



**SYDNEY BOYS HIGH
SCHOOL**
MOORE PARK, SURRY HILLS

2005

YEAR 12

ASSESSMENT TASK #3

Mathematics Extension 1

General Instructions

- Working time – 90 minutes.
- Reading Time – 5 minutes.
- Write using black or blue pen.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may not be awarded for messy or badly arranged work
- Hand in your answer booklets in 3 sections. Section A (Question 1), Section B (Question 2) and Section C (Question 3)

Total Marks - 80

- Attempt questions 1 – 3
- All sections are NOT of equal value.

Examiner: *A. Fuller*

Total marks - 80

Attempt Questions 1 - 3

All questions are NOT of equal value

Answer each SECTION in a SEPARATE writing booklet.

Section A

Question 1 (27 marks)

- | | | Marks |
|-----|---|-------|
| (a) | Evaluate $\log_2 0.125$ | 1 |
| (b) | Expand $\left(e^{\frac{1}{2}x} - e^{-\frac{1}{2}x}\right)^2$ | 2 |
| (c) | Evaluate $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) + \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$ | 2 |
| (d) | Sketch the graph of $y = 3 \sin^{-1} \frac{x}{2}$ | 2 |
| (e) | Prove that $\log_{ab} x = \frac{\log_a x}{1 + \log_a b}$ | 2 |
| (f) | If $\int_0^1 \frac{1}{3+x^2} dx = a\pi$, find the value of a | 2 |
| (g) | Differentiate $\log_e(\sin^3 x)$ writing your answer in the simplest form | 2 |
| (h) | (I) For what values of x is $\sin^{-1} x$ defined? | 1 |
| | (II) Find the maximum value of $2x(1-x)$ | 1 |
| | (III) Find the range of the function given by $f(x) = \sin^{-1}[2x(1-x)]$ | 2 |

$\frac{\log ab}{1 + \log a}$

Section B (Use a SEPARATE writing booklet)

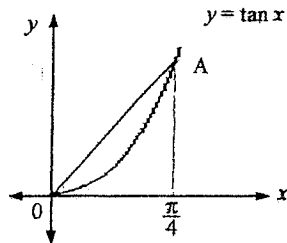
Question 2 (27 marks)

Marks

(i) Use the substitution $u = x^2$ to find $\int_0^{\frac{1}{\sqrt{2}}} \frac{x}{\sqrt{1-x^4}} dx$ 3

(j) If $y = x^n e^{ax}$, show that $\frac{dy}{dx} - ay = \frac{ny}{x}$ 3

(k)



In the diagram, A is a point on the curve $y = \tan x$ with an x coordinate of $\frac{\pi}{4}$. The chord OA has been drawn from the origin to the point A.

Show that the area enclosed by the chord OA and the curve $y = \tan x$ between $x = 0$ and $x = \frac{\pi}{4}$ has a magnitude of $\frac{1}{8}(\pi - 4 \ln 2)$ units² 4

End of Section A

(a) If $y = \sec x$, prove $\frac{dy}{dx} = \sec x \tan x$ 2

(b) A function is defined as $f(x) = 1 + e^{2x}$

(I) Write down the domain and the range of the function 1

(II) Show that the inverse function can be defined as $f^{-1}(x) = \frac{1}{2} \ln(x-1)$ 2

(III) On the same set of axes, sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ 2

(IV) Show that the equation of the normal to $y = f^{-1}(x)$ at the point where $f^{-1}(x) = 0$ is $2x + y - 4 = 0$ 2

(c) If $\frac{dy}{dx} = 1 + y$ and when $x = 0$, $y = 2$. Show that $y = 3e^x - 1$ 3

