SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2002
HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK # 2

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes.
- Working time – 90 minutes
- Write using black or blue pen.
- Board approved calculators may be used.
- *Each section* is to be returned in a *separate* booklet, clearly marked Section A (Questions 1 and 2), Section B (Questions 3 and 4), Section C (Questions 5 – 7)
  Each booklet must also show your name.
- All necessary working should be shown in every question.

Total Marks - 80 marks

- Attempt **ALL** sections
- All questions are **NOT** of equal value.

Examiner: *E. Choy*

SECTION A

Question 1: [12 Marks]

Marks

(a) Use your calculator to find **1**

$$e^{2\cdot 7}$$

correct to 3 decimal places

(b) Express 300° in radians, giving your answer in terms of π **1**

(c) Differentiate

(i) $\tan(3x+1)$ **2**

(ii) e^{4x-1} **2**

(d) Find

(i) $\int \sin 2x \, dx$ **2**

(ii) $\int e^{-x} \, dx$ **2**

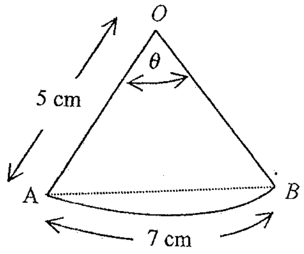
(e) Find **2**

$$\lim_{x\to 0}\frac{\sin 5x}{7x}$$

Show all working.

Question 2: [10 Marks]

Marks



An arc AB of length 7 cm of a circle of radius 5 cm subtends an angle θ at the centre O. If AB is the chord, find:

(i) $\angle AOB$ in radians. **2**

(ii) The area of sector AOB. **3**

(iii) The area of the segment enclosed between arc AB and chord AB. **2**

(iv) The ratio of arc AB to the length of chord AB. **3**

END OF SECTION A

SECTION B (Start a new booklet)

Question 3: [11 Marks]

Marks

(a) Given that

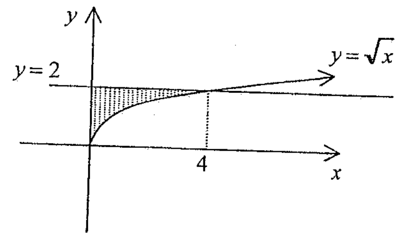
$$\cos 2x = \cos^2 x - \sin^2 x$$

(i) Show that $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ **3**

(ii) Hence evaluate **3**

$$\int_0^{\frac{\pi}{2}} \cos^2 x \, dx$$

(b) The diagram shows the area bounded by the y - axis, the curve $y = \sqrt{x}$ and the line $y = 2$. **5**



Find the area of the shaded area.

Question 4: [15 Marks]

Marks

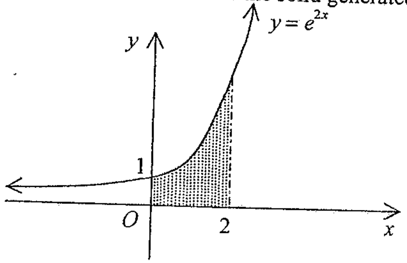
(a) Let $f(x)=3\cos\left(2x+\frac{\pi}{2}\right)$

(i) State the period of $f(x)$. **2**

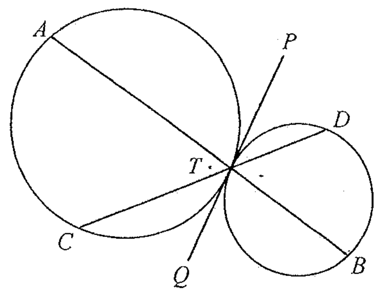
(ii) What is the range of $f(x)$? **2**

(iii) Sketch the curve of $y=f(x)$, for $-\pi\le x\le\pi$ **3**

(b) The diagram shows the region bounded by the curve $y=e^{2x}$, the $x-$axis, the $y-$axis and the line $x=2$. **4**
The shaded area is rotated about the $x-$axis between $x=0$ and $x=2$.
Find the exact value of the volume of the solid generated.



(c) In the diagram below, PQ is the common tangent to the two circles at T.



Copy the diagram in to your answer booklet.

Prove that AC is parallel to DB. **4**

END OF SECTION B

$\frac{1}{n+1}$

SECTION C (Start a new booklet)

Question 5: [8 Marks]

Marks

(i) Copy and complete the table of values for $y=\frac{1}{1+x^2}$. Give answers in exact form. **2**

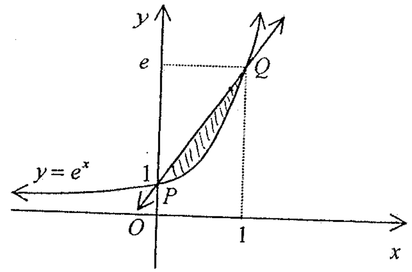
x	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2
y					

(ii) Hence use Simpson's Rule with five function values to estimate **6**

$$\int_0^2 \frac{dx}{1+x^2}$$

Question 6: [8 Marks]

Marks



The sketch above shows the curve $y=e^x$ and the points $P(0,1)$ and $Q(1,e)$ on the curve.

