SYDNEY BOYS HIGH SCHOOL

EXTENSION 2

MATHEMATICS COURSE

May 2001

Assessment Task # 1

Time Allowed: 90 minutes (plus 5 minutes Reading Time)

Total Marks: 80

Examiner: Mr S Parker

INSTRUCTIONS:

- Attempt *all* questions.
- All questions are of equal value.
- All *necessary* working should be shown in every question. Full marks *may not* be awarded if work is careless or badly arranged.
- Standard integrals are provided on the back of this page. Approved calculators may be used.
- Return your answers in **4** sections: Section A, Section B, Section C and Section D. Each booklet **MUST** show your name.
- If required, additional Answer Booklets may be obtained from the Examination Supervisor upon request.

Section 1 **Marks**

(1) (a) Find $\displaystyle\int \frac{x\,dx}{\sqrt{x^2+2}}$, using $u = x^2 + 2$ **2**

$(\sin x)^3$

$3\cos$ $3(\sin^2 x)\cos x$

(b) Find $\displaystyle\int \sin^3 x \cos x\, dx$ $3\cos$ **2**

$-\sin^4 x +$

(2) (a) If $z = -1 + i\sqrt{3}$ find

(i) iz **1**

(ii) $\bar{z}$ **1**

(iii) $|z|$ **1**

(iv) $\arg z$ **2**

(b) If $z = \dfrac{2+i}{(i-1)^2}$, express z in the form $a + ib$ where a and b are real. **3**

(3) If $z = 1 + i\sqrt{3}$, show on the same Argand diagram, the points representing the complex numbers: **5**

$$z, z^2, \frac{1}{z}, z^2 - z$$

(4) Show that the function $f(x) = \tan^{-1} x$ is an increasing function for all real values of x. **3**
Find the two points on the curve at which the tangents are parallel to the line $x - 2y + 1 = 0$

A z

Section 2 (Start a new answer booklet) **Marks**

(5) In the Argand diagram shade the region representing z satisfying **4**

$$|z| \leq 2 \text{ and } -\frac{\pi}{2} \leq \arg(z-i) \leq \frac{\pi}{4}$$

(There is no need to specify the vertices on the boundary)

(6) Given that $z = \sqrt{3} - i$, express

(i) z in modulus argument form **2**

(ii) z^3 in the form $a + ib$, where a and b are real. **2**

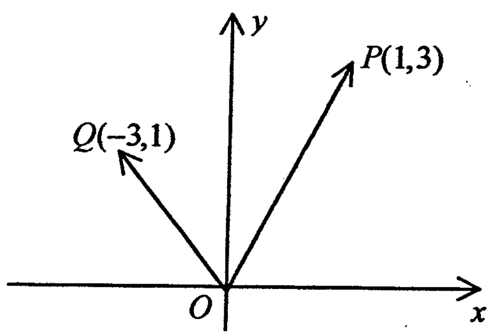
(7) (a) Find a square root of $-3-4i$ **4**

(b) Solve **2**

$$z^2 - 7z + (13+i) = 0$$

giving answers in the form $a + ib$, where a and b are real.

(8) In the figure $P(1,3)$ and $Q(-3,1)$ represent the complex numbers z and w respectively.



(i) Express $\dfrac{w}{z}$ in the form $a + ib$, where a and b are real. **3**

(ii) Find $\angle POQ$ **3**

Section 3 (Start a new answer booklet) **Marks**

(9) α, β and γ are the roots of $x^3 - 7x^2 + 3x + 1 = 0$

Find

(i) $\alpha + \beta + \gamma$ **1**

(ii) $\alpha\beta + \alpha\gamma + \beta\gamma$ **1**

(iii) $\alpha\beta\gamma$ **1**

(iv) $\alpha^2 + \beta^2 + \gamma^2$ **2**

(v) the equation whose roots are $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$ **2**