(d) Find the exact value of $\cos\left[2 \sin^{-1} \frac{3}{4}\right]$ 3

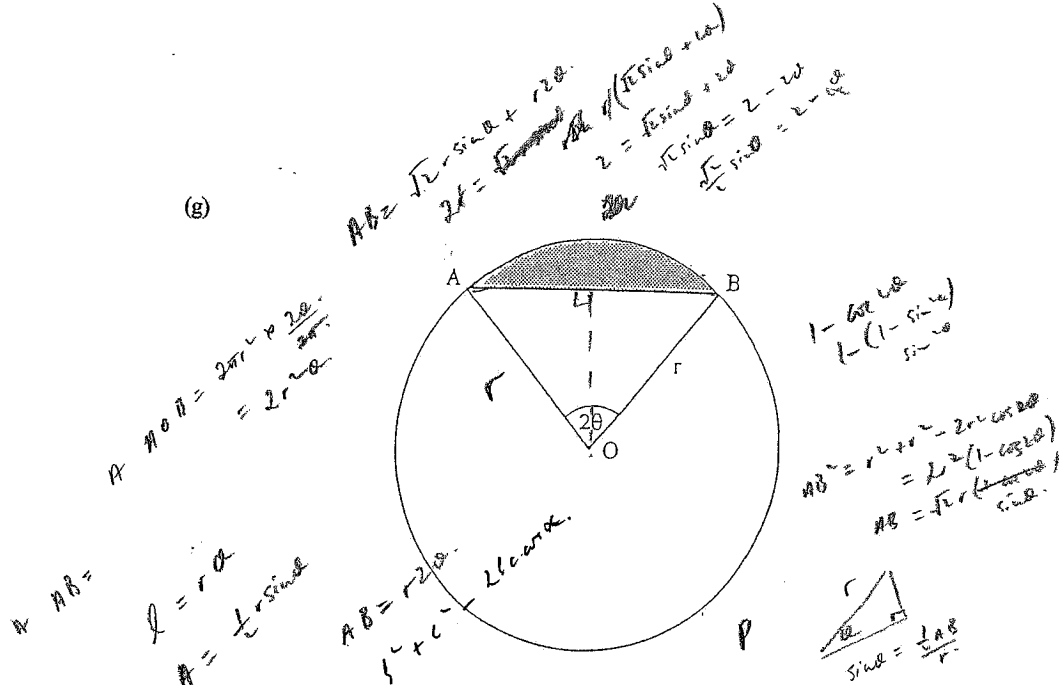
(e) Evaluate $\int \frac{3x}{\sqrt{1+x}} dx$ using the substitution $u = 1+x$ 3

(f) If $f(x) = a \cos^{-1}(bx)$, evaluate a and b if $f(0) = 2$ and $f'(0) = 2$. 3

Section C (Use a SEPARATE writing booklet)

Marks

Question 3 (26 marks)



The diagram above shows a shaded segment which subtends an angle of 2θ radians at the centre O of a circle with radius r . Given that the perimeter of the shaded segment equals twice the diameter of the circle

- $2r =$
- Show that $\sin \theta = 2 - \theta$ 2
 - Show that the equation $\sin \theta + \theta - 2 = 0$ has a root that lies between $\theta = 1$ and $\theta = 1.5$ 1
 - Use one application of Newton's method with an initial approximation of $\theta = 1.25$ to obtain a better approximation of the root of the equation $\sin \theta + \theta - 2 = 0$ 2
 - Using the result found in (III) find to the nearest degree the size of $\angle AOB$ 1

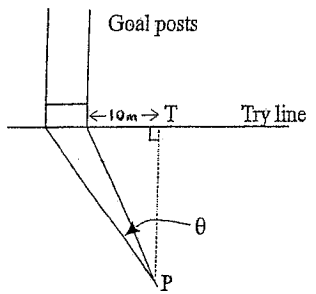
End of Section B

- The velocity v of a particle moving along the x axis starting from $x = 1.8$ is given by $v = e^{-2x} \sqrt{2x^2 - 6}$, $x \geq 1.8$ where x is the displacement of the particle from the origin.
 - Show that the acceleration a of the particle in terms of its displacement can be expressed by $a = -2e^{-4x} (2x^2 - x - 6)$ 2
 - Hence, find the displacement of the particle at which the maximum speed occurs. 1
 - Show that the time T in seconds taken by the particle to move from $x = 2$ to $x = 3$ can be expressed as $T = \int_2^3 \frac{e^{2x}}{\sqrt{2x^2 - 6}} dx$ 2
 - Use Simpson's rule with three function values to obtain an approximate value for T . 2
- A water tank is generated by rotating the curve $y = \frac{x^4}{16}$ around the y -axis.
 - Show that the volume of water, V as a function of its depth h , is given by $V = \frac{8}{3} \pi h^{\frac{3}{2}}$ 2
 - Water drains from the tank through a small hole at the bottom. The rate of change of the volume of water in the tank is proportional to the square root of the water's depth. 3

Use this fact to show that the water level in the tank falls at a constant rate.