(i) Show that the equation of the chord PQ is $(e-1)x-y+1=0$ **4**

(ii) Find the area enclosed between the curve $y=e^x$ and the chord PQ. **4**

Question 7: [16 Marks]

Marks

(a) (i) On the SAME diagram, sketch the graphs of $y = e^{-\frac{1}{2}x}$ and $y = 5 - x^2$, showing all intercepts with the coordinate axes. **3**

(ii) On your diagram, indicate the negative root, α, of the equation **2**

$$x^2 + e^{-\frac{1}{2}x} = 5$$

(iii) Show that $-2 < \alpha < -1$ **2**

(iv) Use one iteration of Newton's Method, starting with $x = -2$ to show that α is approximately **3**

$$-\frac{18}{e+8}$$

(b) Find $\lim_{x \to 0} \frac{1 - \cos 2x}{x^2}$ **3**

(c) Prove $7^n + 13^n + 19^n$ is a multiple of 13, if n is odd. **3**

THIS IS THE END OF THE EXAMINATION

Sydney Boys High School
2002 Assessment Task #2. Ext 1

$\frac{79}{80}$

Excellent work!

QUESTION 1

A) 14.880 (3dp) ✓

B) $\frac{300°}{180°} \times \pi = \frac{5\pi}{3}^c$ ✓

c) i.) $\frac{d}{dx} \tan(3x+1) = 3\sec^2(3x+1)$ ✓✓

ii.) $\frac{d}{dx} e^{4x-1} = 4e^{4x-1}$ ✓✓

D) i.) $\int \sin 2x \, dx = -\frac{1}{2}\cos 2x + C$ ✓✓

ii.) $\int e^{-x} dx = -e^{-x} + C$

12

E) $\lim_{x \to 0} \frac{\sin 5x}{7x} = \lim_{x \to 0} \frac{\sin 5x}{5x} \times \frac{5}{7}$ ✓

$= \frac{1 \times 5}{7} = \frac{5}{7}$ ✓

QUESTION 2

i.) $l = r\theta$

$\theta = \frac{l}{r}$; $\theta = \frac{7}{5}$. ✓

$\therefore \angle AOB = \frac{7}{5}^c$ ✓

ii.) Area of sector AOB $= \frac{\frac{7}{5}}{2\pi} \times \pi(25)$ ✓✓

$= \frac{7}{10} \times 25 = 17.5 \text{ cm}^2$ ✓

iii.) Area of $\Delta AOB = \frac{1}{2} \times 5 \times 5 \times \sin \frac{7}{5}$

$= 12.3\ cm^2$ ✓

Area of segment $= 17.5 cm^2 - 12.3 cm^2$

$= \underline{5.18\ cm^2}$ ✓

10

chord

iv.) $AB^2 = 5^2 + 5^2 - 2(5)(5)\cos\frac{7}{5}$ ✓

chord $AB = 6.44$

Ratio of arc AB : chord AB

$= 7 : 6.44$ (divide throughout by 6.44)

$= \cancel{\frac{7}{6.44}}$

$= \underline{1.09 : 1}$ ✓ ?

QUESTION 3

A) $\cos 2x = \cos^2 x - \sin^2 x$ ~~why not use cos(x+x)~~

i.) R.T. show $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

is (from above) $\cos^2 x = \cos 2x + \sin^2 x$

$\cos 2x = 1 - 2\sin^2 x$

$2\sin^2 x = 1 - \cos 2x$

$\sin^2 x = \frac{1 - \cos 2x}{2}$

$= \cos 2x + \frac{1 - \cos 2x}{2}$ ✓

$\cos^2 x = \frac{2\cos 2x + 1 - \cos 2x}{2}$ ✓

$\cos^2 x = \frac{\cos 2x + 1}{2}$ ✓

$= \frac{1}{2}(\cos 2x + 1) = \underline{RHS}$

ii.) $\int_0^{\frac{\pi}{2}} \cos^2 x\, dx = \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 + \cos 2x)\, dx$ ✓

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} 1 + \cos 2x\, dx$

$= \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right)_0^{\frac{\pi}{2}} = \frac{1}{2}\left[\frac{\pi}{2}\right] = \frac{\pi}{4}$

$\cancel{= \frac{1}{2}\left(1 + \frac{1}{2}\sin\pi - 1 - \frac{1}{2}\sin 0\right)} = \underline{0}$

