SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

JUNE 2005

TASK #3

YEAR 12

Mathematics

General Instructions
- Reading time – 5 minutes.
- Working time – 2 Hours.
- Write using black or blue pen.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may NOT be awarded for messy or badly arranged work.
- Start each NEW section in a separate answer booklet.

Total Marks - 100 Marks
- Attempt Questions 1 - 6
- All questions are NOT of equal value.

Examiner: *P. Bigelow*

Total marks – 100
Attempt Questions 1 - 6
All questions are NOT of equal value

Answer each SECTION in a SEPARATE writing booklet.

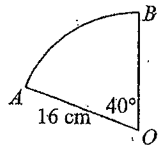
Section A

Marks

Question 1 (15 marks)

(a) Convert 150° to radians — 1

(b) — 2



OAB is a sector of a circle, with radius 16 cm.

Write down the area of the sector *OAB*, correct to the nearest square centimetre.

(c) Write down the value of the following, correct to 2 decimal places.

(i) $\dfrac{1}{\sqrt{e}}$ — 1

(ii) $\cos 3$ — 1

(d) Write down the derivatives of the following

(i) e^{-4x} — 1

(ii) $\tan\left(\dfrac{x}{2}\right)$ — 1

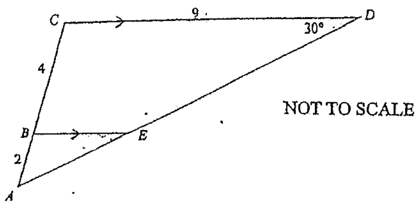
(e) Thirty cards are numbered from 1 to 30.

(i) If one card is selected at random what is the probability that the card has a **4** on it? — 1

(ii) If two cards are chosen at random, with replacement what is the probability that at least one of the cards has a **4** on it? — 2

(f) Find $f(2)$ if $f'(x)=2x+3$ and $f(1)=6$ — 2

(g)



In the diagram, ACD is a triangle where $AB = 2$ cm, $BC = 4$ cm, $CD = 9$ cm and $\angle CDE = 30°$.
Also BE is parallel to CD.

(i) Find the size of $\angle BED$. **1**

(ii) Find the length of BE. Give reasons for your answer. **2**

Question 2 (17 marks)

(a) Differentiate the following

(i) $2x \ln x$ **2**

(ii) $\dfrac{\cos x}{x}$ **2**

(b) Find

(i) $\displaystyle\int_1^4 \frac{dx}{\sqrt{x^3}}$ **2**

(ii) $\displaystyle\int \frac{x^3 + x}{x^2}\, dx$ **2**

(iii) $\displaystyle\int \left(e^x + 1\right)^2 dx$ **2**

Question 2 continues over the page

Question 2 continued

(c) A farmer made the following statement:
"*In 1998 the price of wheat started to fall and it reached its lowest levels in mid 2002. Then from mid 2002, the price began a continual rise and is now, in 2005, at an all time high.*"

If the curve representing the price of wheat, P, is differentiable, what does the above statement imply about:

(i) $\dfrac{dP}{dt}$ between 1998 and 2005 **2**

(ii) $\dfrac{d^2P}{dt^2}$ at mid 2002? **1**

(d) (i) Sketch $y = 3\sin 2x$ for $0 \le x \le 2\pi$ **2**

(ii) Find $\displaystyle\int_0^{\frac{\pi}{4}} 3\sin 2x\, dx$ **2**

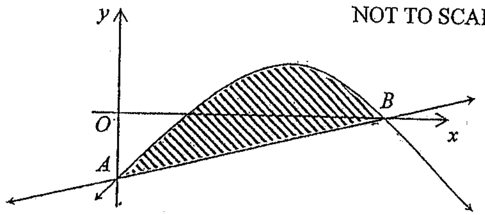
End of Section A

Section B (Use a SEPARATE writing booklet)

Question 3 (17 marks) **Marks**

(a) The graphs of $y = x - 4$ and $y = -x^2 + 5x - 4$ intersect at the points A and B, as shown in the diagram below.

NOT TO SCALE



(i) Find the coordinates of A and B **2**

(ii) Find the area of the shaded part. **2**

(b) Find the equation of the tangent to $y = \ln x$ at the point where $y = \ln x$ crosses the x axis. **2**

(c) Consider the curve given by $y = x^3 - 6x^2 + 9x + 6$

(i) Find the coordinates of the stationary points and determine their nature. **3**

(ii) Find the coordinates of any points of inflexion. **1**

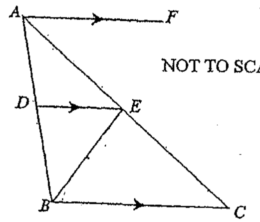
(iii) Sketch the curve for $-1 \le x \le 4$. **2**

(iv) Indicate on your curve, with the letter S, where the curve has the greatest rate of increase. **1**

Question 3 is continued over the page

Question 3 continued

(d)



NOT TO SCALE

The diagram shows ΔABC with D lying on AB and E lying on AC.
The lines AF, DE and BC are parallel and $\angle AED = \angle BED$.

