SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2005
HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK #2

Mathematics Extension 2

General Instructions

- Reading Time – 5 Minutes
- Working time – ~~90 Minutes~~ 2hrs
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- Each Section is to be returned in a separate bundle.
- All necessary working should be shown in every question.

Total Marks – 92

- Attempt questions 1 – 6

Examiner: *C.Kourtesis*

Section A
(Start a new answer sheet.)

Question 1. (18 marks)

Marks

(a) If $z = 1 + 2i$ and $w = 3 - i$ express $z^2 \div \overline{w}$ in the form $a + ib$ (where a and b are real). **3**

(b) Sketch the region in the Argand diagram represented by $|z + i| \leq 2$. **2**

(c) Find all pairs of integers a and b that satisfy $(a + ib)^2 = -3 - 4i$. **3**

(d) If $|z - 1| = \text{Re}(z) + 1$ find the locus of z. **3**

(e) If z and w are two complex numbers, prove that **3**

$$\overline{z - w} = \overline{z} - \overline{w}.$$

(f) (i) Express each of the complex numbers $z = 2i$ and $w = 1 + i\sqrt{3}$ in modulus-argument form. **2**

(ii) Find the exact value of $\arg(z + w)$. **2**

Question 2. (15 marks)

Marks

(a) Find $\int \frac{dx}{x^2+2x+5}$. **2**

(b) Find $\int \frac{\sin x \, dx}{(2+7\cos x)^5}$ using the substitution $u = 2+7\cos x$. **2**

(c) Use integration by parts to find: **3**

$$\int x^3 \ln x \, dx$$

(d) (i) Find real numbers a, b, and c such that **2**

$$\frac{6}{x^2(x+3)} = \frac{ax+b}{x^2} + \frac{c}{x+3}.$$

(ii) Find $\int \frac{6}{x^2(x+3)} dx$ **2**

(e) Use the substitution $t = \tan\frac{x}{2}$ to evaluate **4**

$$\int_0^{\frac{\pi}{3}} \frac{1}{1-\sin x} dx$$

Give your answer in simplest exact form.

Section B
(Start a new answer sheet.)

Question 3. (14 marks)

Marks

(a) Find the equation whose roots are twice those of the equation: **2**

$$x^3 - x - 3 = 0.$$ $y = 2x$

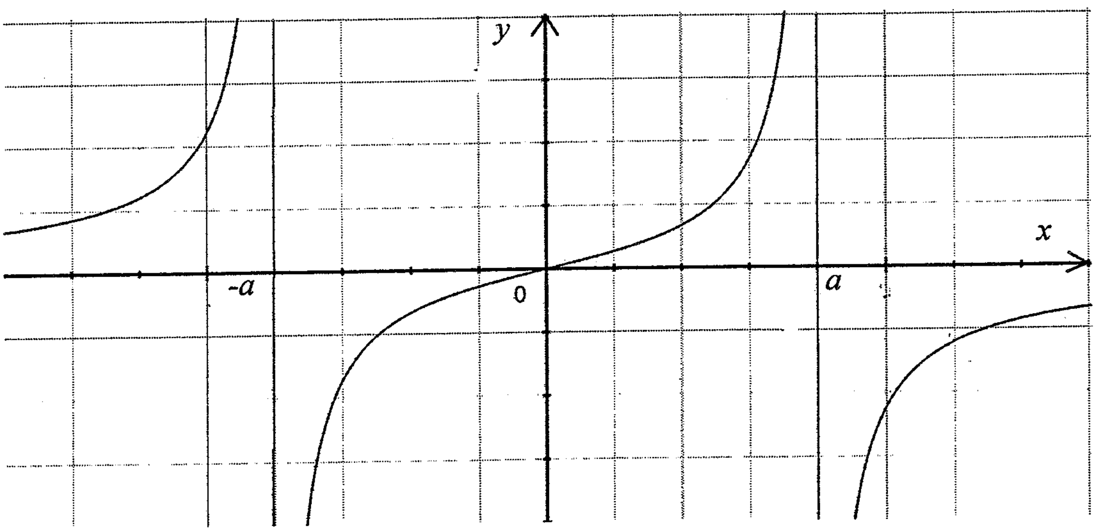
(b) (i) Prove that if α is a double root of the equation $P(x) = 0$ then $P'(\alpha) = 0$. **2**

(ii) Prove that $P(x) = x^3 - 3ax + b$ has a double root if $b^2 = 4a^3$. **3**

(iii) Hence or otherwise solve the equation **2**

$$x^3 + 3x + 2i = 0$$

(c) The graph of $y = f(x)$ is shown:



Draw sketches of the following curves (on separate diagrams):

(i) $y = [f(x)]^2$ **1**

(ii) $y = f'(x)$ **2**

(iii) $y = \dfrac{1}{f(x)}$ **2**

Question 4 (15 marks)

(a) Sketch the graphs of:

(i) $y = \sin^2 2x$ for $-2\pi \le x \le 2\pi$. **2**

(ii) $\sin(x+y) = 0$ **2**

(b) (i) Sketch the graph of $y = |x-1|(x+1)$. **2**

(ii) Hence or otherwise sketch the graph of **2**

$$y = \frac{1}{|x-1|(x+1)}.$$

(c) Consider the function:

$$f(x) = \frac{e^{-x} - e^{x}}{e^{x} + e^{-x}}$$

(i) Show that $f(x)$ is an odd function. **1**

(ii) Find the equations of any asymptotes. **2**

(iii) Show that $f(x)$ is a decreasing function for all real x. **2**

(iv) Sketch the graph of $y = f(x)$. **2**

Section C
(Start a new answer booklet)

Question 5 (13 marks)

(a) Sketch the graph of the function **4**

$$y = x^{\frac{1}{3}} + \tfrac{1}{4}x^{\frac{4}{3}}$$

indicating the nature of any turning points and the co-ordinates of any points of inflexion.

(b) A rectangle is divided by m lines parallel to one pair of opposite sides and by n lines parallel to the other pair of opposite sides. **2**

How many rectangles of any size are formed in the resulting figure? (Leave your answer in unsimplified form.)

(c) Given that $z_1 = f(z) = \dfrac{z+i}{z-i}$, show that $f(z_1) = \left(\dfrac{z+1}{z-1}\right).i$ **3**

(d) If a and b are two roots of the equation $x^3 + 4x - 2 = 0$, show that ab is a root of the equation $x^3 - 4x^2 - 4 = 0$. **4**

Question 6 (17 marks)

(a) Consider the function

$$f(x) = x + \log_e(1-x)$$

(i) Sketch the graph of $y = f(x)$ showing all essential features. **3**

(ii) Hence show that $x \le \log_e(1-x)$ for all $x \le 1$. **1**

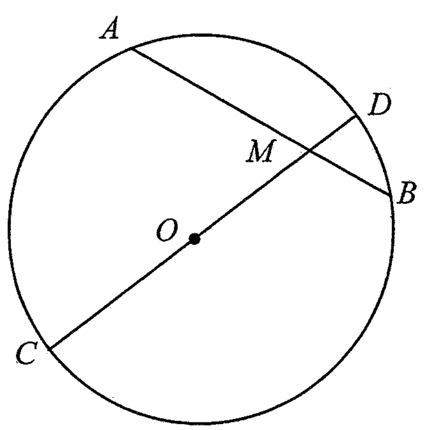
(b) (i) Show that $\int_0^{\frac{\pi}{2}} x\cos x\,dx = \frac{\pi}{2} - 1$. **2**

(ii) If $u_n = \int_0^{\frac{\pi}{2}} x\cos^n x\,dx$ and n is a positive integer greater than 1, prove that **4**

$$u_n = \left(\frac{n-1}{n}\right)u_{n-2} - \frac{1}{n^2}.$$

(iii) Deduce that $u_5 = \frac{4\pi}{15} - \frac{149}{225}$. **2**

(c)



A chord AB and a diameter CD of a circle centre O intersect at a point M within the circle.
(M is not the centre.)

(i) Show that $(CM + MD)^2 > (AM + MB)^2$ **2**

(ii) Deduce that $(CM - MD)^2 > (AM - MB)^2$ **3**

This is the end of the paper.

SBHS HSC 2005 AT 2 Ashley Cheng. Mon 4:30

Question 1 Excellent effort after Question 1?

(a) $z^2 = (1+2i)^2 = -3+4i$ ✓

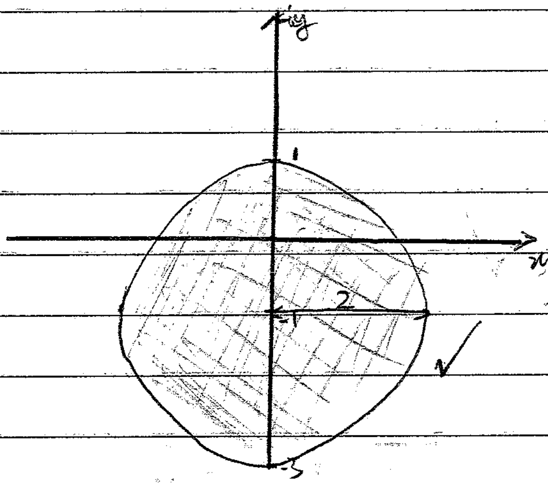
$\bar{\omega} = 3+i$ ✓

$\therefore \frac{z^2}{\bar{\omega}} = \frac{-3+4i}{3+i} \times \frac{3-i}{3-i}$

$= \frac{-9+3i+12i+4}{9+1}$

$= -\frac{1}{2} + \frac{3}{2}i$ ✓

b)



c) $(a+ib)^2 = -3-4i$ Also $z^2 = -3-4i$

$a^2-b^2+2abi = -3-4i$ $|z^2| = 5 \Rightarrow a^2+b^2 = 5$

$a^2-b^2 = -3$ $|b|>|a|$? $a^2-b^2 = -3$

$2ab = -4$ $2a^2 = 2$

$ab = -2$ $\therefore a<0$ $a^2 = 1$

$a = \pm 1$, $b = \mp 2$.