(10) Show that the equation **2**

$$x^3 - x + 3 = 0$$

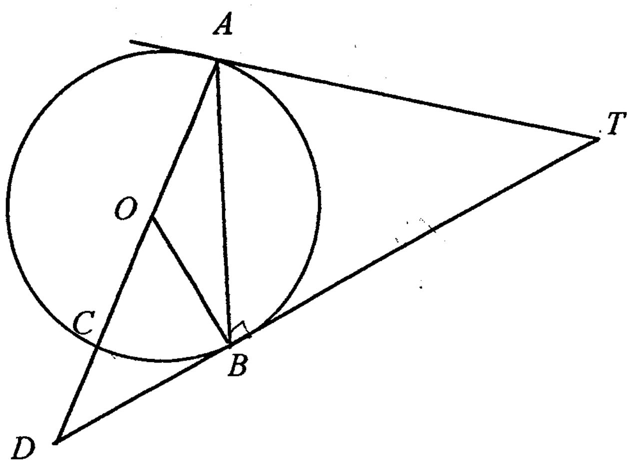
cannot have a double root.

Section 3 is continued on the next page

A 3

(11) (a) In the diagram below, from an external point T, two tangents TA and TB are drawn to touch a circle, centre O, at A and B respectively.

Angle ATB is acute. The diameter AC produced meets the tangent TB produced at D.



(i) Copy the diagram into your Answer Booklet

(ii) Prove that $\angle DBC = \frac{1}{2}\angle ATB$ **2**

(iii) Prove $\Delta ABC \,|||\, \Delta TBO$ **2**

(iv) Deduce $BC \times OT = 2(OA^2)$ **2**

(12) A function $f(n)$ is defined for $n > 0$ and for n being an integer such that

$$f(1) = 1 \text{ and } f(n+1) = f(n) + n + 2$$

(i) Find $f(2)$ and $f(3)$ **2**

(ii) Prove, by mathematical induction, that **3**

$$f(n) = \tfrac{1}{2}\left(n^2 + 3n - 2\right)$$

Section 4 (Start a new answer booklet) **Marks**

(13) Twelve girls and 15 boys attend a school party. In how many ways can 4 pairs be selected to dance? Each pair being two people of the opposite sex. **4**

(14) (a) Write down the general solution to $\tan 4\theta = 1$ **2**

(b) Use De Moivre's Theorem to find

(i) $\cos 4\theta$ in terms of $\cos\theta$ and $\sin\theta$ **2**

(ii) $\sin 4\theta$ in terms of $\cos\theta$ and $\sin\theta$ **2**

(iii) Hence derive the result **3**

$$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$

(c) (i) Find the roots of $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$, where $x = \tan\theta$. **3**

(ii) Hence prove the result **4**

$$\tan^2\tfrac{\pi}{16} + \tan^2\tfrac{3\pi}{16} + \tan^2\tfrac{5\pi}{16} + \tan^2\tfrac{7\pi}{16} = 28$$

THIS IS THE END OF THE PAPER

$x^2(4x$

$x^2(x^2+4x-6 - \frac{4}{x}) + 1$

$x^2(x^2+\frac{4}{x}) + 4(x-6)) + 1$

$x(x^2+16$

$x^2(x+\frac{4}{x})^2$

$x^2 + 8x + \frac{16}{x^2}$

A 4

Sydney Boys High School :- Extension 2 → May 2001 Assessment TASK 1

25072002
Amanda Lui

Section 1

1 a) Find $\int \frac{x}{\sqrt{x^2+2}}dx$ using $u = x^2+2$

let $u = x^2+2$.

$\therefore du = 2x\,dx$.

let $I = \int \frac{x}{\sqrt{x^2+2}}dx$.

$\therefore I = \frac{1}{2}\int \frac{1}{\sqrt{u}}du$

$= \frac{1}{2}\int u^{-\frac{1}{2}}du$

$= \frac{1}{2}\left[\frac{2u^{\frac{1}{2}}}{1}\right] + c$.

$= \sqrt{u} + c$

$= \sqrt{x^2+2} + c$ //

b). Find $I = \int \sin^3 x \cos x\, dx$.

let $u = \sin x$

$du = \cos x\, dx$.

$\therefore I = \int u^3 du$

$= \frac{1u^4}{4} + c$.

$= \frac{1}{4}\sin^4 x + c$.