B) Area $= (4 \times 2) - \int_0^4 \sqrt{x}\, dx$ ✓

$= 8 - \left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right)_0^4$ ✓

10

$= 8 - \left(\frac{2x^{\frac{3}{2}}}{3}\right)_0^4$ ✓

$= 8 - \frac{2}{3} \cdot 4^{\frac{3}{2}}$ ✓

$= 8 - \frac{16}{3} = \underline{\underline{\frac{8}{3}}}\ u^2$ ✓

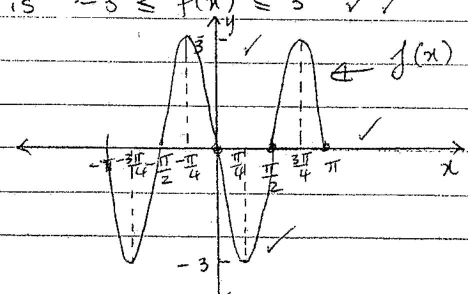
QUESTION 4

A) $f(x) = 3\cos\left(2x + \frac{\pi}{2}\right)$

i.) Period $= \frac{2\pi}{n}$ ✓ $= \frac{2\pi}{2} = \underline{\underline{\pi}}$ ✓

ii.) Range is $-3 \le f(x) \le 3$ ✓✓

iii.)



$f(x) = 3\cos\left(2x + \frac{\pi}{2}\right)$

b) $V = \pi \int_0^2 e^{4x}\,dx$

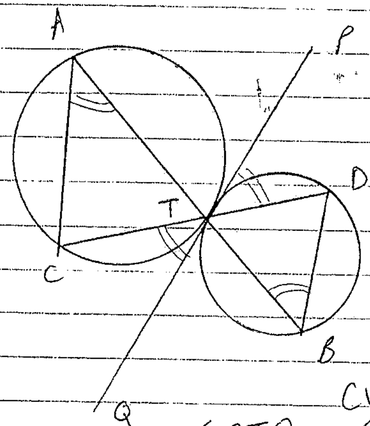
$= \pi \left(\frac{1}{4}e^{4x}\right)_0^2$

$= \pi \left(\frac{1}{4}e^{8} - \frac{1}{4}\right)$

$= \frac{\pi}{4}(e^8 - 1)\ u^3$

15

c)



Construct AC and DB

Let $\angle PTD = \alpha$.

$\angle PTD = \angle TBD = \alpha$

($\angle$ made by tgt and chord equals $\angle$ in alt segment).

Also, $\angle PTD = \angle CTQ = \alpha$

(vert. opp $\angle$ equal)

$\angle CTQ = \angle CAT = \alpha$ ($\angle$ made by tgt and chord equals $\angle$ in alt. segment)

$\therefore \angle CAT = \angle TBD$

$\therefore AC \parallel DB$ (alt $\angle$ on $\parallel$ lines)

QUESTION 5

8

i.)

x	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
$y = \frac{1}{1+x^2}$	1	$\frac{4}{5}$	$\frac{1}{2}$	$\frac{4}{13}$	$\frac{1}{5}$

ii.) $\int_0^2 \frac{dx}{1+x^2} \doteq \frac{\frac{1}{2}}{3}\left(1 + 4\left(\frac{4}{5}\right) + 2\left(\frac{1}{2}\right) + 4\left(\frac{4}{13}\right) + \frac{1}{5}\right)$

$\doteq \frac{1}{6}\left(1 + \frac{16}{5} + 1 + \frac{16}{13} + \frac{1}{5}\right)$