- (c) Given $y = \cos^{-1}(\sin x)$
- (I) Show that $\frac{dy}{dx}$ has two values 3
- (II) Hence, or otherwise sketch the graph of $y = \cos^{-1}(\sin x)$ for $-\pi \leq x \leq \pi$ 2

- (e) In rugby league, teams score points by placing the ball on (or over) the try line at the end of the field. A kicker may then convert the try by taking the ball back at right angles from the point T on the try line where the try was scored and attempt to kick the ball between the goal posts which are 6 metres apart.



In the diagram above, a try has been scored 10 metres to the right of the goal posts. The kicker has brought the ball back x metres to a point P to attempt the conversion. The kicker wants to maximise θ , the angle of his view of the goal posts.

- (I) Show that $\tan \theta = \frac{6x}{160 + x^2}$. 3
- (II) Letting $E = \tan \theta$, find the value of x for which E is a maximum. 2
- (III) Hence show that the maximum angle, θ , is given by 2
- $$\theta = \tan^{-1}\left(\frac{3}{\sqrt{160}}\right)$$

- (IV) Find the maximum value of θ (to the nearest minute) and the corresponding value of x (to the nearest centimetre).

End of paper



Sydney Boys' High School

Name: 12496308

Maths Class: _____ Teacher: _____

Paper: _____

Section: 1

Sheet No. _____ of _____ for this Section:

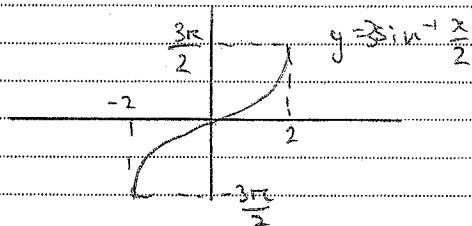
Q.No	Tick	Mark
1		27
2		
3		
4		
5		
6		
7		
8		
9		
10		74

Q1 a) $\log_2 0.125 = \log_2 2^{-3} = -3 \log_2 2 = -3$

b) $(e^{\frac{1}{2}x} - e^{-\frac{1}{2}x})^2 = e^x - 2e^{\frac{1}{2}x} \cdot e^{-\frac{1}{2}x} + e^{-x}$
 $= e^x - 2e^0 + e^{-x}$
 $= e^x + e^{-x} - 2$

c) $\sin^{-1}(\frac{1}{\sqrt{2}}) + \tan^{-1}(\frac{-1}{\sqrt{3}})$
 $= \frac{\pi}{4} + -\frac{\pi}{6} = \frac{2\pi}{24} = \frac{\pi}{12}$

d) R: $\{y: -\frac{3\pi}{2} < y < \frac{3\pi}{2}\}$ D: $\{x: -2 < x < 2\}$



e) $\log_{ab} x = \frac{\log_a x}{1 + \log_a b} \Rightarrow ab^{\frac{\log_a x}{1 + \log_a b}} = x$

$\frac{\log_a x}{\log_a b + 1} = \frac{\log_a x}{\log_a a + \log_a b}$

$= \frac{\log_a x}{\log_a ab}$

$ab^{\frac{\log_a x}{\log_a ab}} = b^{\frac{\log_a x}{\log_a ab}} \cdot a^{\frac{\log_a x}{\log_a ab}} = \frac{x}{ab} \cdot b^{\frac{\log_a x - \log_a ab}{\log_a ab}}$

$\frac{\log_a x}{\log_a ab} = \log_{ab} x$
 $\frac{\log_a x}{\log_a b} = \log_b x$
 $\log_{ab} x = \log_b x - b$

f) $\int_0^1 \frac{dx}{\sqrt{3+x^2}} = \left[\frac{1}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}} \right]_0^1 = \frac{\pi}{6} \times \frac{1}{\sqrt{3}} = \frac{\pi\sqrt{3}}{18} \therefore a = \frac{\sqrt{3}}{18}$

g) $\frac{d}{dx} \ln(\sin^3 x) = \frac{3 \cos x \sin^2 x}{\sin^3 x} = \frac{3 \cot x}{\sin x}$

h) i) $-1 \leq x \leq 1$

ii) $\frac{d}{dx} (2x - 2x^2) = 2 - 4x$
 $2 - 4x = 0 \Rightarrow x = \frac{1}{2}$

