SYDNEY BOYS' HIGH SCHOOL

AUGUST 1996 TRIAL HSC

MATHEMATICS

3 UNIT (ADDITIONAL) AND 3/4 UNIT (COMMON)

Time allowed - 2 hours
(Plus 5 minutes reading time)

Examiner: PS Parker

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- ALL questions are of equal value.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board-approved calculators may be used.
- Standard integrals are provided at the back of the examination paper
- Each section is to be returned in a separate Writing Booklet clearly marked with the section and the questions on the cover. Start each question on a new page, clearly showing your name, class and teacher's name. Second and subsequent Writing Booklets are to be inserted in the first Writing Booklet for the section
- If required, additional Writing Booklets may be obtained from the Examination Supervisor upon request
- This is a trial paper and does not necessarily reflect the format or content of the HSC examination for this subject

Question 1 (Start a new page) **Marks**

(a) The point $P(8,-2)$ divides the interval joining $Q(2,7)$ and $R(6,1)$ externally in the ratio $k:1$. What is the value of k? **2**

(b) A tank is emptied by a tap from which water flows so that, until the flow ceases, the rate after t minutes is R litres/minute where **4**

$$R=(t-3)^2$$

(i) What is the initial rate of flow?

(ii) How long does it take to empty the tank?

(iii) How long will it take (to the nearest second) for the flow to drop to 20 litres/minute?

(iv) How much water was in the tank initially?

(c) A circle has equation $x^2+y^2+4x-6y=0$ **4**

(i) Find the centre and the radius of the circle.

(ii) The line $3x+2y=0$ meets this circle in two points, A and B.

(α) Find the coordinates of A and B.

(β) Calculate the distance AB.

(d) Evaluate $\lim_{x\to 0}\frac{\sin 7x}{6x}$ **2**

(e) $P(x)=10x^4-33x^3-7x^2+45x+9$ **3**

Given $P(-1)=P(3)=0$. Find all the zeros of $P(x)$.

Question 2 (Start a new page) Marks

(a) A subcommittee of seven persons is chosen at random from 7 men and 5 women. Find the probability that the subcommittee **4**

(i) consists entirely of men.

(ii) included all the women.

(iii) includes a majority of women.

(b) Let $f(x) = 2x^3 + 2x - 1$ **4**

(i) Show that $f(x)$ has a root between $x = 0$ and $x = 1$.

(ii) By considering $f'(x)$, explain why this is the only root of $f(x)$.

(iii) Taking $x = 0$ as an initial approximation, use Newton's Method to find a closer approximation.

(c) Let $F(x) = 3\sin^{-1}(4x)$ **3**

(i) Write down the domain and range of $F(x)$.

(ii) Sketch $F(x)$.

(d) Find the indefinite integral $\int \frac{4x+9}{4+9x^2}dx$ **4**

Question 4 (Start a new page) Marks

(a) $P(x, y)$ is a variable point on the line $x = 2$ **4**

(i) Sketch a diagram of this situation.

(ii) Show that $\theta = \tan^{-1}\left(\frac{y}{2}\right)$, where θ is the angle between OP and the positive direction of the x axis. Hence find $\frac{d\theta}{dy}$.

(b) (i) Show that $\int_0^{\frac{\pi}{4}} \cos^2\theta \, d\theta = \frac{\pi+2}{8}$ **8**

(ii) Hence using the substitution $x = 2\sin\theta$, or otherwise, evaluate $\int_0^{\sqrt{2}} \sqrt{4-x^2}\, dx$.

(c) If α, β and γ are the roots of $8x^3 - 6x + 1 = 0$ then evaluate $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$ **3**

Question 5 (Start a new page)

Marks

(a) $f(x) = g(x) - \ln[g(x)+1]$ **5**

(i) Prove that $f'(x) = \dfrac{g(x).g'(x)}{g(x)+1}$

(ii) Hence evaluate $\displaystyle\int \frac{\sin 2x \cos 2x}{\sin 2x + 1}\,dx$

(b) By using the substitution $u^2 = x+1$, find the volume of the solid formed by rotating the area bounded by the curve $y = \dfrac{x-1}{\sqrt{x+1}}$, the x axis and the lines $x = 3$ and $x = 8$. **6**
Leave your answer as an exact value.

