SYDNEY BOYS HIGH SCHOOL

MOORE PARK, SURRY HILLS

2005

YEAR 12

ASSESSMENT TASK #2

Mathematics Extension 1

General Instructions

- Working time – 90 minutes.
- Reading Time – 5 minutes.
- Write using black or blue pen.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may not be awarded for messy or badly arranged work
- Hand in your answer booklets in 3 sections. Section A (Question 1), Section B (Question 2) and Section C (Question 3)

Total Marks - 76

- Attempt questions 1 – 3
- All sections are NOT of equal value.

Examiner: *A. Fuller*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate

Total marks - 76
Attempt Questions 1 - 3
All questions are NOT of equal value

Answer each SECTION in a SEPARATE writing booklet.

Section A

Question 1 (22 marks) **Marks**

(a) Differentiate the following:

i) $\tan 2x$ **1**

ii) $\dfrac{1}{e^{\frac{x}{2}}}$ **2**

iii) $\ln(1-2x)^2$ **2**

iv) $\sin^2(3x)$ **2**

(b) Find the following:

i) $\int \cos\frac{x}{2}\,dx$ **1**

ii) $\int e^{1-4x}\,dx$ **1**

iii) $\int \cot x\,dx$ **2**

iv) $\int 3x.e^{x^2}\,dx$ **2**

(c) Find $\lim\limits_{x\to 0}\dfrac{\sin 2x}{x}$ **1**

(d) i) Show that the equation $x.\ln x - 1 = 0$ has a root between $x = 1$ and $x = 2$. **2**

ii) Using $x = 2$ as the first approximation, apply Newton's method once to obtain a better approximation of this root. (correct to two decimal places) **2**

(e) i) Show that $\frac{2x+1}{x+2}=2-\frac{3}{x+2}$ 1

ii) Hence or otherwise, find the exact value of $\int_0^1 \frac{2x+1}{x+2}dx$. 3

End of Section A

Section B (Use a SEPARATE writing booklet)

Marks

Question 2 (28 marks)

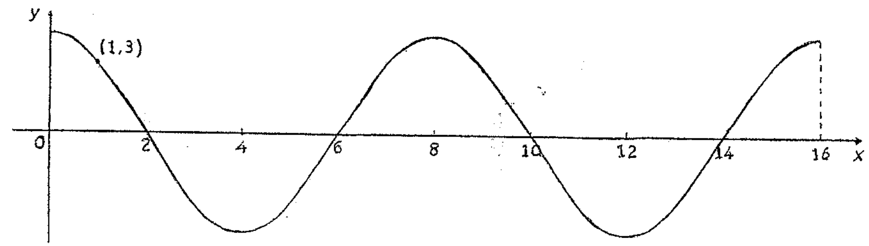
(a) Consider the function $f(x)=\log_e(3x-6)$

i) State the largest possible domain of $f(x)$. 1

ii) Sketch the curve $y=f(x)$. 2

iii) Find the equation of the normal to the curve at the point where $x=4$ 3

(b)



Given that the equation of the above curve is of the form $y=a\cos(bx)$ find the values of a and b. 2

(c) Consider the function $f(x)=xe^{-x}$

i) Find the coordinates of any stationary points and determine their nature. 2

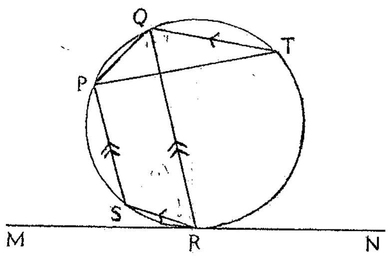
ii) Show that there is a point of inflection at $x=2$ 2

iii) Hence sketch the graph of $y=f(x)$ showing all essential features. 3

(d) i) Evaluate $\int_0^{\frac{\pi}{8}} \sin^2 3x\,dx$ 3

ii) Show that $\lim_{h\to 0}\frac{1-\cos 2h}{h^2}=2$ 2

(e) MN is a tangent at R and PQRS is a cyclic quadrilateral.
PS is parallel to QR and QT is parallel to SR