$\therefore a = -1$, $b = 2$ ✓

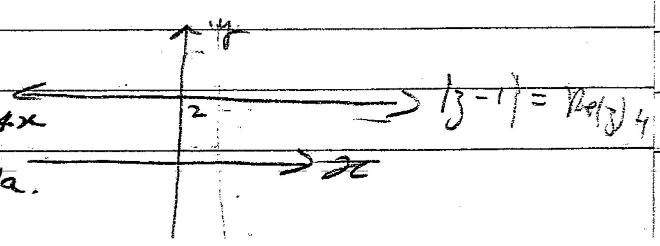
or $a = 1$, $b = -2$

d) $|z-1| = Re(z) + 1$ let $z = x+iy$

$|(x-1)+iy| = x+1$

$\sqrt{(x-1)^2+y^2} = (x+1) \Rightarrow y^2 = -4x$

$x-1+iy = x+1$ Parabola.



$iy = 2$ ...

e) n/p $\overline{z-w} = \bar{z} - \bar{w}$

RHS $= \bar{z} - \bar{w}$ let $z = x+iy$, $w = u+vi$

$= x - iy - (u - vi)$

$= x - u - i(y - v)$

$= \overline{x - u + i(y-v)}$

LHS $= \overline{z - w}$

$= \overline{x+iy-u-vi}$

$= \overline{(x-u)+i(y-v)}$

= RHS.

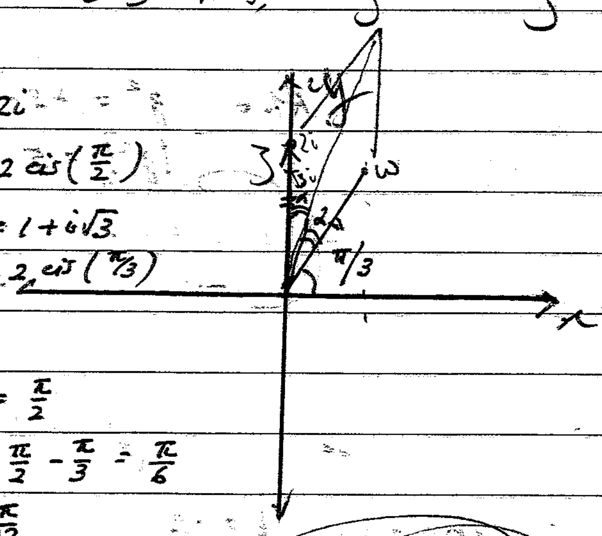
∴ Since LHS = RHS, $\overline{z-w} = \bar{z} - \bar{w}$

f) (i) $z = 2i$

$= 2\,\text{cis}\left(\frac{\pi}{2}\right)$

$w = 1 + i\sqrt{3}$

$= 2\,\text{cis}\left(\frac{\pi}{3}\right)$



$2\alpha + \frac{\pi}{3} = \frac{\pi}{2}$

$2\alpha = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$

$\alpha = \frac{\pi}{12}$

(ii) $\arg(z+w) = \arg(z) \times \arg(w)$

$\arg(z) = \frac{\pi}{2}$

$\arg(w) = \frac{\pi}{3}$

∴ $\cancel{\arg(z+w) = \frac{\pi^2}{6}}$

Question 2

a) $\int \frac{1}{(x^2+2x+5)}\,dx = \int \frac{1}{(x+1)^2+4}\,dx$ ✓

$= \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C$ ✓

b) $\int \frac{\sin x\,dx}{(2+7\cos x)^5}$ let $u = 2+7\cos x$

$\frac{du}{dx} = -7\sin x$ ✓

$-\frac{1}{7}\int \frac{1}{u^5}\,du = -\frac{1}{7}\int u^{-5}\,du$

$= -\frac{1}{7}\left(\frac{u^{-4}}{-4}\right) + c$

$= -\frac{+1}{28u^4} + c$

$= \frac{+1}{28(2+7\cos x)^4} + C$

(c) $\int x^3 \ln x\,dx$

$u = \ln x$ $\quad v' = x^3$

$u' = \frac{1}{x}$ $\quad v = \frac{1}{4}x^4$

$\int x^3 \ln x\,dx = \frac{1}{4}x^4\ln x - \frac{1}{4}\int x^3$ ✓

$= \frac{1}{4}\left(x^4\ln x - \frac{1}{4}x^4\right) + c$ ✓

d) (i) $6 = (ax+b)(x+3) + cx^2$

let $x = -3$

$6 = 9c$

$c = \frac{2}{3}$ ✓

let $x = 1$

$6 = 4(a+2) + \frac{2}{3}$

$6 = 4a + 8 + \frac{2}{3}$ ✓

$a = -\frac{2}{3}$ ✓

let $x = 0$

$6 = 3b$

$b = 2$ ✓

d) (i) cont'd)

$$\frac{6}{x^2(x+3)} = \frac{-\frac{2}{3}x+2}{x^2} + \frac{\frac{2}{3}}{x+3}$$

(ii) $$\int \frac{6}{x^2(x+3)}\,dx = \int -\frac{1}{3}\left(\frac{2x-6}{x^2}\right)dx + \int \frac{2}{3}\left(\frac{1}{x+3}\right)dx$$