2 a). $z = -1 + i\sqrt{3}$

i) $iz = i(-1 + i\sqrt{3})$

$= -i - \sqrt{3}$

ii) $\bar{z} = -1 - i\sqrt{3}$

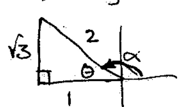
iii) $|z| = \sqrt{x^2+y^2}$

$= \sqrt{(-1)^2 + (\sqrt{3})^2}$

$= \sqrt{1+3}$

$= 2$.

iv) arg z



$\sin^{-1}\frac{\sqrt{3}}{2} = \theta$.

$\therefore \theta = \frac{\pi}{3}$.

$\therefore \alpha = \frac{2\pi}{3}$

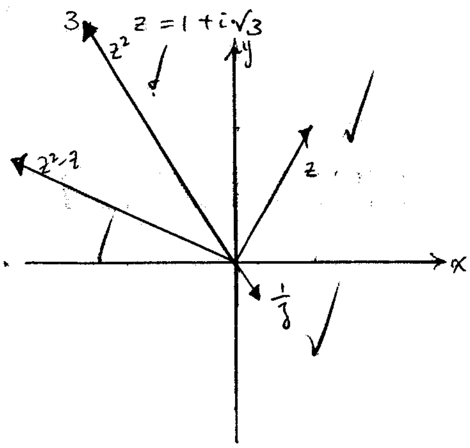
b) $z = \frac{2+i}{(i-1)^2}$

$= \frac{2+i}{i^2-2i+1}$

$= \frac{2+i}{-2i}$

$= \frac{2i-1}{2}$

$= i - \frac{1}{2}$ //



z

$z^2 = (1+i\sqrt{3})(1+i\sqrt{3})$

$= 1 - 3 + 2\sqrt{3}i = -2 + 2\sqrt{3}i$

$\frac{1}{z} = \frac{1}{1+i\sqrt{3}} = 1 - i\sqrt{3} \div 1+3$

$= \frac{1-\sqrt{3}i}{4} = \frac{1}{4} - \frac{\sqrt{3}}{4}i$.

$z^2 - z$. graph

4. $f(x) = \tan^{-1}x$.

$f'(x) = \frac{1}{1+x^2}$

Since for all x, positive and negative, the value for $f'(x)$ will be positive, the gradient will always be positive.

$\therefore$ for all real x, the function will be increasing.

$x - 2y + 1 = 0$.

$2y = 1 + x$

$y = \frac{1}{2}x + \frac{1}{2}$.

$\therefore m = \frac{1}{2}$.

for lines to be parallel, $m_1 = m_2$

$\therefore m = \frac{1}{2}$

$\therefore \frac{1}{2} = \frac{1}{1+x^2}$

$2 = 1 + x^2$

$2 - 1 = x^2$

$\therefore x = \pm 1$.

Sub $x = 1$ into $\tan^{-1}x = f(x)$

$f(1) = \tan^{-1}1$

$= \frac{\pi}{4}$. $\therefore (1, \frac{\pi}{4})$

$f(-1) = \tan^{-1}-1$

$= -\frac{\pi}{4}$. $\therefore (-1, -\frac{\pi}{4})$

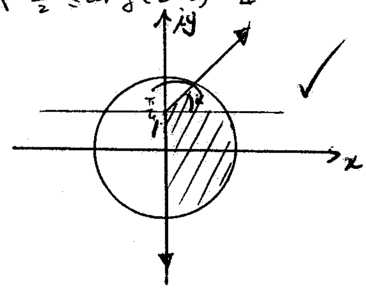
$\therefore$ two points are

$(1, \frac{\pi}{4})(-1, -\frac{\pi}{4})$

Section 2

5. $|z| \le 2$ $\therefore$ circle centre O.