Prove that:

i) QR bisects $\angle PQT$ **2**

ii) MN $\parallel$ PT **2**

(f) Prove by Mathematical Induction that $2^{3n-1}+3$ is divisible by 7 for all positive integers $n \ge 1$. **4**

End of Section B

Section C (Use a SEPARATE writing booklet)

Marks

Question 3 (26 marks)

(a) An eight-person committee is to be formed from a group of 10 women and 15 men. In how many ways can the committee be chosen if the committee must contain:

i) 4 men and 4 women **1**

ii) more women than men **2**

iii) at least 2 women **2**

(b) Using the results

$$\frac{d(e^x \sin x)}{dx} = e^x \cos x + e^x \sin x$$

and $$\frac{d(e^x \cos x)}{dx} = e^x \cos x - e^x \sin x$$

evaluate $\int_0^{\frac{\pi}{2}} e^x \cos x dx$ **2**

(c) A car can hold six people, three in the front and three in the back. Only two of the six people can drive. In how many different seating arrangements can they complete the journey? **2**

(d) The portion of the curve $y = \sin x + \cos x$ between $x = 0$ and $x = \frac{\pi}{2}$ is rotated about the x-axis. Show that the volume of the solid of revolution generated is $\frac{\pi}{2}(\pi + 2)$ cubic units. **3**

(e) Using a first approximation $x_1 \neq 0$, show that the second approximation, x_2, is such that $|x_2| > |x_1|$ when using Newton's method to obtain the zero of $\sqrt[3]{x}$ **2**

(f) Prove by Mathematical Induction that **4**

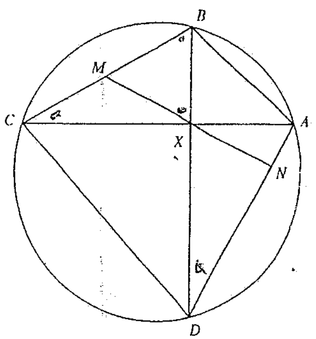
$$\cos x + \cos 3x + \cos 5x + \ldots\ldots + \cos(2n-1)x = \frac{\sin 2nx}{2\sin x},$$

for $n \geq 1$ (where n is an integer).

$2(k+1)$

(g) If $y = \frac{1}{2}\left(e^{x} - e^{-x}\right)$, prove that $x = \log_e\left(y + \sqrt{y^2+1}\right)$ **3**

(h)



ABCD is a cyclic quadrilateral. The diagonals AC and BD intersect at right angles at X. M is the midpoint of BC. MX produced meets AD at N. Prove that:

i) $\angle MBX = \angle MXB$ **2**

ii) MN is perpendicular to AD **3**

End of paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\ x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right)\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\ x > 0$