$$= -\frac{2}{3}\ln(x) - \frac{2}{x} + \frac{2}{3}\ln(x+3) + C$$

e) $$\int_0^{\frac{\pi}{3}} \frac{1}{1-\sin x}\,dx$$ let $t = \tan\frac{x}{2}$, $dx = \frac{2\,dt}{1+t^2}$

$$= \int_0^{\frac{\pi}{3}} \frac{1}{1-\left(\frac{2t}{1+t^2}\right)} \cdot \frac{2\,dt}{1+t^2}$$

$$= \int_0^{\frac{\pi}{3}} \frac{2}{1+t^2-2t}\,dt$$

$$= \int_0^{\frac{\pi}{3}} \frac{2}{(t-1)^2}\,dt$$ let $u = t-1$, $du = dt$

$$= \int_0^{\frac{\pi}{3}} \frac{2}{u^2}\,du$$

$$= \left[-\frac{2}{u}\right]_0^{\frac{\pi}{3}} = \left[-\frac{2}{t-1}\right]_0^{\frac{\pi}{3}} = \left[-\frac{2}{\tan\frac{x}{2}-1}\right]$$

$$= \left[-\frac{2}{\frac{1}{\sqrt{3}}-1} - 2\right]$$

$$= \left[-\frac{2\sqrt{3}}{1-\sqrt{3}} - 2\right] = \frac{-2\sqrt{3}-2+2\sqrt{3}}{1-\sqrt{3}} = \frac{-2}{1-\sqrt{3}}$$

$$= \frac{-2(1+\sqrt{3})}{4}$$

$$= \frac{-(1-\sqrt{3})}{2}$$

Question 3 SBHS AT2 ME2 Monday Mon 7:30

a) Let $x = \frac{y}{2}$

$$\left(\frac{y}{2}\right)^3 - \left(\frac{y}{2}\right) - 3 = 0$$ ✓

$$y^3 - 4y - 24 = 0$$

$$x^3 - 4x - 24 = 0$$ ✓

b) (i) Let $P(x) = (x-\alpha)^2 Q(x)$

$$P'(x) = (x-\alpha)^2 Q'(x) + 2(x-\alpha)Q(x)$$ ✓
$$= (x-\alpha)\left[(x-\alpha)Q'(x) + 2Q(x)\right]$$

$$\therefore P'(\alpha) = 0$$ ✓

(ii) $P(x) = x^3 - 3ax + b$
$P'(x) = 3x^2 - 3a$

$$\therefore 3x^2 - 3a = 0$$
$$x^2 = a$$ ✓
$$x = \sqrt{a}$$

$$P(\sqrt{a}) = (\sqrt{a})^3 - 3a\sqrt{a} + b = 0 \quad \left\{\text{since } P(\sqrt{a}) = 0\right\}$$
$$= a\sqrt{a} - 3a\sqrt{a} + b = 0$$ ✓
$$b = -2a\sqrt{a}$$
$$b^2 = 4a^3$$ ✓

(iii) let $P(x) = x^3 + 3x + 2i$
If $b^2 = 4a^3$ then $P(x)$ has a double root.