$\frac{2}{2} - 2 \times \frac{1}{4} = 1 - \frac{1}{2} = \frac{1}{2}$

iii) $-\frac{\pi}{2} < y < \frac{\pi}{6}$



Sydney Boys' High School

Name: 12496308

Maths Class: _____ Teacher: _____

Paper: _____

Section: 2

Sheet No. _____ of _____ for this Section:

Q.No	Tick	Mark
1		
2		26 1/2
3		
4		
5		
6		
7		
8		
9		
10		

i) $\int_0^{\frac{1}{\sqrt{2}}} \frac{x}{\sqrt{1-x^2}} dx$ let $u = 1-x^2$ $du = -2x dx$
 $x = \frac{1}{\sqrt{2}}$ $u = \frac{1}{2}$
 $x=0$ $u=0$
 $= \int_0^{\frac{1}{2}} \frac{1}{2} \frac{du}{\sqrt{1-u}}$ $= \left[\frac{1}{2} \sin^{-1} u \right]_0^{\frac{1}{2}} = \frac{1}{2} \times \frac{\pi}{6} = \frac{\pi}{12}$

j) $\frac{dy}{dx} = nx^{n-1}e^{ax} + ae^{ax}x^n$
 $= e^{ax}x^{n-1}(n+ax)$

ay = ax^n e^{ax} — (2)

$\frac{ny}{n} = \frac{nx^n e^{ax}}{n} = nx^{n-1}e^{ax}$ — (3)

(1) + (3) = LHS = $ae^{ax}x^{n-1} + ae^{ax}x^{n-1} = 2ae^{ax}x^{n-1}$
 $= ne^{ax}x^{n-1} =$ (3) = RHS

k) gradient of OA $\frac{y_1 - y_0}{x_1 - x_0} = \frac{1-0}{\frac{\pi}{4}-0} = \frac{1}{\frac{\pi}{4}} = \frac{4}{\pi}$

$\frac{4}{\pi}(x - \frac{\pi}{4}) = (y-1)$ $4x - \pi = \pi y - \pi$

$\frac{4x}{\pi} - 1 + 1 = y$, $y = \frac{4}{\pi}x$

Area between curves = $\int_0^{\frac{\pi}{4}} \left[\frac{4}{\pi}x - \tan x \right] dx$

$= \left[\frac{2}{\pi}x^2 + \ln(\cos x) \right]_0^{\frac{\pi}{4}}$ $\int \tan x dx = \int \frac{\sin x}{\cos x} dx = -\ln|\cos x|$

$= \left(\frac{2}{\pi} \times \frac{\pi^2}{16} + \ln \frac{1}{\sqrt{2}} \right) - (0 + \ln 1)$
 $= \frac{\pi}{8} - \frac{1}{2} \ln 2 = \frac{1}{8}(\pi - 4 \ln 2)$ units²
= RHS

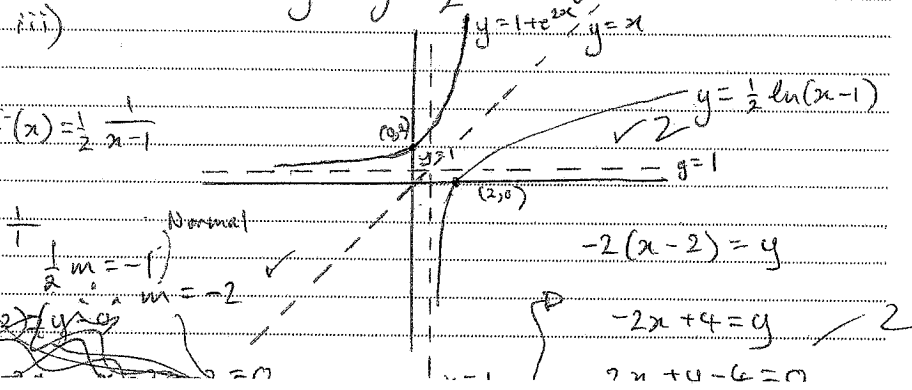
2) a) $y = \sec x = (\cos x)^{-1}$ $\frac{dy}{dx} = -1 \times -\sin x (\cos x)^{-2}$
 $= \frac{\sin x}{\cos^2 x} = \tan x \sec x$

b) $f(x) = 1 + e^{2x}$

i) D: {x: all real x} R: {y: y ≥ 1}

ii) switch x & y $x = 1 + e^{2y}$ $x-1 = e^{2y}$

$\ln(x-1) = 2y$ $y = \frac{1}{2} \ln(x-1)$ = RHS



iv) $\frac{d}{dx} f(x) = \frac{1}{2} \frac{1}{x-1}$

$x=2$
 $= \frac{1}{2} \times \frac{1}{1}$
 $= \frac{1}{2}$
 Normal $m = -1$
 $m = -2$