(i) Show that ΔBEC is isosceles. **2**

(ii) State the reason why $AD : DB = AE : EC$. **1**

(iii) Show that $AD : DB = AE : EB$ **1**

Question 4 (16 marks)

(a) The part of the curve $y = \dfrac{1}{\sqrt{2x+1}}$ between $x = 0$ and $x = 1$ is rotated about the x axis. **3**

Find the volume of the solid obtained

(b) Five marbles are numbered 1, 2, 3, 4 and 5 and placed in bag. Two marbles are taken out in succession, the first marble not being replaced before the second is withdrawn.

(i) Find the probability that

(α) the 4 is selected. **1**

(β) the 3 is NOT selected. **1**

(γ) the 1 is the second marble chosen. **1**

(ii) If the first marble is replaced before the second is selected, find the probability that at least one 2 is drawn. **2**

Question 4 continued

(c) (i) Differentiate $y=\ln(\cos x)$ 2

(ii) Hence find $\int \tan x \, dx$ 1

(d) Find $\int_0^{\frac{\pi}{6}} \sec^2 2x \, dx$ 2

(e) For $a>0$, it is given that 3

$$\int_1^4 \frac{4x}{4+x^2}\,dx=\ln a$$

By evaluating the integral, write down the value of a.

End of Section B

Section C (Use a SEPARATE writing booklet)

Question 5 (16 marks)

Marks

(a) Find the second derivative of $x\sin x$. 2

(b) Find $\int_0^1 \frac{e^x}{1+e^x}\,dx$ 2

(c) A continuous curve $y=f(x)$ has the following properties over the interval $a\le x\le b$:

$$f(x)>0,\ f'(x)>0,\ f''(x)>0$$

(i) Sketch a curve satisfying these properties. 2

(ii) State the least value of $f(x)$ over this interval. 1

(d) At every point on a curve $\frac{d^2y}{dx^2}=2x$. 4

The point $(3,6)$ lies on the curve and its tangent at this point is inclined at $45°$ to the positive direction of the x axis.

Find the equation of the curve

(e) (i) For the function $f(x)=e^{x^2}$, copy and complete the following table, using 4 significant figures where necessary. 2

x	-1	$-0\cdot5$	0	$0\cdot5$	1
$f(x)$					

(ii) Hence, use Simpsons rule with 5 function values to approximate 3

$\int_{-1}^{1} e^{x^2}\,dx$.

Leave your answer correct to 3 significant figures.

Question 6 (19 marks) — Marks

(a) (i) Sketch $y=\log_e x$ and $y=x$ on the same diagram. **2**

(ii) Use your diagram in (i) to indicate the number of solutions to the equation $\log_e x = x$ **1**

(b) Given the curve $y=2xe^{\frac{x}{2}}$

(i) Show that $\frac{dy}{dx}=(x+2)e^{\frac{x}{2}}$ **2**

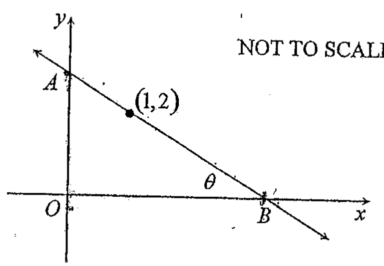
(ii) Find $\frac{d^2y}{dx^2}$ **2**

(iii) Find the minimum value of $2xe^{\frac{x}{2}}$ **2**

(iv) For what values of x is the curve concave down? **1**

(v) For what values of c does the equation $2xe^{\frac{x}{2}}=c$ have two unequal real roots? **1**

(c) In the diagram AB meets the x and y axes at B and A respectively and it passes through the point $(1,2)$. The angle OBA is θ in radians.



(i) Show that the equation of AB is given by **2**

$$y=-(\tan\theta)x+2+\tan\theta$$

(ii) Find the area of ΔOAB in terms of $\tan\theta$. **2**

(iii) Find the value of θ, correct to 2 decimal places, for which the area is a minimum. **4**

End of paper

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n\neq -1;\ x\neq 0, \text{if } n<0$$

$$\int \frac{1}{x}dx = \ln x,\ x>0$$

$$\int e^{ax}dx = \frac{1}{a}e^{ax},\ a\neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a\neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ a\neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ a\neq 0$$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a\neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a},\ a>0,\ -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}dx = \ln\left(x+\sqrt{x^2-a^2}\right),\ x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\ x>0$

SBHS Task 3 Year 12 2-UNIT

Q1 (a) $5\pi/6$ (b) $A=\frac{1}{2}r^2\theta=\frac{1}{2}\times16^2\times\frac{2\pi}{9}=89\ldots$