$x.70$

b) (iii) $b^2 = (2i)^2 = 4i^2$ $\quad 4a^3 = 4(-1)^3$

$= -4 \qquad\qquad = -4$

$\therefore b^2 = 4a^3$

$\therefore$ double root

$P'(x) = 3x^2 + 3 = 0$

$x^2 = -1$

$x = -i$ ✓

$\therefore P(x) = x^3 + 3x + 2i = (x+i)^2 Q(x)$

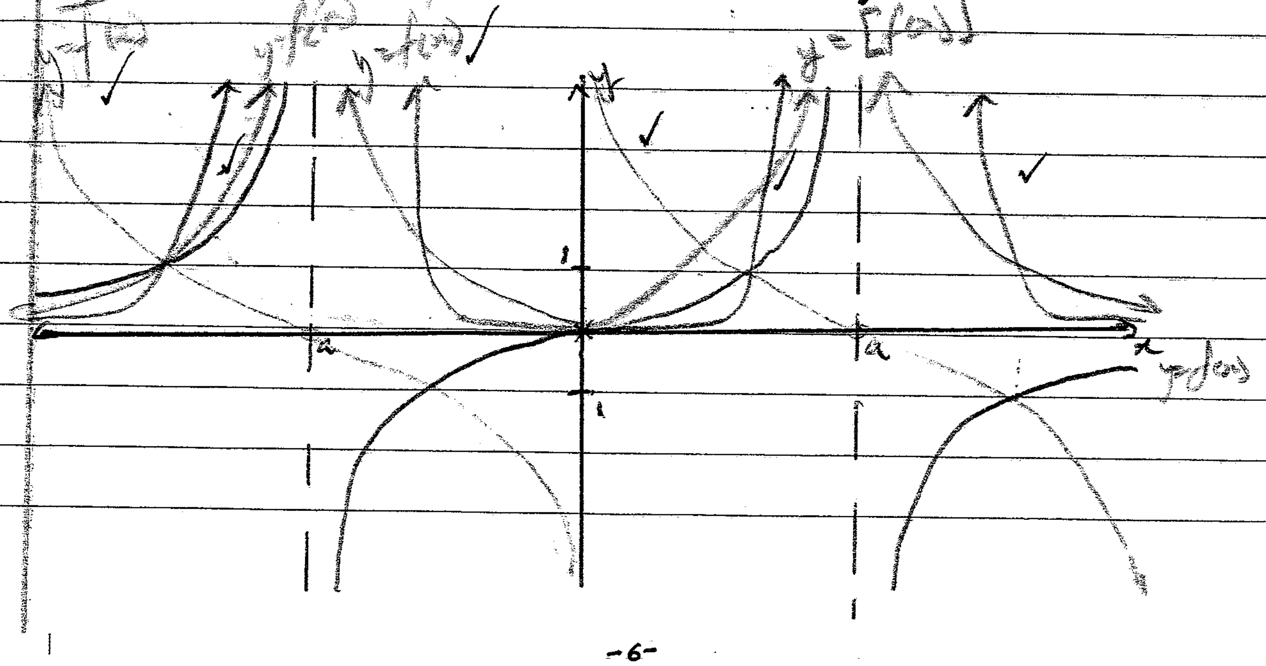
$(x+i)^2 = x^2 + 2ix - 1$ ✓

$$\begin{array}{r l}
 & x - 2i \\
x^2 + 2ix - 1 \,\big)\!\! & x^3 + 3x + 2i \\
 & x^3 + 2ix^2 - x \\
\hline
 & -2ix^2 + 4x + 2i \\
 & -2ix^2 + 4x + 2i
\end{array}$$

$\therefore P(x) = x^3 + 3x + 2i = (x+i)^2(x-2i)$ ✓

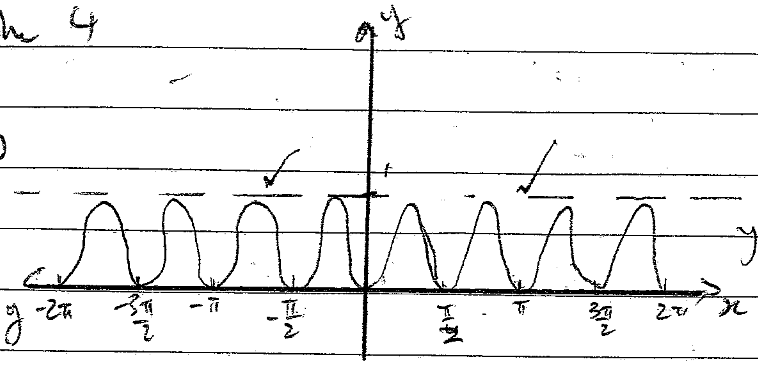
$\therefore x = -i, -i; 2i$

c) (i), (ii), (iii)



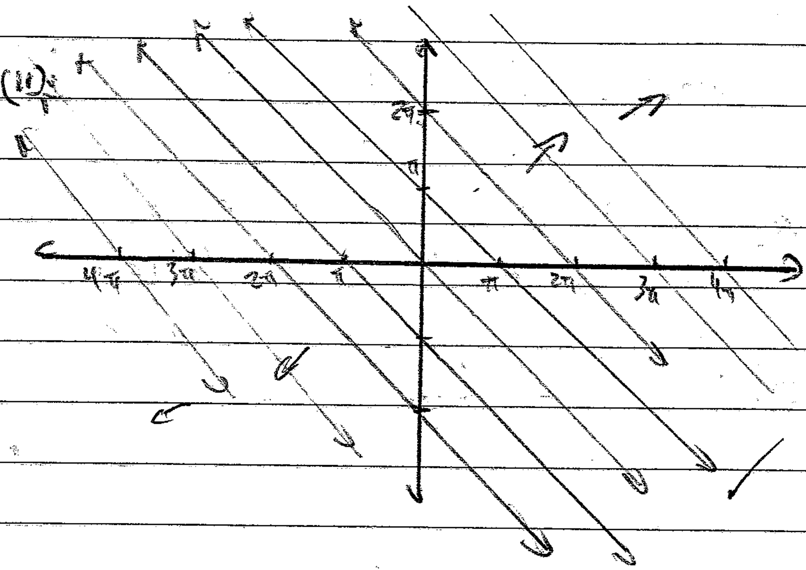
Question 4

a) (i)



$y = \sin^2 2x \quad (-2\pi \le x \le 2\pi)$

(ii)