$-2(x-2) = y$

$-2x + 4 = y$

$2x + y - 4 = 0$

$$c) \frac{dy}{dx} = 1+y \quad \frac{dx}{dy} = \frac{1}{1+y} \quad \int \frac{1}{1+y} dy = x$$

$$= \ln(1+y) + C = x, \quad x=0, y=2 \text{ (given)}$$

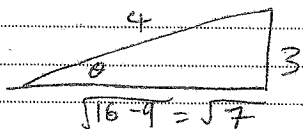
$$\ln 3 + C = 0$$

$$C = -\ln 3$$

$$x = \ln(1+y) - \ln 3 = \ln\left(\frac{1+y}{3}\right) = x$$

$$e^x = \frac{1+y}{3}, \quad 3e^x - 1 = y = \text{RHS}$$

$$d) \cos\left[2\sin^{-1}\frac{3}{4}\right]$$



$$= \cos\left[2\cos^{-1}\frac{\sqrt{7}}{4}\right]$$

$$= \left(\cos\left(\cos^{-1}\frac{\sqrt{7}}{4}\right)\right)^2 - \left(\sin\left(\sin^{-1}\frac{3}{4}\right)\right)^2$$

$$= \frac{7}{16} - \frac{9}{16} = \frac{-2}{16} = \frac{-1}{8}$$

$$e) \int \frac{3x}{\sqrt{1+x}} dx \quad \text{let } u=1+x \quad du=dx$$

$$= 3 \int \frac{(u-1)du}{\sqrt{u}}$$

$$= 3 \int \sqrt{u} du - 3 \int u^{-\frac{1}{2}} du$$

$$= 3 \times \frac{2}{3} u^{\frac{3}{2}} - 3 \times 2 \times u^{\frac{1}{2}} + C_1 + C_2$$

$$= 2(x+1)^{\frac{3}{2}} - 6\sqrt{x+1} + C$$

$$f) f(0) = 2, \quad \cos^{-1} a x = 0$$

$$\cos^{-1} b x = \frac{\pi}{2}$$

$$b = \frac{1}{2}$$

$$2 = a \cos^{-1} 0 b$$

$$= a \cos^{-1} 1 \rightarrow \cos^{-1} 1 = 0$$

$$f'(x) = a \frac{-b}{\sqrt{1-b^2x}} \quad a = \frac{4}{\pi}$$

$$= \frac{-4b}{\pi \sqrt{1-b^2x}} \quad x=0, f'(0) = 2$$

$$2 = \frac{-4b}{\pi \sqrt{1}} \quad 2\pi = -4b \quad b = -\frac{\pi}{2}$$

$$g) \text{ Perimeter of shaded seg} = 2r + r \sin \theta + r \sin \theta$$

$$= 2r(\theta + \sin \theta)$$

is equal to twice the diameter

$$= 2 \times 2r$$

$$= 4r$$

$$2r(\theta + \sin \theta) = 4r$$

$$\theta + \sin \theta = 2$$

$$i) \sin \theta = 2 - \theta$$

$$ii) P(x) = \sin \theta + \theta - 2 = 0$$

$$P(1) = -0.1585 \text{ (-ve)}$$

$$P(1.5) = 0.4975 \text{ (+ve)}$$

$\therefore P(x)$ cuts the x -axis between 1 and 1.5 and there is a root between x and 1.5

$$iii) x_2 = 1.25 - \frac{f(1.25)}{f'(1.25)} \quad f'(x) = \cos \theta + 1$$

$$= 1.25 - \frac{0.148985}{1.56532}$$

$$\theta \approx 1.12288 \quad P(1.12288) = 0.02423$$

(closer than

0.128985) \therefore

better approximation

$$\angle AOB = 2\theta$$

$$\theta \approx 1.12288$$

$$\therefore 2\theta \approx 2.24576$$

$$\approx 128^\circ 40'$$

$$\approx 129^\circ$$

$$\pi^\circ = 180^\circ$$

$$1^\circ = \frac{180^\circ}{\pi}$$

$$180 \quad |$$



Sydney Boys' High School

Name: 12496308

Maths Class: _____ Teacher: _____

Paper: _____

Section: 3

Sheet No. _____ of _____ for this Section:

Q.No	Tick	Mark
1		
2		
3		20 $\frac{1}{2}$
4		
5		
6		
7		
8		
9		
10		

3) a) i) $x = 1.8$ $v = e^{-2x} \sqrt{2x^2 - 6}$ $x \geq 1.8$

$a = \frac{d}{dx} \frac{1}{2} v^2$ $v^2 = e^{-4x} (2x^2 - 6)$

$\frac{1}{2} v^2 = e^{-4x} (x^2 - 3)$

$\frac{d}{dx} \frac{1}{2} v^2 = -4e^{-4x} (x^2 - 3) + e^{-4x} (2x)$

$= e^{-4x} (12 - 4x^2 + 2x) = -2e^{-4x} (2x^2 - x - 6)$
 $= R.H.S$

ii) max speed occurs when $a = 0$

$a = -2e^{-4x} (2x^2 - x - 6)$

$-2e^{-4x} \neq 0 \therefore 2x^2 - x - 6 = 0$

$\frac{1}{2} (2x - 4)(2x + 3) = 0$

$(x - 2)(2x + 3) = 0$

since $x \geq 1.8$, $x \neq -\frac{3}{2}$

$\therefore x = 2$

$v = e^{-2x} \sqrt{2x^2 - 6} = \frac{\sqrt{2x^2 - 6}}{e^{2x}} = \frac{dx}{dt}$

$\frac{dt}{dx} = \frac{e^{2x}}{\sqrt{2x^2 - 6}} \therefore T = \int \frac{e^{2x}}{\sqrt{2x^2 - 6}} dx$

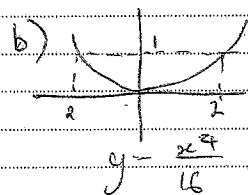
and T between 2 and 3

$= \int_2^3 \frac{e^{2x}}{\sqrt{2x^2 - 6}} dx = \underline{R.H.S}$

iv) $\frac{1}{2} \times \frac{1}{3} (f(3) + f(2) + 4[f(2.5)])$

$= \frac{1}{6} (116.46 + 38.61 + 4 \times (58.21))$

$= \frac{1}{6} (387.91) = 64.65 \text{ 2 seconds}$



$V = \pi \int (2\sqrt{y})^2 dy = \pi \int 4y dy$
 $= 4\pi \frac{2}{3} y^{\frac{3}{2}} = \frac{8\pi}{3} y^{\frac{3}{2}}$

$y =$ the depth as $x=0$, $y=0$

$\therefore V = \frac{8\pi}{3} h^{\frac{3}{2}} = R.H.S$

ii) $\frac{dV}{dt} \propto \sqrt{h} \therefore \frac{dV}{dt} = k \sqrt{h} \quad k h^{\frac{1}{2}}$

$V = \frac{8\pi}{3} h^{\frac{3}{2}} \quad \frac{dV}{dh} = 4\pi h^{\frac{1}{2}}$

$\frac{dV}{dh} \times \frac{dh}{dt} = \frac{dV}{dt} = k h^{\frac{1}{2}} \quad \frac{d}{dh} = \frac{k}{2} h^{-\frac{1}{2}}$

$\frac{d}{dh} \left(\frac{dV}{dt} \right) = \frac{k}{2} h^{-\frac{1}{2}}, \quad \frac{dV}{dh} = 4\pi h^{\frac{1}{2}}$

$\frac{dh}{dt} \times \frac{d}{dh} \left(\frac{dV}{dt} \right) = \frac{1}{4\pi h} \times \frac{k}{2\sqrt{h}} \frac{dh}{dt} = \frac{1}{4\pi \sqrt{h}}$

$$\frac{dV}{dt} = k\sqrt{h} \quad \frac{d}{dt} \left(\frac{1}{2} k \sqrt{h} \right)$$