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Mathematics Extension 1

Sample Solutions

Section	Marker
A	AMG
B	FN
C	EC

Question 1

(a) (i) $\frac{d}{dx}\tan 2x = 2\sec^2 2x$ 1

(ii) $\frac{d}{dx}\left(\frac{1}{e^{x/2}}\right) = \frac{d}{dx} e^{-x/2}$

$= -\frac{1}{2}e^{-x/2}$

$= \frac{-1}{2e^{x/2}}$ 2

(iii) $\frac{d}{dx}\ln(1-2x)^2$

$= \frac{1}{(1-2x)^2} \times \frac{d}{dx}(1-2x)^2$

$= \frac{2\cancel{(1-2x)}\times -2}{(1-2x)^{\cancel{2}}}$

$= \frac{-4}{1-2x}$ 2

(iv) $\frac{d}{dx}\sin^2(3x)$

$= 2\cos(3x)\sin(3x)\times 3$

$= 3\sin(6x)$ 2

(b) (i) $\int \cos\frac{x}{2}\,dx$

$= 2\int \frac{1}{2}\cos\frac{x}{2}\,dx$ $\boxed{1}$

$= 2\sin\frac{x}{2} + C$

(ii) $\int e^{1-4x}\,dx = -\frac{1}{4}e^{1-4x} + C$ $\boxed{1}$

(iii) $\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx$

$= \ln \sin x + C$ $\boxed{2}$

(iv) $\int 3x \cdot e^{x^2}\,dx = \frac{3}{2}\int 2x e^{x^2}\,dx$

$= \frac{3}{2}e^{x^2} + C$ $\boxed{2}$

(c) $\lim_{x\to 0}\frac{\sin 2x}{x} = 2\lim_{x\to 0}\frac{\sin 2x}{2x}$

$= 2\lim_{u\to 0}\frac{\sin u}{u}$

(Since if $u=2x$, then $u\to 0$ as $x\to 0$)

$= 2\times 1$

$= 2$ $\boxed{1}$

Qn 1 contd

(d) (i) $x \cdot \ln x - 1 = 0$

When $x=1$ LHS $= 1 \cdot \ln 1 - 1$

$= 1 \times 0 - 1$

$= -1$

When $x=2$ LHS $= 2\ln 2 - 1$

$\doteq 0.386$

Since curve is continuous, and sign changes, $\therefore$ there is a root in $[1, 2]$ $\boxed{2}$

(ii) $f(x) = x\ln x - 1$

$f'(x) = x \cdot \frac{1}{x} + 1 \cdot \ln x$

$= 1 + \ln x$

Now $z_2 = z_1 - \frac{f(z_1)}{f'(z_1)}$

Let $z_1 = 2$

$\therefore z_2 = 2 - \frac{f(2)}{f'(2)}$

$= 2 - \frac{2\ln 2 - 1}{1 + \ln 2}$

$\doteq 1.77$

$\boxed{2}$

(2)

(e) (i) $\frac{2x+1}{x+2} = \frac{2x+4-3}{x+2}$

$= \frac{2(x+2)-3}{x+2}$

$= 2 - \frac{3}{x+2}$ $\boxed{1}$

(ii) $\int_0^1 \frac{2x+1}{x+2}\,dx = \int_0^1 \left[2 - \frac{3}{x+2}\right] dx$

$= \left[2x - 3\ln(x+2)\right]$

$= (2 - 3\ln 3) - (0 - 3\ln 2)$

$= 2 - 3\ln 3 + 3\ln 2$

$= 2 + 3\ln\frac{2}{3}$

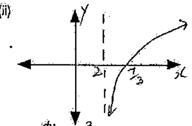
$\boxed{3}$

QUESTION 2

(a) (i) 3x-6>0

x>2 (1)

(ii)



(2)

(iii) $\frac{dy}{dx} = \frac{3}{3x-6}$

gradient of tangent = $\frac{1}{x-2}$

gradient of normal = $-x+2$

at $x=4$ grad of normal = -2 and $y = \log_e 6$

eqn. of normal is $y - \log_e 6 = -2(x-4)$

$2x + y - 8 - \ln 6 = 0$ (3)

(b) period = $\frac{2\pi}{b} = 8$

$b = \frac{\pi}{4}$

$3 = a\cos(b \times 1)$

$3 = a\cos\frac{\pi}{4}$ (2)

$3 = a \times \frac{1}{\sqrt{2}}$

$a = 3\sqrt{2}$

(c) (i)

$y = xe^{-x}$

$y' = e^{-x} - xe^{-x}$

$y' = e^{-x}(1-x)$

$y'' = e^{-x} - e^{-x} + xe^{-x}$

$y'' = e^{-x}(-1-1+x)$

$y'' = e^{-x}(-2+x)$

$y' = 0$ when $x = 1$ and at $x = 1$ $y' = \frac{1}{e}(-1), y'' < 0$

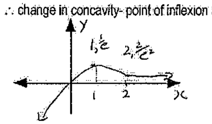
$\therefore$ max. turning point at x=1 (2)