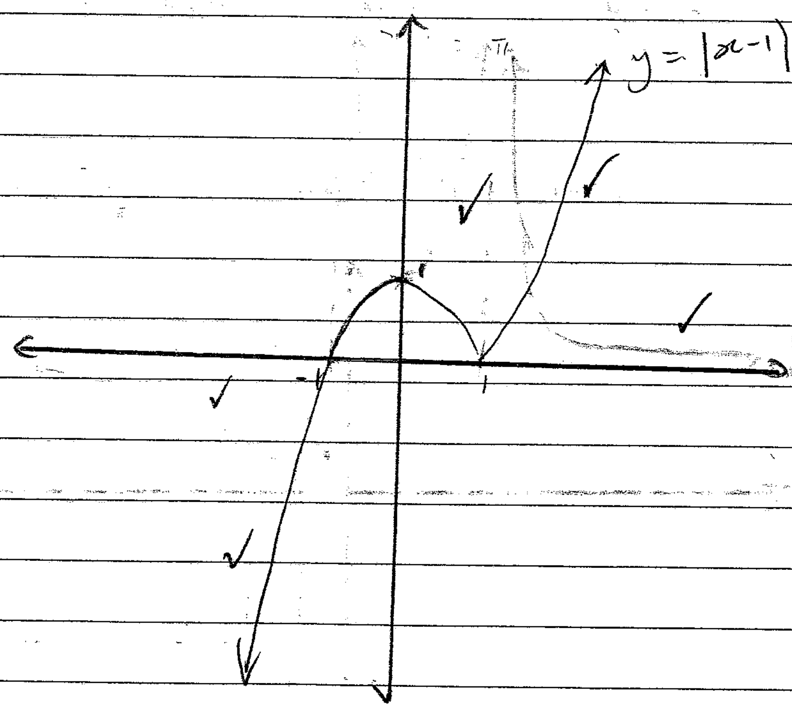


$\sin(x+y) = 0$

$x + y = 0, \pi, 2\pi, \ldots \quad \pm\pi \pm 2\pi$

$y = -x \pm n\pi$

b) (i)



$y = |x-1|(x+1)$

c) (i) for odd func $f(x) = -f(-x)$

$$RHS = \frac{e^{-(-x)} - e^{(-x)}}{e^{(-x)} + e^{-(-x)}} = -\frac{e^{x} - e^{-x}}{e^{-x} + e^{x}}$$

$$= \frac{e^{-x} - e^{x}}{e^{-x} + e^{x}} = LHS$$

Hence f(x) is odd

c) (ii) $\lim_{x \to \infty} \frac{e^{-x} - e^{x}}{e^{x} + e^{-x}} = -1$ ✓

$\lim_{x \to -\infty} \frac{e^{-x} - e^{x}}{e^{x} + e^{-x}} = 1$ ✓

$\therefore$ asymptotes at $y = 1$ & $y = -1$ ✓

(iii) $f(x) = \frac{e^{-x} - e^{x}}{e^{x} + e^{-x}}$ $\quad u = e^{-x} - e^{x}$ $\quad v = e^{x} + e^{-x}$

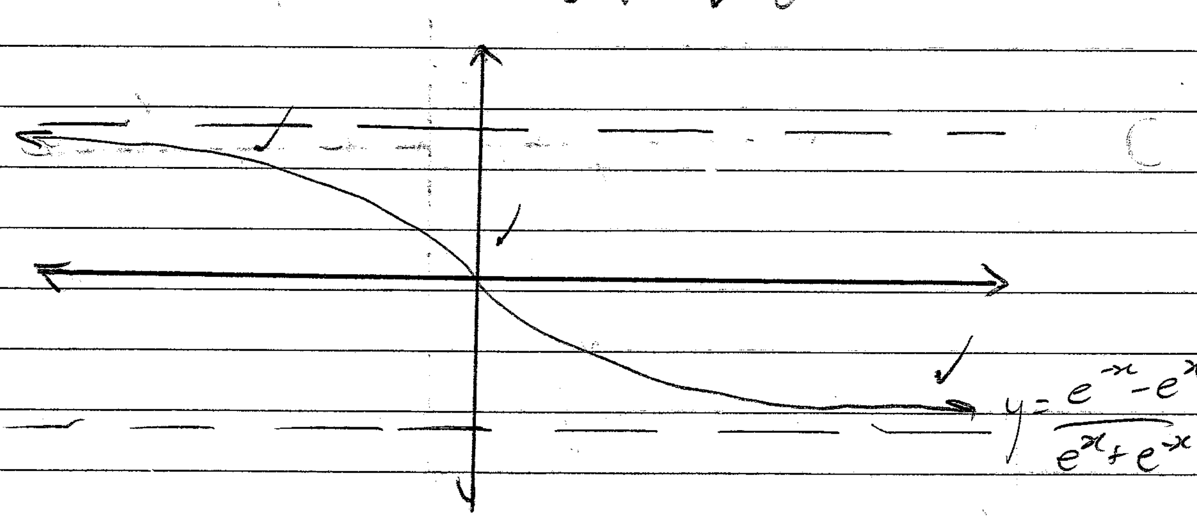
$u' = -(e^{-x} + e^{x})$ $\quad v' = e^{x} - e^{-x}$

$f'(x) = \frac{-(e^{-x} + e^{x})(e^{x} + e^{-x}) - (e^{-x} - e^{x})(e^{x} - e^{-x})}{(e^{x} + e^{-x})^2}$ ✓

$f'(x) < 0$ ✓ since numerator < 0
denominator > 0

$\therefore f(x)$ is a decreasing function ✓ for $x \in \mathbb{R}$

(iv)