$\cap -\frac{\pi}{2} \le \arg(z-i) \le \frac{\pi}{4}$



6. $z = \sqrt{3} - i$

i) $|z| = \sqrt{x^2+y^2}$

$= \sqrt{-1^2 + (\sqrt{3})^2}$

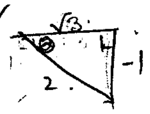
$= \sqrt{4}$

$= 2$ //

$\arg z = \sin^{-1}\frac{y}{x}$

$= \sin^{-1}\frac{-1}{\sqrt{3}}$

$= -\frac{\pi}{6}$



$2\,\text{cis}(-\frac{\pi}{6})$

ii) $z^3 = \left[2\,\text{cis}(-\frac{\pi}{6})\right]^3$

$= 2^3\,\text{cis}\left(\frac{3\pi}{-6}\right)$

$= 8\,\text{cis}(-\frac{\pi}{2})$ //

$= 8(-i) = -8i$.

$(x+iy)^2 = -3-4i$

$x^2 - y^2 + 2xiy = -3-4i$

$\therefore x^2 - y^2 = -3$ — ①

& $2xy = -4$.

$xy = -2$ — ②

$\therefore y = \frac{-2}{x}$. — ③

$\therefore x^2 - (\frac{-2}{x})^2 = -3$

$x^4 - 4 + 3x^2 = 0$.

$(x^2+4)(x^2-1) = 0$.

$x = \pm 1$, sub into ②.

$\therefore y = \pm 2$ ∓ 2

$\therefore$ root $= \pm(1-2i)$

b). $z^2 - 7z + (13+i) = 0$

$z = \frac{7 \pm \sqrt{49 - 4(13+i)}}{2}$

$= \frac{7 \pm \sqrt{-3-4i}}{2}$

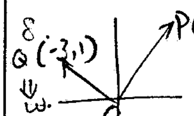
using answer from a.

$z = \frac{7 \pm (1-2i)}{2}$

$= \frac{8-2i}{2}$ or $\frac{6+2i}{2}$

$= 4 - i$ or $3 + i$

in a+ib form.



). $\frac{w}{z}$ in form a+ib.

$w = -3 + i$

$z = 1 + 3i$

$\therefore \frac{w}{z} = \frac{-3+i}{1+3i}$

$= \frac{(-3+i)(1-3i)}{(1+3i)(1-3i)}$

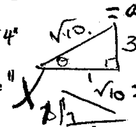
$= \frac{-3+3+9i+i}{1+9}$

$= \frac{10i}{10} = i$

Note: arg $\frac{w}{z}$ = arg w

ii) $P = \sqrt{10}\,\text{cis}\,71°33'54"$

$Q = \sqrt{10}\,\text{cis}\,108°26'6"$



= arg

$\therefore \angle POQ = Q - P$

$= 36°52'11"$

$= 90$

9. α, β, γ roots of

$x^3 - 7x^2 + 3x + 1 = 0$.

i) $\alpha + \beta + \gamma = \frac{7}{1} = 7$.

ii) $\alpha\beta + \alpha\gamma + \beta\gamma = 3$

iii) $\alpha\beta\gamma = -1$

iv) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma)$

$= (7)^2 - 2(3)$

$= 43$

v). let $a = \frac{1}{x}$ $\therefore x = \frac{1}{a}$.

$\therefore (\frac{1}{a})^3 - 7(\frac{1}{a})^2 + 3(\frac{1}{a}) + 1 = 0$.

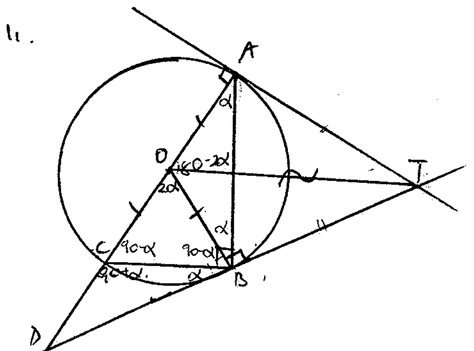
$\frac{1}{a^3} - 7\cdot\frac{1}{a^2} + 3\cdot\frac{1}{a} + 1 = 0$ $\times a^3$