(c) (i) $0\cdot61$ (ii) $\cos(3\text{ radians})\approx-1\cdot00$

(d) (i) $-4e^{-4x}$ (ii) $\frac{1}{2}\sec^2(\frac{x}{2})$

(e) (i) $\frac{1}{30}$ (ii) $P=1-P(\text{No }4)=1-\frac{29}{30}\times\frac{29}{30}=\frac{59}{900}$

(f) $f(x)=x^2+3x+k \Rightarrow f(2)=(2)^2+3(2)+2=12$

(g) (i) $\angle BED=30°$

(ii) $\angle BED=\angle CDE$ (Corresp'ng $\angle$'s $CD\parallel BE$)

& $\angle CAD$ in common $\therefore \Delta ABE \parallel\!| \Delta CAD$ (equiangular)

$\therefore \frac{BE}{CD}=\frac{2}{6}$ (corres. sides in similar Δ are in the same ratio)

$\therefore BE=3$

Q2

[a] (i) $f'(x)=2\ln x+2x.\frac{1}{x}=2\ln x+2.$

(ii) $f'(x)=\frac{x(-\sin x)-1.(\cos x)}{x^2}=\frac{-x\sin x-\cos x}{x^2}$

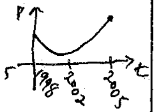
[b] (i) $=\int_1^4 x^{-3/2}\,dx=\left[\frac{-2}{\sqrt{x}}\right]_1^4=(-1)-(-2)=1$

(ii) $=\int x+\frac{1}{x}\,dx=\left[\frac{x^2}{2}+\ln x+c\right]$

(iii) $=\int e^{2x}+2e^x+1\,dx=\frac{1}{2}e^{2x}+2e^x+x+c$

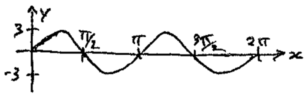
[c] (i) $\frac{dP}{dt}<0$ from 1998 → 2002

$\frac{dP}{dt}>0$ from 2002 → 2005



(ii) $\frac{d^2P}{dt^2}>0$ (always)

[d] (i)



(ii) $=\frac{3}{2}\left[-\cos 2x\right]_0^{\pi/4}=\frac{3}{2}\left((-0)-(-1)\right)=\frac{3}{2}$

Qu 3

[a] (i) Solve simult: $x-4=-x^2+5x-4$

$x^2-4x=0$

$x(x-4)=0 \quad \therefore x=0,4$

$\therefore A=(0,-4) \quad B=(4,0)$

(ii) Area $=\int_0^4(-x^2+5x-4)-(x-4)\,dx$

$=\int_0^4 4x-x^2\,dx=\left[2x^2-\frac{x^3}{3}\right]_0^4=\frac{32}{3}$

[b] $y'=\frac{1}{x} \quad y'(1)=1 \leftarrow m$ at $(1,0)$

$\therefore y-0=1(x-1)$

or $y=x-1 \quad (x-y-1=0)$

Q3[c] $y=x^3-6x^2+9x+6$

$y'=3x^2-12x+9$

$y''=6x-12$

(i) $y'=0$ when $3(x-3)(x-1)=0 \Rightarrow \begin{cases}x=1\\y=10\end{cases}$ & $\begin{cases}x=3\\y=6\end{cases}$

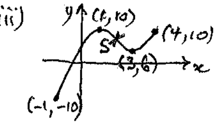
$y''(1)=-6<0$ R.Max ; $y''(3)=+6>0$ R.MIN.

(ii) Possible P.O.I. at $y''=0 \Rightarrow (2,8)$

x	2^-	2	2^+
y''	$-$	0	$+$

y'' changes sign at $x=2 \therefore$ P.O.I.

(iii)



(iv) greatest rate of increase is when $\frac{dy}{dx}$ is max

ie at $y''=0$ at $S(2,8)$

[d] (i) $\angle AED=\angle ACB$ (Corresp. $\angle$'s for $DE\parallel BC$)

$\angle DEB=\angle EBC$ (Alternate $\angle$'s " ")

but $\angle AED=\angle DEB$ (given)

$\therefore \angle ACB=\angle EBC$

$\therefore \Delta EBC$ is isosceles (base $\angle$'s are equal)

(ii) Transversals cut $\parallel$ lines in equal ratios.

