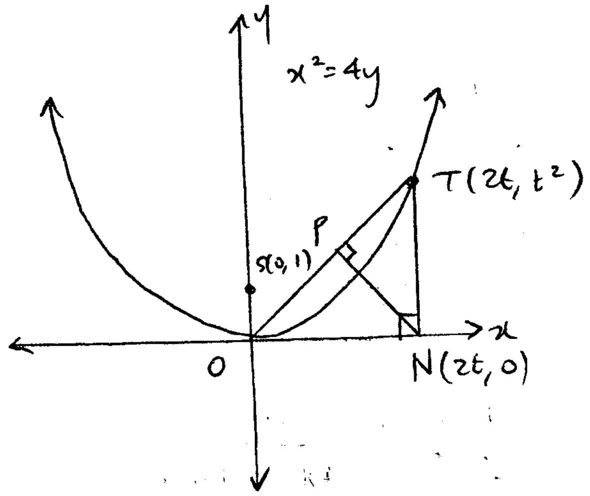
✓

$= 2\pi \displaystyle\int_2^3 u^3 - 4u + \frac{4}{u}\, du = 2\pi\left(\frac{u^4}{4} - 2u^2 + 4\ln u\right)_2^3$ ✓

$= 2\pi\left(\frac{81}{4} - 18 + 4\ln 3 - 4 + 8 - 4\ln 2\right)$

$= 2\pi\left(\frac{25}{4} + \ln\frac{81}{16}\right)$ ✓

$= \left(\frac{25\pi}{2} + 2\pi \ln\frac{81}{16}\right) u^3$ ✓




c) Grad. of OT is $\frac{t^2}{2t} = \frac{t}{2}$

Eqt of OT is:

$(y-t^2) = \frac{t}{2}(x-2t)$

$2y - \cancel{2t^2} = tx - \cancel{2t^2}$

$2y = tx$

$y = \frac{tx}{2}$ — ① ✓

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Since PN ⊥ OT, grad. of PN $= -\frac{2}{t}$ $(m_1 m_2 = -1)$

Eqt. of is $y - 0 = -\frac{2}{t}(x - 2t)$

$ty = 4t - 2x$

$y = \frac{4t-2x}{t}$ — ② ✓

To find P, solve ① and ② simult.

$\frac{tx}{2} = \frac{4t - 2x}{t}$

$t^2 x = 8t - 4x$

$t^2 x + 4x = 8t$

$x = \frac{8t}{t^2+4}$, $y = t\left(\frac{\overset{4}{\cancel{8}}t}{\cancel{2}(t^2+4)}\right) = \frac{4t^2}{t^2+4}$ ✓

$x = \frac{8t}{t^2+4}$ ✓ ; $y = \frac{4t^2}{t^2+4}$

$t^2 + 4 = \frac{8t}{x}$

$t^2 = \frac{8t}{x} - 4$

$y = \frac{4\left(\frac{8t}{x} - 4\right)}{\frac{8t}{x}} = \frac{32t - 16x}{x} \times \frac{x}{8t}$

$= \frac{4t - 2x}{t}$

$ty = 4t - 2x$

$ty - 4t = -2x$; $2x = 4t - ty$

$t = \frac{2x}{4-y}$ (sub into x and y coord of P)

Locus of P :

$$x = \frac{8\left(\frac{2x}{4-y}\right)}{\frac{4x^2}{(4-y)^2}+4} \qquad y = \frac{4\left(\frac{4x^2}{(4-y)^2}\right)}{\frac{4x^2}{(4-y)^2}+4}$$