(c) $T(2t, t^2)$ is a variable point on the parabola $x^2 = 4y$ whose vertex is O. N is the foot of the ordinate from T and the perpendicular from N to OT meets OT at P. Prove that the locus of P is a circle and state its centre and radius. **4**

Question 6 (Start a new page)

Marks

(a) In an acute angled triangle ABC, angle B > angle C. The line BD is drawn so that $\angle DBC = \angle ACB$ and $BD = AC$. If this line cuts AC in O and AD and DC are joined, prove that: **5**

(i) $AO = OD$

(ii) $\Delta ADB \equiv \Delta DAC$

(iii) $AD \parallel BC$

(b) A particle, P, is moving in a straight line, with its motion given by $\ddot{x} = -9x$ where x is the displacement of P from O. **7**
Initially P is 4 m on the right side of O and is moving towards O with velocity 12 m/s.

(i) Show that $\ddot{x} = \dfrac{d}{dx}\left(\tfrac{1}{2}v^2\right)$

(ii) Show that its speed at position x is $3\sqrt{32 - x^2}$ m/s

(iii) Verify that $x = 4\sqrt{2}\cos\left(\frac{\pi}{4} + 3t\right)$ and hence find its velocity, v, as a function of t.

(iv) Find the greatest (α) speed of P

(β) acceleration of P

(δ) displacement of P from O

(v) Find the period of the motion

(c) (i) Simplify the following expression $\dfrac{\sin(x - \frac{\pi}{6}) + \sin(x + \frac{\pi}{6})}{\cos(x - \frac{\pi}{6}) - \cos(x + \frac{\pi}{6})}$ **3**

(ii) If $f(x) = \dfrac{\sin(x - \frac{\pi}{6}) + \sin(x + \frac{\pi}{6})}{\cos(x - \frac{\pi}{6}) - \cos(x + \frac{\pi}{6})}$, for what values of x is $f(x)$ independent of x. Hence sketch the function.

Question 7 (Start a new page) **Marks**

(a) Solve $2\cos(x - \frac{7\pi}{18}) + 1 = 0$ for $0 \le x \le 2\pi$ **3**

(b) (i) Prove that $1^2 + 2^2 + 3^2 + \ldots + n^2 = \sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$, by the process of mathematical induction. **6**

(ii) Draw the graph of $y = x^2$ and construct n trapezia between the curve and the x axis from $x = 0$ to $x = 1$. The width of each trapezium is $\frac{1}{n}$ units.

(α) If S denotes the sum of the areas of these trapezia, show that

$$S = \frac{1}{2}\cdot\frac{1}{n}\left\{(0+1) + \frac{2}{n^2}\left(1^2 + 2^2 + 3^2 + \ldots + (n-1)^2\right)\right\}$$

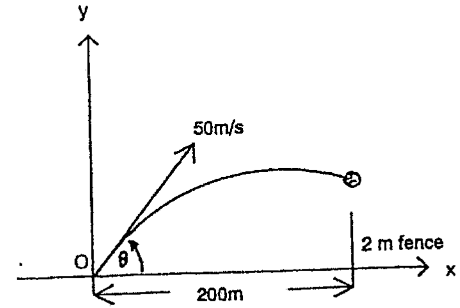
(β) Using the result from part (i) above, show that

$$S = \frac{1}{2n}\left\{1 + \frac{1}{3n}(n+1)(2n+1) - 2\right\}$$

(δ) If A denotes the exact area under the curve show that $A = \lim_{n\to\infty} S$ and hence evaluate A.

Question 7 Continued **Marks**

(c) A method to score a home run in a baseball game is to hit the ball over the boundary fence on the full. **6**

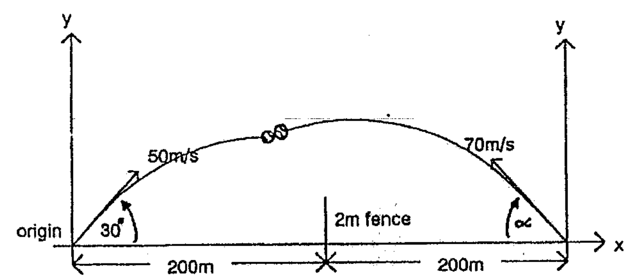


A ball is hit at 50 metres per second. The fence 200 metres away is 2 metres high. You may neglect air resistance and acceleration due to gravity can be taken as 10 metres per second per second and you may assume the following equations of motion: $x = 50t\cos\theta$ and $y = 50t\sin\theta - 5t^2$

(i) Show that if ball just clears the 2 metre boundary fence then $80\tan^2\theta - 200\tan\theta + 82 = 0$, where θ is the angle of projection.

(ii) In what range of values must θ lie to score a home run by this method?

(iii) In an adjacent field another ball is hit at the same instant at 70 metres per second and the balls collide. Assuming that $\theta = 30^\circ$, find the angle of projection, α, of the second ball and the time and position where the balls collide.