(ii) $y'' = 0$ when $x = 2$,

$y'' < 0$ when $x = 1$ and > 0 when $x = 3$ $\left(\frac{1}{e} \times 1\right)$

$\therefore$ change in concavity- point of inflexion at x=2 (2)

(iii) (3)



(d) (i) $\int_0^{\frac{\pi}{8}} \sin^2 3x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{8}} 1 - \cos 6x\,dx$

$= \frac{1}{2}\left[x - \frac{1}{6}\sin 6x\right]_0^{\frac{\pi}{8}}$

$= \frac{1}{2}\left[\frac{\pi}{8} - \frac{1}{6}\sin\frac{3\pi}{4}\right] - \frac{1}{2}[0-0]$

$= \frac{1}{2}\left[\frac{\pi}{8} - \frac{1}{6\sqrt{2}}\right]$

$= \frac{\pi}{16} - \frac{\sqrt{2}}{24}$ (3)

(ii) $\lim_{h \to 0} \frac{1 - \cos 2h}{h^2}$

$= \lim_{h \to 0} \frac{2\sin^2 h}{h^2}$

$= 2 \times \lim_{h \to 0} \frac{\sin h}{h} \times \frac{\sin h}{h}$

$= 2 \times 1 \times 1$

$= 2$ (2)

(e) (i) Let ∠TQR = a

∠QRS = a(alternate ∠)

∠PSR = π - a(cointerior)

∠PQR = a(opp. ∠ in cyclic quad)

so PQ bisects ∠PQT (2)

(ii) Join PT

from (i) PQR = ∠PTR = a (angles on arc PR)

∠RQT = ∠TRN = a (∠in opp. segment)

so $PT \| MN$ as ∠PTR = ∠TRN (alt. ∠s) (2)

(f) When $n = 1$, $2^{3\times1-1} + 3 = 7$ (4)

Statement true for $n = 1$

Assume statement true for $n = k$

$2^{3k-1} + 3 = 7A$ where A is a positive integer.

If $n = k+1$

$2^{3(k+1)-1} + 3 = 2^{3k-1} \times 2^3 + 3$

$= 8(7A - 3) + 3$

$= 56A - 21$ which is divisible by 7

$2^{3(k+1)-1} + 3$ is divisible by 7

If the statement is true for *n=k*, it is true for *n=k+1*. So, by Mathematical Induction it is true for any integer $n \geq 1$

[Section C]

Question (3) (a)

(i)

M (15)	W (10)

(i) $\binom{15}{4} \times \binom{10}{4} = 1365 \times 210 = \boxed{286650}$

(ii)

5W 3M $\quad \binom{15}{3}\binom{10}{5} = 455 \times 252$

114660

6W 2M $\quad \binom{15}{2}\binom{10}{6} = 22050$

7W 1M $\quad \binom{15}{1}\binom{10}{7} = 1800$

8W $\quad \binom{10}{8}$ $\quad$ Total = 138555

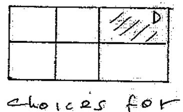
(iii) 6M 2W, 5M 3W, 4M 4W, 3M 5W, 2M 6W, 1M 7W, 8W.

$\binom{15}{6}\binom{10}{2} + \binom{15}{5}\binom{10}{3} + \binom{15}{4}\binom{10}{4} + \binom{15}{3}\binom{10}{5} + \binom{15}{2}\binom{10}{6} + \binom{15}{1}\binom{10}{7} + \binom{10}{8}$

$= 225225 + \cdots + 45$

$= \boxed{1010790}$

(c)



2 choices for the driver and the rest of the passengers can arrange themselves in 5! ways

i.e $= \boxed{2 \times 5!}$
$= \boxed{240}$.

(b)

$\int (e^x \cos x + e^x \sin x) = e^x \sin x$

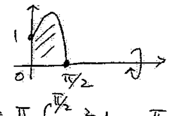
$\int (e^x \cos x - e^x \sin x) = e^x \cos x$

$\therefore \int_0^{\pi/2} e^x \cos x \, dx = \left[\frac{e^x}{2}(\sin x + \cos x)\right]_0^{\pi/2}$

$= \frac{e^{\pi/2}}{2} - \frac{1}{2}$

$= \boxed{\frac{1}{2}(e^{\pi/2} - 1)}$.