$$= \frac{16x}{\cancel{4-y}} \times \frac{(4-y)^2}{4x^2+4(4-y)^2}$$

$$x = \frac{16x(4-y)}{4x^2+4(4-y)^2}$$

$$x\left(4x^2+4(16-8y+y^2)\right) = 16x(4-y)$$

$$\cancel{x}(4x^2+64-32y+4y^2) = 64\cancel{x}-16\cancel{x}y$$

$$4x^2+\cancel{64}-3\cancel{2}y+4y^2-\cancel{64}+1\cancel{6}y=0$$

$$4x^2-16y+4y^2=0$$

$$x^2+y^2-4y=0 \quad \text{(complete the square)}$$

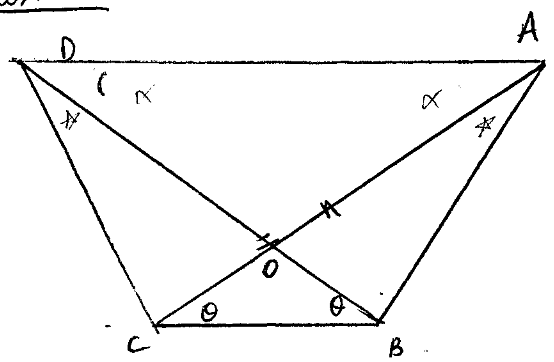
$$x^2+(y^2-4y+4)-4=0$$

$$x^2+(y-2)^2=4$$

$\therefore$ Locus of P is a circle with centre (0,2), radius = 2

Question 6

A)




i) $\Delta OCB$ is an isos $\Delta$ ( $\angle OCB = \angle OBC$. base $\angle$ of isos $\Delta$ equal)

$\therefore$ $OC = OB$ (2 sides of isos $\Delta$ equal) ✓

Also, we are given that $DB = AC$

$\therefore$ $OD = AO$

ii) In $\Delta ADB$ and $\Delta DAC$,

① $\angle DAC = \angle ADB = \theta$ (base $\angle$ of isos $\Delta$ DOA are equal) ✓

② $DB = AC$ (given)

③ Since $\Delta DCB \equiv \Delta ABC$ (SAS), $\angle ABC = \angle DCB$ (corresp $\angle$ of cong $\Delta$ equa

$\angle DBA = \angle ABC - \theta$

$\angle DCA = \angle DCB - \theta$ $\quad \therefore \angle DBA = \angle DCA$ ✓

$\therefore$ $\Delta ADB \equiv \Delta DAC$ (AAS)

iii) ~~$\angle COB = \angle DOA$ (vert. opp $\angle$ equal)~~

In $\Delta DAO$ and $\Delta OCB$,

① $\angle COB = \angle DOA$ (vert. opp $\angle$ equal)

② $\dfrac{OC}{CA} = \dfrac{OB}{DB}$ (Given: $DB = CA$ and $OC = OB$ (~~two~~ sides of isos $\Delta OCB$ equal)) ✓

$\therefore$ $\Delta DAO \parallel\!| \Delta OCB$ (~~eq~~ one $\angle$ equal and 2 sides in same proportion

$\therefore$ $\angle DAO = \angle OCB = \theta$ and $\angle ADO = \angle OBC = \theta$ (corresp. $\angle$ of $\parallel\!|\Delta$ equal ✓

$\therefore$ $AD \parallel BC$ (alt. $\angle$ on $\parallel$ lines)

B) $\ddot{x} = -9x$

i.) Rqd. to show: $\ddot{x} = \dfrac{d}{dx}\left(\tfrac{1}{2}v^2\right)$

RHS $\dfrac{d}{dx}\left(\tfrac{1}{2}v^2\right) = \dfrac{d}{dv}\left(\tfrac{1}{2}v^2\right) \times \dfrac{dv}{dx} = v \times \dfrac{dv}{dx} = \dfrac{\cancel{dx}}{dt} \times \dfrac{dV}{\cancel{dx}}$

✓ $= \dfrac{dV}{dt} = \ddot{x} = \underline{\underline{LHS}}$

ii.) $\frac{1}{2}v^2 = \int -9x \, dx$

$v^2 = -\frac{9x^2}{2} \times 2 + C$

$v^2 = -9x^2 + C$

when $x = 4$, $v = 12$.

$144 = -9(16) + C$ ; $144 = -144 + C$ ; $C = \underline{288}$

$\therefore \underline{v^2 = -9x^2 + 288}$

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$v = \pm\sqrt{288 - 9x^2}$

speed $= |v|$

$\therefore$ speed $= \sqrt{9(32 - x^2)} = \underline{3\sqrt{32 - x^2} \text{ m/s}}$

iii.) $\frac{dx}{dt} = -3\sqrt{32 - x^2}$ when $t = 0$, $v = -12$m/s (towards 0).