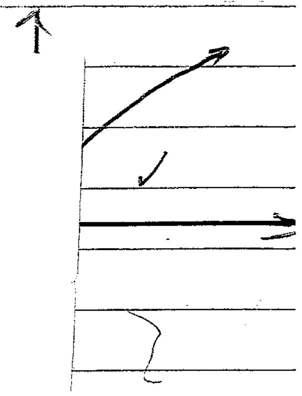
Questn 5

(a)

$$y'' = 0 = \frac{1}{3}x^{-\frac{2}{3}} + \frac{1}{3}x^{\frac{1}{3}}$$

$$= \frac{1}{3}x^{\frac{1}{3}}\left[x^{-1} + 1\right] ✓$$

$$\therefore x = 0 \ \& \ x = -1 ✓$$



$$y'' = -\frac{2}{9}x^{-\frac{5}{3}} + \frac{1}{9}x^{-\frac{2}{3}}$$

test at $x = 0$

$$y'' = 0$$

$\therefore$ possible POI

test at $x = -1$

$$y'' = \frac{1}{3}$$

concave up

$\therefore$ min tp. ✓

?

test y'' at $x = 0$

x	-0.5	0	0.5
y''	$+$	0	$-$

$\therefore$ POI ✓

$\therefore$ see graph on original page.

-9-

$$z + i - i(z - i)$$

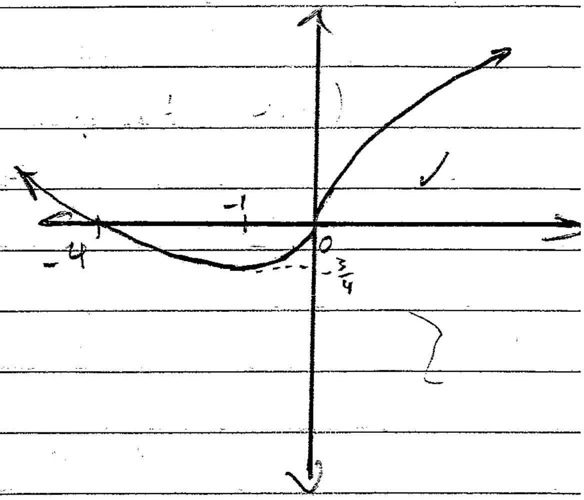
$$= \frac{z + 1 + i(z + 1)}{z - 1 - i(z - 1)}$$

$$= \frac{z+1}{z-1}\left(\frac{1+i}{1-i}\right)$$

but $\frac{1+i}{1-i} = i$ ✓

$$\therefore \left(\frac{z+1}{z-1}\right)i = f(z_1)$$

$$y = x^{\frac{1}{3}} + \frac{1}{4}x^{\frac{4}{3}}$$



$$y' = \frac{1}{3}x^{-\frac{2}{3}} + \frac{1}{3}x^{\frac{1}{3}}$$

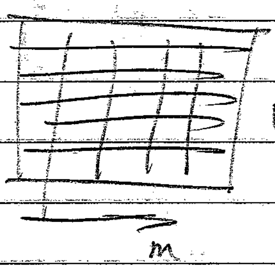
$$y'' = -\frac{2}{9}x^{-\frac{5}{3}} + \frac{1}{9}x^{-\frac{2}{3}}$$

$$0 = x^{\frac{1}{3}} + \frac{1}{4}x^{\frac{4}{3}}$$

$$= x^{\frac{1}{3}}\left(1 + \frac{1}{4}x\right)$$

$$\therefore x = 0, \quad x = -4$$

b)



$$\therefore \; {}^{m+2}C_2 \cdot {}^{m+2}C_2 \; ?$$

c) $$f(z) = \frac{\left(\frac{z+i}{z-i}\right) + i}{\left(\frac{z+i}{z-i}\right) - i}$$

$$= \frac{z+i + i(z-i)}{z+i - i(z-i)}$$

$$= \frac{z+1 + i(z+1)}{z-1 - i(z-1)}$$

$$= \frac{z+1}{z-1}\left(\frac{1+i}{1-i}\right) \qquad \text{but } \frac{1+i}{1-i} = i$$

$$\therefore \left(\frac{z+1}{z-1}\right)i = f(z)$$

d) from root let γ be the other root of $x^3+4x-2=0$

$\therefore \; ab\gamma = 2$

$ab = \frac{2}{\gamma}$ $\therefore$ we must prove $\frac{2}{\gamma}$ is a root of $x^3-4x^2-4=0$