END OF THE PAPER

High 96 R. Ong

1. (a) P(8, −2), Q: (2, 7) R(6, 1) k : 1

$8 = 6k - 2$ −2:

$k - 1$

$8k - 8 = 6k - 2$

$2k = 6$

$k = 3.$

(b). $R = (t-3)^2$

when t = 0

(i) R = 9. litres/min

(ii). ie when R = 0. Need to find V first i.e. $\frac{dV}{dt} = (t-3)^2$

$(t-3)^2 = 0.$ V = ...

$t = 3.$ then let V = 0.

(iii) $20 = (t-3)^2$

$20 = t^2 - 6t + 9$

$t^2 - 6t - 11 = 0.$

$t = \frac{6 \pm \sqrt{36 + 44}}{2}$

$= \frac{6 \pm \sqrt{80}}{2}$

$\doteqdot 7.47$

= 7 sec.

(iv). $\int \frac{dV}{dt} = \int (t-3)^2 dt$

$V = \left[\frac{(t-3)^3}{3}\right]_0^3$

t = 0, V = 9 litres.

(c).(i) $x^2 + y^2 + 4x - 6y = 0.$

$x^2 + 4x + 4 + y^2 - 6y + 9 = 13$

$(x+2)^2 + (y-3)^2 = 13.$

Centre: (−2, 3). Radius = $\sqrt{13}$.

(ii). $2y = -3x$

$y = \frac{-3x}{2}.$

and $x^2 + \frac{9x^2}{4} + 4x + 9x = 0.$

$4x^2 + 9x^2 + 16x + 36x = 0.$

$13x^2 + 52x = 0.$ x = 0, x = 4.

x = 0, y = 0.

x = 4, y = −6

(0, 0), (4, −6)

(iii). $\sqrt{4^2 + 6^2}$

$= \sqrt{52}$

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(d). $\lim_{x \to 0} \frac{\sin 7x}{6x}$

$= \frac{\sin 7x}{7x} \times \frac{7}{6}$

= 7/6.

(e). $10x^4 - 33x^3 - 7x^2 + 45x + 9$

$(x+1)(x-3)$ is a factor.

$x^2 - 2x - 3$

$$\begin{array}{r} 10x^2 - 13x - 3 \\ x^2 - 2x - 3 \overline{) 10x^4 - 33x^3 - 7x^2 + 45x + 9} \\ 10x^4 - 20x^3 - 30x^2 \\ \hline -13x^3 + 23x^2 + 45x \\ -13x^3 + 26x^2 + 39x \\ \hline -3x^2 + 6x + 9 \\ -3x^2 + 6x + 9 \\ \hline 0. \end{array}$$

$f(x) = (x^2 - 2x - 3)(10x^2 - 13x - 3)$

$= (x+1)(x-3)(\cancel{3x-2}\ 2x-3)(5x+1)$

$\therefore x = -1, 3, 3/2, -1/5$

2. (a) (i) $\frac{\binom{7}{7}}{\binom{12}{7}} = \frac{1}{792}.$

(ii). $\frac{\binom{5}{4}\binom{7}{3}}{\binom{12}{7}} = \frac{7}{264}.$

(iii). 4 Women + 5 Women

$\frac{\binom{5}{4}\binom{7}{3}}{\binom{12}{7}} + \frac{7}{264}.$

$= \frac{175}{792} + \frac{7}{264}$

$= \frac{49}{198}.$

(b). $f(x) = 2x^3 + 2x - 1$

f(0) = −1 −ve

f(1) = 3 +ve

∴ Change in sign. Root between 0, 1.

ii). $f'(x) = 6x^2 + 2$

$\Delta < 0.$

$\therefore$ No Turn Points.

$\therefore$ Only one Root.

iii). $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$= 0 - \dfrac{f(0)}{f'(0)}$

$= \dfrac{+1}{2}.$

$= 1/2.$

(c) (i) $F(x) = 3\sin^{-1}4x$

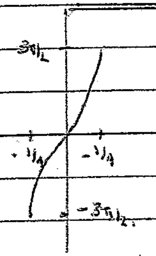
D: $-1 \le 4x \le 1$

$-1/4 \le x \le 1/4$

R: $-\pi/2 \le y/3 \le \pi/2$

$-3\pi/2 \le y \le 3\pi/2$

(ii)



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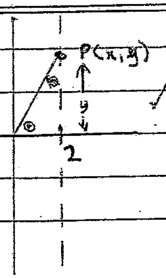
(d). $\displaystyle\int \frac{4x+9}{4+9x^2}$