$\frac{dt}{dx} = \frac{-1}{3\sqrt{32 - x^2}}$ ; $t = \frac{1}{3} \cancel{\sin}^{-1} \cos^{-1}\left(\frac{x}{4\sqrt{2}}\right) + C$

when $t = 0$, $x = 4$.

$0 = \frac{1}{3} \cancel{\sin}^{-1} \cos^{-1} \frac{1}{\sqrt{2}} + C$

$0 = \frac{\pi}{12} + C$ ; $C = -\frac{\pi}{12}$

$\therefore t = \frac{1}{3} \cancel{\sin}^{-1} \cos^{-1}\left(\frac{x}{4\sqrt{2}}\right) - \frac{\pi}{12}$.

$\frac{1}{3} \cancel{\sin}^{-1} \cos^{-1}\left(\frac{x}{4\sqrt{2}}\right) = t + \frac{\pi}{12}$

$\cancel{\sin}^{-1} \cos^{-1}\left(\frac{x}{4\sqrt{2}}\right) = 3\left(t + \frac{\pi}{12}\right)$

$\frac{x}{4\sqrt{2}} = \cancel{\sin} \cos\left(3t + \frac{\pi}{4}\right)$

$x = 4\sqrt{2} \cancel{\sin} \cos\left(3t + \frac{\pi}{4}\right)$

$$v = \frac{dx}{dt} = 12\sqrt{2}\cos\left(3t + \frac{\pi}{4}\right)$$

iv.) α) greatest speed is when $\cos\left(3t + \frac{\pi}{4}\right) = 1$.

∴ greatest speed $= 12\sqrt{2} \times 1$

$= \underline{12\sqrt{2}\ m/s}$ ✓

β)

$$\frac{d^2x}{dt^2} = -36\sqrt{2}\sin\left(3t + \frac{\pi}{4}\right).$$

Greatest acceleration is when $\sin\left(3t + \frac{\pi}{4}\right) = 1$.

∴ greatest acceleration $= \left|-36\sqrt{2}\right|$

$= \underline{36\sqrt{2}\ m/s^2}$ ✓

δ) Greatest displacement is when $v=0$ (ie endpt)

$$3\sqrt{32 - x^2} = 0$$

$32 - x^2 = 0$ ; $x^2 = 32$

$x = \pm 4\sqrt{2}$ ✓

∴ greatest displacement is $4\sqrt{2}$m away from centre of motio

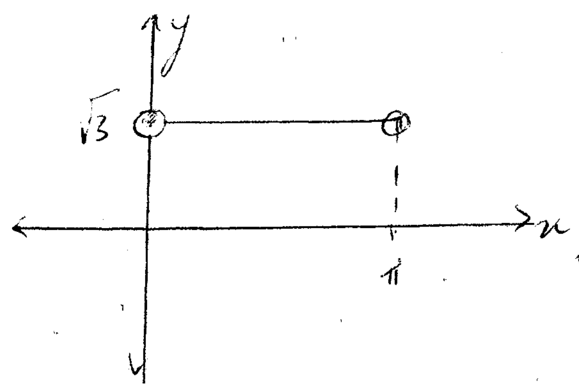
v.) Period of motion $= \frac{2\pi}{n} = \underline{\frac{2\pi}{3}\ \text{sec}}$ ✓

c) i.)

$$\frac{\sin\left(x - \frac{\pi}{6}\right) + \sin\left(x + \frac{\pi}{6}\right)}{\cos\left(x - \frac{\pi}{6}\right) - \cos\left(x + \frac{\pi}{6}\right)} = \frac{\sin x\cos\frac{\pi}{6} - \cancel{\cos x\sin\frac{\pi}{6}} + \sin x\cos\frac{\pi}{6} + \cancel{\cos x\sin\frac{\pi}{6}}}{\cancel{\cos x\cos\frac{\pi}{6}} + \sin x\sin\frac{\pi}{6} - \cancel{\cos x\cos\frac{\pi}{6}} + \sin x\sin\frac{\pi}{6}}$$

$$= \frac{\cancel{2}\sin x \cdot \frac{\sqrt{3}}{\cancel{2}}}{2\sin x \cdot \frac{1}{2}} = \frac{\sqrt{3}\cancel{\sin x}}{\cancel{\sin x}} = \boxed{\sqrt{3}.}$$ ✓