$1 - 7a + 3a^2 + a^3 = 0$

$x^3 + 3x^2 - 7x + 1 = 0$ //

10. $x^3 - x + 3 = 0$.
double root
$P(\alpha) = 0$ & $P'(\alpha) = 0$.
$\therefore$ double root at $(x-\alpha)$
let $P(x) = x^3 - x + 3$
$P'(x) = 3x^2 - 1$
presume double root
$\therefore \quad 3x^2 - 1 = 0$
$\therefore x^2 = \frac{1}{3}$.
$\therefore x = \pm\frac{1}{3}$.
but double root
is both $P'(\alpha) = 0$ & $P(\alpha) = 0$.
sub $x = \pm\frac{1}{3}$
$\therefore P(\frac{1}{3}) \neq 0$
$P(-\frac{1}{3}) \neq 0$
$\therefore$ no double root possible.

11.



ii) $\angle DBC = \frac{1}{2}\angle ATB \Rightarrow$ RTP
construct BC
let $\angle CAB = \alpha$.
$\therefore \angle ABO = \alpha$ (Base $\angle$s in isos Δs)
and $OB \perp DT$ (tangent and normal $\perp$)
$\angle ABC$ is $90°$ ($\angle$ in a semicircle)
$\angle OBC + OBA = 90°$ ($\angle ABC$)
$\therefore \angle OBC = 90° - \alpha$
~~$180° = (90° - \alpha) + (90 - \alpha) + \angle COB$ (/ OCB)~~
~~$180° = 180 - 2\alpha + \angle COB$~~
~~$\angle COB = 2\alpha$~~
~~$\angle B = \angle CBO + \angle COB$ (ext $\angle$)~~
~~$= 2\alpha + 90 - \alpha$~~
~~$= 90 + \alpha$.~~
$\angle OBD = 90°$ ($\perp$)
$\angle OBD = \angle OBC + \angle CBD$
$90 = 90 - \alpha + \angle CBD$
$\therefore \angle CBD = \alpha$.
$\angle BAT = 90 - \alpha$
$\angle ABT = 90 - \alpha$.
in Δ TAB
$(90 - \alpha) + (90 - \alpha) + \angle ATB = 180°$ ($\angle$ sum)
$90° 90° - 2\alpha + \angle ATB = 180°$
$\angle ATB = 2\alpha$
$\therefore \angle ATB = 2\angle CBD$
or $\angle DBC = \frac{1}{2}\angle ATB$

iii) construct TO.
RTP $\Delta ABC \parallel\!\!\mid \Delta TBO$
In ΔABC & ΔTBO,
1. $\angle ABC = \angle TBO = 90°$ (from ii)
2. OT bisects $\angle BTA$, $\therefore \angle OTB = \alpha = \angle CAB$
$\therefore \Delta ABC \parallel\!\!\mid \Delta TBO$ (equiangular)

iv). deduce $BC \times OT = 2(OA^2)$
~~in Δs OTA and OTB.~~
~~1. OT is common~~
~~2. AT = BT (tangents to an external pt are equal)~~
3. OA = OB (radii of circle).
~~$\therefore \Delta OTA \equiv \Delta OTB$ (SSS)~~
~~$\therefore OA = OB$ (corresponding sides of equilateral triangle)~~
in ΔOAT
$\sin\alpha = \frac{OA}{OT}$ and,
In ΔBCA
$\sin\alpha = \frac{BC}{AC}$, now $AC = 2OA$
$\therefore \sin\alpha = \frac{BC}{2OA}$
$\therefore \frac{OA}{OT} = \frac{BC}{2OA}$
$\therefore 2(OA)^2 = BC \cdot OT$ //
as required