(d)



$V = \pi \int_0^{\pi/2} y^2 dx = \pi \int_0^{\pi/2} (1 + \sin 2x) \, dx$

$= \pi \left[x - \frac{\cos 2x}{2}\right]_0^{\pi/2}$

$= \left(\frac{\pi}{2} + \frac{1}{2}\right) - \left(-\frac{1}{2}\right)$

$= \boxed{\pi/2 \, (\pi + 2)}$

Question (3).

(e) $f(x) = x^{1/3}$

$f'(x) = \frac{1}{3} x^{-2/3}$

If x_1 is the 1^{st} approx then a better approx x_2 is given by

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= x_1 - \frac{x_1^{1/3}}{(1/3 \, x^{-2/3})}$

$= x_1 - 3x_1 = -2x_1$

$\therefore |x_2| = |-2x_1| = 2|x_1|$

i.e $|x_2| > |x_1|$.

(f) Let $S(n)$ be the statement that

$\cos x + \cos 3x + \cdots + \cos(2n-1)x = \frac{\sin 2nx}{2 \sin x}$.

For $n = 1$,

LHS $= \cos x$

RHS $= \frac{\sin 2x}{2 \sin x} = \frac{2 \sin x \cos x}{2 \sin x} = \cos x$

$\therefore S(1)$ is true

Assume $S(k)$ is true

Consider $n = k+1$

Now $S(k+1) = \cos x + \cos 3x + \cdots + \cos(2k+1)x$

$= \frac{\sin 2kx}{2 \sin x} + \cos(2k+1)x$

$= \frac{\sin 2kx + 2 \sin x \cos(2k+1)x}{2 \sin x}$

$= \frac{\sin 2kx + 2 \sin x [\cos 2kx \cos x - \sin 2kx \sin x]}{2 \sin x}$

$= \frac{\sin 2kx (1 - 2\sin^2 x) + \cos 2kx \sin 2x}{2 \sin x}$

$= \frac{\sin 2kx (\cos 2x) + \cos(2kx) \sin 2x}{2 \sin x}$

$= \frac{\sin(2kx + 2x)}{2 \sin x}$

$= \frac{\sin 2(k+1)x}{2 \sin x}$

$S(k+1)$ is true if $S(k)$ is true. $\therefore$ By M.I. $S(n)$ is true $\forall n \geq 1$ where $n \in Z^+$

(g) $2y = e^x - \frac{1}{e^x}$.

$2y e^x = e^{2x} - 1$

$\therefore (e^x)^2 - (2y) e^x - 1 = 0$

$\Rightarrow e^x = \frac{2y \pm \sqrt{4y^2 + 4}}{2}$

$e^x = y \pm \sqrt{y^2 + 1}$

$\because e^x > 0 \quad \therefore e^x = y + \sqrt{y^2 + 1}$

$\therefore \boxed{x = \ln(y + \sqrt{y^2 + 1})}$.

(h) ΔCXB is a rt. $\angle \, \Delta$. therefore B, C, X are concyclic pts. and $\because BC$ subtends a right $\angle \Rightarrow BC$ is a diameter.

$\therefore MC = MB$

$\therefore M$ is the centre and MX is also a radius.

i.e ΔMBX is isosceles.

$\therefore \angle MBX = \angle MXB$.

(ii) In ΔBCX Let $\angle B = \alpha$.

$\therefore \angle BCA = 90 - \alpha$ ($\angle$ sum of Δ)

and $\angle BCA = \angle BDA = 90 - \alpha$ ($\angle$ in same seg.). Also $\angle BXM = \angle DXN = \alpha$ (vert. opp)

$\therefore \angle XND = 90^\circ \Rightarrow MN \perp AD$.