ii.) $f(x)$ is independent of $x$ for $0 < x < \pi$ ✓




✓

Question 7

A) $2\cos\left(x - \frac{7\pi}{18}\right) + 1 = 0 \qquad 0 \le x \le 2\pi$

$\cos\left(x - \frac{7\pi}{18}\right) = -\frac{1}{2}$

$-\frac{7\pi}{18} \le x - \frac{7\pi}{18} \le \frac{29}{18}\pi$

$x - \frac{7\pi}{18} = \frac{2\pi}{3}, \frac{4\pi}{3}$

$x = \frac{19\pi}{18}, \frac{31\pi}{18}$

B) i.) Step 1

let $n=1$

LHS $n^2 = 1^2 = 1$

RHS $\frac{1}{6}n(n+1)(2n+1) = \frac{1}{6}.1.\cancel{2.3} = 1 = LHS$

$\therefore$ true for $n=1$

Step 2

Assume true for $n=k$

$1^2+2^2+3^2+\ldots k^2 = \frac{1}{6}.k(k+1)(2k+1)$

R.T.P. also true for $n=k+1$

$1^2+2^2+\ldots k^2+(k+1)^2 = \frac{1}{6}(k+1)(k+2)(2k+3)$

LHS $1^2+2^2+\ldots k^2+(k+1)^2 = \underbrace{\frac{1}{6}.k(k+1)(2k+1)}_{\text{from assumption}} + (k+1)^2$

$= \frac{(k+1)}{6}\left[k(2k+1)+6(k+1)\right]$

$= \frac{(k+1)}{6}\left[2k^2+7k+6\right]$

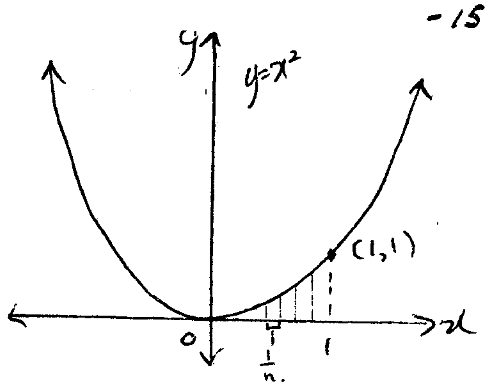
$= \frac{(k+1)}{6}(2k+3)(k+2) = RHS$

Step 3

If true for $n=k$ and $n=k+1$ and also true for $n=1$, then it is true for $n=1+1=2$, $n=2+1=3$ and so on.

$\therefore$ by the principle of mathematical induction, it is true for all integers $n$.

ii.)




α) Width of trapezium = height = $\frac{1}{n}$ units.
top and base of trapezium is $0, \frac{1}{n^2}, \frac{4}{n^2}, \frac{9}{n^2}, \frac{16}{n^2} \dots \frac{(n-1)^2}{n^2}, 1$

Sum of areas of trapezium is: $\frac{\frac{1}{n}}{2}\left(0+\frac{1}{n^2}\right) + \frac{1}{2n}\left(\frac{1}{n^2}+\frac{2^2}{n^2}\right) + \frac{1}{2n}\left(\frac{2^2}{n^2}+\frac{3^2}{n^2}\right)$
$\dots \frac{1}{2n}(1)$

$$= \frac{1}{2n}\left\{\frac{1}{n^2} + \frac{(1+2^2)}{n^2} + \frac{(2^2+3^2)}{n^2} + \dots 1\right\}$$

$$= \frac{1}{2n}\left\{\frac{1+1+2^2+2^2+3^2+3^2+\dots}{n^2} \quad 1\right\}$$

$$= \frac{1}{2n}\left\{\frac{2}{n^2}(1^2+2^2+3^2+4^2+\dots (n-1)^2) + 1\right\}$$

$$\therefore S = \frac{1}{2}\cdot\frac{1}{n}\left\{(0+1) + \frac{2}{n^2}\left(1^2+2^2+3^2+\dots (n-1)^2\right)\right\}$$

β) $S = \frac{1}{2n}\left\{1 + \frac{2}{n^{\cancel{2}}}\left(\frac{1}{\cancel{6}_3}\cancel{n}(n+1)(2n+1) - n^2\right)\right\}$ ⟸ from part (i.)