$\doteq 1\frac{41}{390}$

QUESTION 6

8

i.) Grad. of PQ $= \frac{e-1}{1-0} = e-1$

Eqt of PQ is $(y-e) = (e-1)(x-1)$

$y = ex - e - x + 1 + e$

$y = ex - x + 1$

$y = x(e-1) + 1$

$\therefore$ eqt. of PQ is $x(e-1) + 1 - y = 0$

ii.) Area $= \int_0^1 x(e-1) + 1 - e^x\,dx$

$= \int_0^1 ex - x + 1 - e^x\,dx$

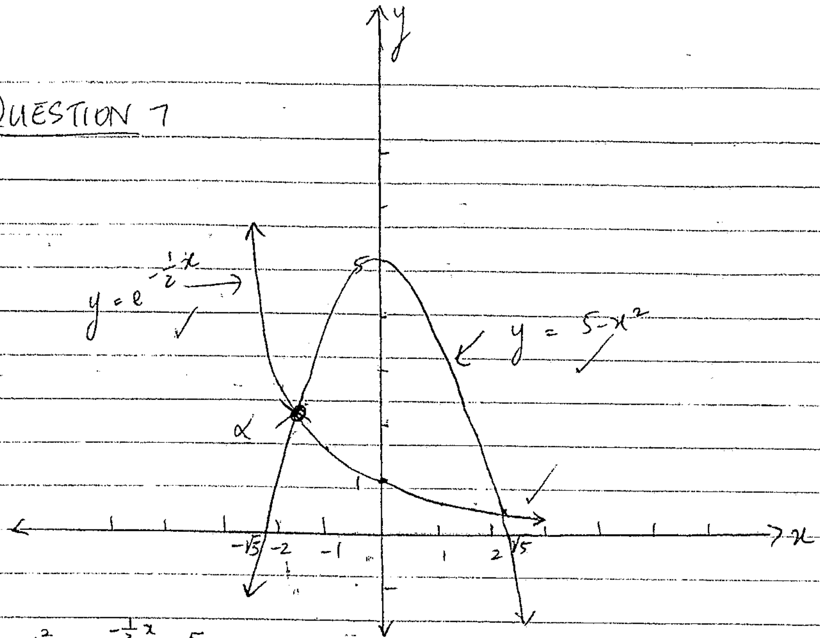
$= \left(\frac{ex^2}{2} - \frac{x^2}{2} + x - e^x\right)_0^1$

$= \left(\frac{e}{2} - \frac{1}{2} + 1 - e\right) - (-1) = \frac{e}{2} + \frac{1}{2} - e + 1 = \frac{3}{2} - \frac{e}{2}$

$= \boxed{\frac{3-e}{2}\ u^2}$

QUESTION 7

A) i.)



ii.) $x^2 + e^{-\frac{1}{2}x} = 5$

$e^{-\frac{1}{2}x} = 5 - x^2$

Find the NEGATIVE intersection btw the two graphs (done on diagram)

iii.) $x^2 + e^{-\frac{1}{2}x} - 5 = 0$

Let $P(x) = x^2 + e^{-\frac{1}{2}x} - 5$

$P(-2) = 4 + e - 5 = 1.170$

$P(-1) = 1 + e - 5 = -1.28 < 0$

$P(x)$ is a continuous function

$\therefore$ a root lies between $x = -2$ and $x = -1$

Let root be $x = \alpha$

$\therefore \; -2 < \alpha < -1$

iv.) $P(x) = x^2 + e^{-\frac{1}{2}x} - 5$

$P'(x) = 2x - \frac{1}{2}e^{-\frac{1}{2}x}$

$x = -2$

$x_1 = x - \frac{P(x)}{P'(x)}$

$x_1 = -2 - \frac{(e-1)}{-4 - \frac{1}{2}e}$ $\cancel{= -2 - \frac{(e-1)}{\frac{-8-e}{2}} = -2 - \frac{(e-1) \times 2}{-(8+e)}}$

$x_1 = -2 - \frac{(e-1)}{\frac{-8-e}{2}}$

$= -2 - \frac{(e-1) \times 2}{-8-e}$

$= \frac{-2(-8-e) - 2(e-1)}{-(8+e)}$

$= \frac{2(8+e) - 2e + 2}{-(8+e)}$

$= \frac{16 + \cancel{2e} - \cancel{2e} + 2}{-(8+e)}$

$= \frac{-18}{8+e}$

B) $\lim_{x \to 0} \frac{1 - \cos 2x}{x^2}$

$= \lim_{x \to 0} \frac{1 - (1 - 2\sin^2 x)}{x^2}$

$= \lim_{x \to 0} \frac{2\sin^2 x}{x^2}$

$= 2 \times \lim_{x \to 0} \frac{\sin x}{x} \times \frac{\sin x}{x}$

$= 2 \times 1 \times 1 = \boxed{2}$

16

c) Step 1

Let $n=1$.

$7^n+13^n+19^n = 7+13+19 = 39 = 3(13)$

which is a multiple of 13.

$\therefore$ true for $n=1$

Step 2

Assume true for $n=k$ (k is an odd integer)

$7^k+13^k+19^k = 13M$ (M is an integer)

R.T.P. also true for $n=k+2$

$7^{k+2}+13^{k+2}+19^{k+2} = 13N$ (N is another integer)

LHS $7^{k+2}+13^{k+2}+19^{k+2} = 7^2(7^k+13^k+19^k)+120.13^k+312.19^k$

$= 7^2(13M)+120.13^k+312.19^k$

$= 637M+120.13^k+312.19^k$

$= 13(49M+120.13^{k-1}+24.19^k) = 13N = RHS$

Step 3

If true for $n=k$ and $n=k+2$ and also true for $n=1$, then it is true for $n=1$, $n=1+2=3$, $n=3+2=5$ and so on. $\therefore$ by the Principle of Mathematical Induction, $7^n+13^n+19^n$ is a multiple of 13 for all odd numbers, n.