(iii) Since ΔEBC is isosceles $EC=EB$

Substitute into result part ii)

Q4

[a] $V=\pi\int_0^1\frac{1}{2x+1}\,dx=\frac{\pi}{2}\left[\ln(2x+1)\right]_0^1=\frac{\pi}{2}\ln 3$

[b] (i) (α) $\frac{1}{5}+\frac{4}{5}\times\frac{1}{4}=\frac{2}{5}$ (γ) $\frac{1}{5}$

(β) $\frac{4}{5}\times\frac{3}{4}=\frac{3}{5}$

(ii) $P=1-P(\text{No 2 drawn})=1-\frac{4}{5}\times\ldots=\frac{9}{25}$

[c] (i) $y'=\frac{-\sin x}{\cos x}=-\tan x$

(ii) $\therefore \int\tan x\,dx=-\ln(\cos x)+c$

[d] $I=\frac{1}{2}\left[\tan 2x\right]_0^{\pi/6}=\frac{1}{2}\left\{\tan\frac{\pi}{3}-0\right\}=\frac{\sqrt{3}}{2}$

[e] $I=2\left[\ln(4+x^2)\right]_1^4=2\left\{\ln(20)-\ln(5)\right\}$

$=2\ln\frac{20}{5}$

$=\ln 4^2$

$=\ln a$

$\therefore a=16$

Q5

[a] $f'(x)=x\cos x+\sin x$

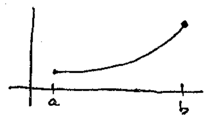
$f''(x)=\cos x-x\sin x+\cos x$

$=2\cos x-x\sin x.$

[b] $I=\left[\ln(e^x+1)\right]_0^1=\ln(e+1)-\ln 2$

$=\ln(\frac{e+1}{2}) \approx (\approx 0\cdot62)$

[c]



(i) Sketch

(ii) $f(a)$ is least value

[d] At $x=3$, $y=6$, & $y'=1$ (45° inclination)

$y''=2x$

$y'=x^2+c \leftarrow$ sub $x=3$ $y'=1 \Rightarrow c=-8$

$\therefore y=\frac{x^3}{3}-8x+k \leftarrow$ sub $x=3, y=6 \rightarrow k=21$

[e] (i)

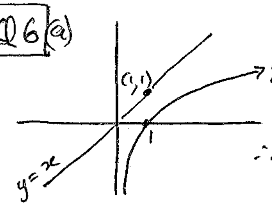
x	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$+1$
$f(x)$	$2\cdot718$	$1\cdot284$	1	$1\cdot284$	$2\cdot718$

$h=0.5$

(ii) Area $\approx\frac{0\cdot5}{3}\left\{2\cdot718+2\cdot718+4(1\cdot284+1\cdot284)+2(1)\right\}$

$\approx\frac{0\cdot5}{3}\left\{17\cdot708\right\}\approx 2\cdot951$ units²

Q6 (a)



No intersection

$\therefore x=\ln x$

No solutions.

[b]

(i) $y'=2e^{\frac{x}{2}}+2x.\frac{1}{2}e^{\frac{x}{2}}=2e^{\frac{x}{2}}+xe^{\frac{x}{2}}=$ RHS

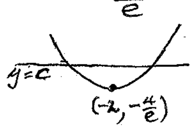
(ii) $y''=1.e^{\frac{x}{2}}+(x+2).\frac{1}{2}e^{\frac{x}{2}}=\frac{1}{2}e^{\frac{x}{2}}(x+4)$

(iii) $y'=0$ for $\begin{cases}x=-2\\y=-\frac{4}{e}\end{cases}$ $y''(-2)=e^{-1}=\frac{1}{e}>0$

$\therefore$ Min Value $\frac{-4}{e}$

(iv) $y''<0$ if $x<-4$

(v) $y=c$ must be above min. point



$\therefore c>-\frac{4}{e}$

Q6 [c] gradient is $\tan(\pi-\theta)=-\tan\theta$

(i) $y-2=-\tan\theta(x-1)$

$\therefore y=-x\tan\theta+\tan\theta+2$

(ii) at $x=0$ $y=2+\tan\theta$ $A=(0,2+\tan\theta)$

at $y=0$ $x=\frac{2+\tan\theta}{\tan\theta}$ $B=\left(\frac{2+\tan\theta}{\tan\theta},0\right)$

Area $=\frac{1}{2}(2+\tan\theta)\times\left(\frac{2+\tan\theta}{\tan\theta}\right)$

$A=\frac{(2+\tan\theta)^2}{2\tan\theta}$

(iii) $A=\frac{4+4\tan\theta+\tan^2\theta}{2\tan\theta}$

$=2(\tan\theta)^{-1}+2+\frac{1}{2}\tan\theta$

$\therefore \frac{dA}{d\theta}=\frac{-2\sec^2\theta}{\tan^2\theta}+\frac{1}{2}\sec^2\theta$

$=-\frac{2}{\sin^2\theta}+\frac{1}{2\cos^2\theta}=0$ for Min. Area.

$\Rightarrow \frac{1}{2\cos^2\theta}=\frac{2}{\sin^2\theta}$

$\therefore \tan^2\theta=4$

$\therefore \tan\theta=\pm2$ (θ is acute)

$\therefore \theta=1.107$ radians.