~~$\frac{dV}{dt}$~~

$$V = \frac{8}{3} \pi r^2 h^{\frac{3}{2}}$$

$$\frac{dV}{dh} = 4\pi r^2 h^{\frac{1}{2}}$$

$$\frac{dh}{dV} = \frac{1}{4\pi r^2 h^{\frac{1}{2}}} \quad \frac{dh}{dV} \times \frac{dV}{dt} = \frac{dh}{dt}$$

= fall rate of water level

$$= k\sqrt{h} \times \frac{1}{4\pi r^2 h^{\frac{1}{2}}} = \frac{k}{4\pi r^2} = \text{Constant}$$

$$\therefore \frac{dh}{dt} (\text{rate of change of water level}) = \text{constant}$$

c) $y = \cos^{-1} \sin 2x \quad \frac{dy}{dx} = \frac{-\cos 2x}{\sqrt{1 - \sin^2 2x}}$

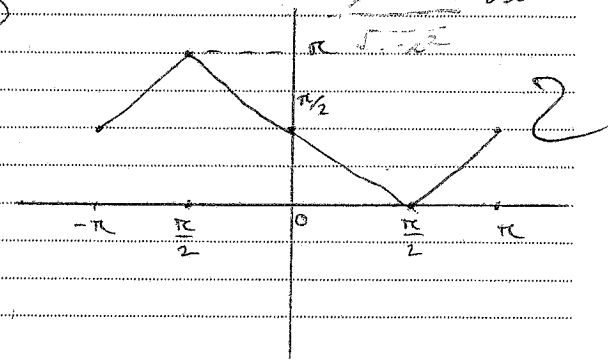
$$= \frac{-\cos 2x}{\sqrt{\cos^2 2x}} = \frac{-\cos 2x}{|\cos 2x|} = -1$$

$$y = \cos^{-1} \left(\cos \left(\frac{\pi}{2} - 2x \right) \right) \quad \frac{dy}{dx} = \frac{-\sin \left(\frac{\pi}{2} - 2x \right)}{\sqrt{1 - \cos^2 \left(\frac{\pi}{2} - 2x \right)}}$$

$$= \frac{-\sin \left(\frac{\pi}{2} - 2x \right)}{\sin \left(\frac{\pi}{2} - 2x \right)} = -1 \quad \therefore \text{two values for } \frac{dy}{dx} = 1, -1$$

$$y = \cos^{-1}(\cos x)$$

ii) $\sin \theta(x) = -1$



e) $\tan \theta = \tan \left(\theta + \tan^{-1} \frac{10}{x} \right)$

$$= \frac{\tan \theta + \frac{10}{x}}{1 - \frac{10 \tan \theta}{x}} \quad \tan \theta \left(\frac{x - 10 \tan \theta}{x} \right)$$

$$\tan \theta (x - 10 \tan \theta) = x \tan \theta + 10$$

$$x \tan \theta - 10 \tan^2 \theta = x \tan \theta + 10$$

$$\tan^2 \theta = -1$$

$$\tan \theta = \dots$$

$$\tan \theta = \tan \left(\tan^{-1} \frac{16}{x} - \tan^{-1} \frac{10}{x} \right)$$

$$= \tan \left(\tan^{-1} \frac{16}{x} - \tan^{-1} \frac{10}{x} \right)$$

$$\tan \theta = \frac{\frac{16}{x} - \frac{10}{x}}{1 + \frac{160}{x^2}} = \frac{6x}{x^2 + 160} = \frac{6x}{x^2 + 160}$$

ii) $\frac{d}{dx} \left(\frac{6x}{x^2 + 160} \right) = \frac{6(x^2 + 160) - 6x(2x)}{(x^2 + 160)^2}$

E max occurs when $d = 0 \therefore 6x^2 + 96 - 12x^2 = 0$
 $96 - 6x^2 = 0$
 $x^2 = 16$
 $x = \pm 4$

iii) $\tan \theta = \frac{6x}{x^2 + 160}, x = 4$

$$= \frac{24}{16 + 160}$$

$$\frac{24}{176} \quad \frac{dy}{dx} = \dots$$

iv) $\max \theta = \tan^{-1} \left(\frac{24}{176} \right) = \tan^{-1} \left(\frac{3}{22} \right)$

$\tan \theta = \frac{3}{22} = \tan \left(\tan^{-1} \frac{16}{x} + \tan^{-1} \frac{10}{x} \right)$ when θ max occurs