$$= \frac{1}{2n}\left\{1 + \frac{1}{3n}(n+1)(2n+1) - 2\right\}$$

δ) As $n \to \infty$,
$S \to A$

This is because as the number of trapezia increases, the sum of their areas ~~also~~ edges closer to the area under the curve. ~~and b~~

~~As n→∞~~, $S = \frac{\cancel{1}}{\cancel{2}n} + \frac{1}{6n^2}(2n^2+3n+1) - \frac{\cancel{2}}{\cancel{2}n}$

$S = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}$

As $n \to \infty$, $S \to \frac{1}{3} + 0 + 0 = \frac{1}{3}$ $\therefore \lim_{n\to\infty} S = A = \frac{1}{3} u^2$

c) Given: $x = 50t\cos\theta$ $\quad y = 50t\sin\theta - 5t^2$

$\dot{x} = 50\cos\theta$ $\quad \dot{y} = 50\sin\theta - 10t$

i.) Find cartesian eqt of motion.

$x = 50t\cos\theta$

$t = \dfrac{x}{50\cos\theta}$ (sub into y-eqt)

$y = 50\sin\theta\left(\dfrac{x}{50\cos\theta}\right) - 5\left(\dfrac{x^2}{2500\cos^2\theta}\right)$

$y = x\tan\theta - \dfrac{x^2}{500}(1+\tan^2\theta)$ ✓

(If ball just clears fence, $x = 200$ and $y = 2$)

$2 = 200\tan\theta - \dfrac{40000}{500}(1+\tan^2\theta)$

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$2 = 200\tan\theta - 80 - 80\tan^2\theta$

$2 - 200\tan\theta + 80 + 80\tan^2\theta = 0$ ✓

$\therefore 80\tan^2\theta - 200\tan\theta + 82 = 0$

ii.) ~~For a score home run, $\Delta > 0$~~

$40\tan^2\theta - 100\tan\theta \cancel{+41} = 0$

$\tan\theta = \dfrac{100 \pm \sqrt{10,000 - 4(40)(41)}}{80}$

$= \dfrac{100 \pm \sqrt{3440}}{80}$

$\tan\theta = 1.983$ or $\tan\theta = 0.517$

$\theta = 63°14'$ ✓ $\quad \theta = 27°20'$ ✓

$\therefore \theta$ must lie between $27°20' \leq \theta \leq 63°14'$ to score a home run.

iii.) Eqts of motion of another ball:

$\dot{x}_1 = 70\cos\alpha$ $\quad \dot{y}_1 = 70\sin\alpha - 10t$

$x_1 = 70t\cos\alpha$ $\quad y_1 = 70t\sin\alpha - 5t^2$

Eqts of motion of ball at origin

$\dot{x}_2 = 50\cos 30 = 25\sqrt{3}$ $\quad \dot{y}_2 = 25 - 10t$

$x_2 = 25\sqrt{3}t$ $\quad y_2 = 25t - 5t^2$

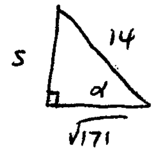
iii.) Both balls collide at same time, $t$ and same height $y$.

$$y_1 = y_2$$

$$70\cancel{t}\sin\alpha - \cancel{5t^2} = 25\cancel{t} - \cancel{5t^2}$$

$$70\sin\alpha = 25$$

$$\sin\alpha = \frac{5}{14}$$




$$\underline{\alpha = 20°55'}$$

~~when~~ At time $t$, $x_1 + x_2 = 400m$.

$$70t\cos\sin^{-1}\left(\tfrac{5}{14}\right) + 25\sqrt{3}\,t = 400$$

$$t\left(\cancel{70}^{5} \cdot \frac{\sqrt{171}}{\cancel{14}} + 25\sqrt{3}\right) = 400$$

$$t = \frac{400}{5\sqrt{171} + 25\sqrt{3}} = \underline{3.68\,\text{sec}}$$

when $t = 3.68$ sec,

$$x_1 = 70(3.68)\cos 20°55'$$

$$= 240m$$

$\therefore$ they collide 40m BEFORE on the left hand side the 2m fence.