$$LHS = \left(\frac{2}{\gamma}\right)^3 - 4\left(\frac{2}{\gamma}\right)^2 - 4$$

$$= \frac{8}{\gamma^3} - \frac{16}{\gamma^2} - 4$$

$$= \frac{8 - 16\gamma - 4\gamma^3}{\gamma^3}$$

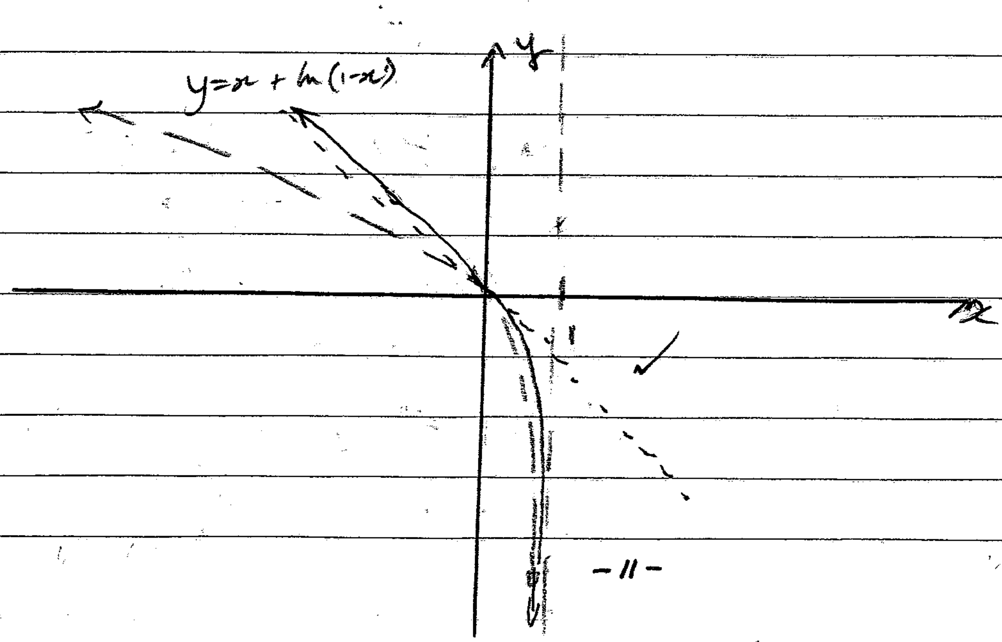
$$= \frac{4(2 - 4\gamma + \gamma^3)}{\gamma^3} \quad \text{but } \gamma^3 - 4\gamma + 2 = 0$$

$$= 0$$

$\therefore$ ab is a root of x^3-4x^2-4

Question 6

a)



Question 6

a) (ii) $y = \ln(1-x)$ is always above $y = x$

$\therefore \quad x \le \ln(1-x)$

b) (i) $\int_0^{\frac{\pi}{2}} x\cos x \, dx$ $\quad u = x \quad v' = \cos x$

$u' = 1 \quad v = \sin x$

$= \left[x\sin x \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x \, dx$

$= \frac{\pi}{2} + \left[\cos x \right]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{2} - 1$

(ii) $U_n = \int_0^{\frac{\pi}{2}} x\cos^n x \, dx = \int_0^{\frac{\pi}{2}} x\cos^{n-1} x \cos x \, dx$ $\quad u = x\cos^{n-1} x$

$u' = \cos^{n-1} x - (n-1)x(\cos^{n-2} x)\sin x$

$v' = \cos x$

$v = \sin x$

$= \left[x\sin x \cos^{n-1} x \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x \left[\cos^{n-1} x - x(n-1)\cos^{n-2} \sin x \right]$

$= -\int_0^{\frac{\pi}{2}} \sin x \cos^{n-1} x \, dx + \int_0^{\frac{\pi}{2}} x(n-1)\cos^{n-2} x \sin^2 x \, dx$

$= \int_0^{\frac{\pi}{2}} u^{n-1} \, du + (n-1)\int_0^{\frac{\pi}{2}} x\cos^{n-2} x(1 - \cos^2 x) \, dx$

$= \left[\frac{u^n}{n} \right]_0^{\frac{\pi}{2}} + (n-1)\left[\int_0^{\frac{\pi}{2}} x\cos^{n-2} x - \int_0^{\frac{\pi}{2}} x\cos^n x \right]$

$= \left[\frac{\cos^n x}{n} \right]_0^{\frac{\pi}{2}} + (n-1)\left[U_{n-2} - U_n \right]$

$= -\frac{1}{n} + (n-1)U_{n-2} - (n-1)U_n$

Questn 6

c) (ii) $(CM + MD)^2 > (AM + MB)^2$

$CM^2 + 2(CM)(MD) + MD^2 > AM^2 + 2(AM)(MB) + MB^2$

In ΔMCB & ΔMAD

$\measuredangle AMD = \measuredangle CMB$ (vertically opposite)

$\measuredangle DAB = \measuredangle DCB$ (angle on same chord)

$\measuredangle ADC = \measuredangle ABC$ (" same as above ")

$\therefore \Delta MCB \ ||| \ \Delta MAD$

$$\therefore \frac{CM}{AM} = \frac{MB}{MD}$$

or quote

$(CM)(MD) = (AM)(MB)$ (Intersecting chords theorem)

$\therefore (CM)^2 + 2(CM)(MD) + (MD)^2 > (AM)^2 + 2(AM)(MB) + (MB)^2$

$(CM)^2 + 2(CM)(MD) + (MD)^2 - 4(CM)(MD) > (AM)^2 + 2(AM)(MB) + (MB)^2 - 4(AM)M$

$\therefore (CM - MB)^2 > (AM - MB)^2$