12.
$f(1) = 1$
$f(n+1) = f(n) + n + 2$
$f(2) = f(1) + 1 + 2$
$= 1 + 1 + 2$
$= 4$
$f(3) = f(2) + 2 + 2$
$= 4 + 2 + 2$
$= 8$.
i). $f(2) = 4$, $f(3) = 8$
ii) $f(n) = \frac{1}{2}(n^2 + 3n - 2) \ldots$ (a)
let $n = 1$
$\therefore f(1) = \frac{1}{2}(1 + 3 - 2)$
$= 1$
$\therefore$ true for $n = 1$
Assume true for $n = k$
$f(k) = \frac{1}{2}(k^2 + 3k - 2)$
RTP: true for $n = k+1$
$f(k+1) = \frac{1}{2}[(k+1)^2 + 3(k+1) - 2]$
$= \frac{1}{2}[k^2 + 2k + 1 + 3k + 3 - 2]$
$= \frac{1}{2}[k^2 + 5k + 2]$
Proof:
LHS $= f(k+1)$
$= f(k) + k + 2$ (from a)
$= \frac{1}{2}(k^2 + 3k - 2) + k + 2$
$= \frac{1}{2}k^2 + \frac{3k}{2} + k + 1$
$= \frac{1}{2}k^2 + \frac{5k}{2} + 1$
$= \frac{1}{2}(k^2 + 5k + 2)$
$=$ RHS //
as required
$\therefore$ if $f(n) = \frac{1}{2}(n^2 + 3n - 2)$
is true for k, then it is true for $k+1$.
$\therefore$ since true for $n = 1$,
then it is true for $n = 2$.
since true for $n = 2$, then
it is true for $n = 3$ and so on
for all positive integers n.

13.
Void - can't do yet.

14. a)
let $4\theta = \alpha$.
$\tan.\ \alpha = \frac{\pi}{4}, \pi + \frac{\pi}{4}$
got values before & after.
$\therefore \ldots -2\pi + \frac{\pi}{4}, -\pi + \frac{\pi}{4}, \frac{\pi}{4}, \pi + \frac{\pi}{4}, 2\pi + \frac{\pi}{4}$
$\therefore \quad \alpha = n\pi + \frac{\pi}{4}$
$\therefore 4\theta = n\pi + \frac{\pi}{4}$
$\theta = n\frac{\pi}{4} + \frac{\pi}{16}$
b) $(\text{cis}\,\theta)^4$
$(\cos\theta + i\sin\theta)^4 = \cos^4\theta + 4\cos^3\theta i\ldots$
$+ 6\cos^2\theta(-1)\sin^2\theta$
$+ 4\cos\theta(i)\sin^3\theta$
$+ (1)\sin^4\theta$
$\cos 4\theta + i\sin 4\theta = (\cos^4\theta - 6\cos^2\ldots$
$+ \sin^4\theta)$
$i(4\cos^3\sin\theta - \ldots$
i) equate real parts
$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$
ii) equate imaginary parts.
$\sin 4\theta = 4\cos^3\sin\theta - 4\cos\theta\sin^3\theta$ //
iii) $\tan 4\theta = \frac{\sin 4\theta}{\cos 4\theta}$
$\frac{\sin 4\theta}{\cos 4\theta} = \frac{4\cos^3\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin\ldots}$
$\div \cos^4\theta$.
$= \dfrac{4\frac{\sin\theta}{\cos\theta} - 4\frac{\sin^3\theta}{\cos^3\theta}}{1 - 6\frac{\sin^2\theta}{\cos^2\theta} + \frac{\sin^4\theta}{\cos^4\theta}}$
$= \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan\ldots}$
(as required).
c) i).
sub $x = \tan\theta$ into b)iii)
$\therefore \tan 4\theta = \frac{4x - 4x^3}{1 - 6x^2 + x^4}$
let $\tan 4\theta = 1$
$\therefore 4x - 4x^3 = 1 - 6x^2 + x^4$
$\therefore x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$
$\therefore \tan 4\theta = 1$ is solution.
$\theta_1 = \frac{\pi}{4} + \frac{\pi}{16}$
$= \frac{5\pi}{16}$ //
$\theta_2 = \frac{2\pi}{4} + \frac{\pi}{16} = \frac{\pi}{2} + \frac{\pi}{16}$
$= \frac{9\pi}{16}$ //
$\theta_3 = \frac{3\pi}{4} + \frac{\pi}{16}$
$= \frac{13\pi}{16}$ //
$\theta_4 = \frac{4\pi}{4} + \frac{\pi}{16} = \pi + \frac{\pi}{16}$
$= \frac{17\pi}{16} = \frac{\pi}{16}$ //.