$= \displaystyle\int \frac{4x}{4+9x^2}\,dx + \int \frac{9}{4+9x^2}\,dx$

$= \dfrac{4}{18}\displaystyle\int \frac{18x}{4+9x^2} + \frac{9}{9}\int \frac{dx}{4/9+x^2} \rightarrow 3\cdot\frac{1}{?}\left(\frac{1}{\frac{2}{3}}\tan^{-1}\left(\frac{x}{\frac{2}{3}}\right)\right)$

$= \dfrac{2}{9}\ln(4+9x^2) + \dfrac{1}{6}\left(\dfrac{3}{3}\right)\tan^{-1}\left(\dfrac{3x}{2}\right) + c.$

4. (a)



$P(2, y)$

$\therefore \tan\theta = y/2$

$\theta = \tan^{-1}(y/2)$

(ii). $\dfrac{d\theta}{dy} = \dfrac{\frac{1}{2}}{1+\frac{y^2}{4}} = \dfrac{2}{4+y^2}$

i) $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$\cos^2\theta = \dfrac{\cos 2\theta + 1}{2}$

$\displaystyle\int_0^{\pi/4} \cos^2\theta = \frac{1}{2}\int_0^{\pi/4} \cos 2\theta + 1$

$= \dfrac{1}{2}\left[\dfrac{\sin 2\theta}{2} + \theta\right]_0^{\pi/4}$

$= \dfrac{1}{2}\left[\dfrac{1}{2} + \dfrac{\pi}{4}\right]$

$= \dfrac{1}{4} + \dfrac{\pi}{8}.$

$= \dfrac{2+\pi}{8}.$

(ii). $\displaystyle\int_0^{\sqrt{2}} \sqrt{4 - x^2}\,dx$

At $x = \sqrt{2}$, $\theta = \pi/4$.
$x = 0$, $\theta = 0$.

Let $x = 2\sin\theta$

$dx = 2\cos\theta \cdot d\theta$

$I = \displaystyle\int_0^{\pi/4} \sqrt{4 - 4\sin^2\theta} \cdot 2\cos\theta \cdot d\theta$

$= \displaystyle\int_0^{\pi/4} 4\cos^2\theta \cdot d\theta$

$= 4\left[\dfrac{2+\pi}{8}\right]$

$= \dfrac{2+\pi}{2}.$

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c). $8x^3 - 6x + 1 = 0.$

$\Sigma\alpha = 0.$

$\Sigma\alpha\beta = -\cancel{6}/\cancel{8} = -\dfrac{3}{4}$

$\alpha\beta\gamma = -1/8.$

$\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2} + \dfrac{1}{\gamma^2}$

$= \dfrac{\beta^2\gamma^2 + \alpha^2\gamma^2 + \beta^2\alpha^2}{\alpha^2\beta^2\gamma^2}$

$= \dfrac{(\Sigma\alpha\beta)^2 - 2(\alpha\beta\gamma)(\alpha+\beta+\gamma)}{(\alpha\beta\gamma)^2}$

$= \dfrac{(\Sigma\alpha\beta)^2 - 2\alpha\beta\gamma(\Sigma\alpha)}{(\alpha\beta\gamma)^2}$ ← Careless!

$= \dfrac{\cancel{3/4}\ \frac{9}{16}}{\cancel{1/64}\ \frac{1}{64}} = 36$

$= \cancel{16}.$

(a) (i) $f(x) = g(x) - \ln(g(x)+1)$

$f'(x) = g'(x) - \dfrac{g'(x)}{g(x)+1}$

$= \dfrac{g(x)g'(x) + g'(x) - g'(x)}{g(x)+1}$

$= \dfrac{g(x)g'(x)}{g(x)+1}$

(ii) $g(x) = \sin 2x$

$g'(x) = 2\cos 2x$

$\therefore \displaystyle\int \frac{\sin 2x \cos 2x}{\sin 2x + 1}$

$= \dfrac{1}{2}\displaystyle\int \frac{2\sin 2x \cos 2x}{\sin 2x + 1}$

$= \dfrac{1}{2}\left[\sin 2x - \ln(\sin 2x + 1)\right] + c.$

(b) $y = \dfrac{x-1}{\sqrt{x+1}}$

$y^2 = \dfrac{(x-1)^2}{x+1}$

$V = \pi \displaystyle\int_3^8 \frac{(x-1)^2}{x+1}$

Let $u^2 = x+1$ At $x = 8, u = 3$; $x = 3, u = 2.$

$2u = \dfrac{dx}{du}$

$2u\,du = dx$

$V = \pi \displaystyle\int_2^3 \frac{(u^2-2)^2}{u^2} 2u\,du$

$= 2\pi \displaystyle\int_2^3 \frac{u^4 - 2u^2 + 4}{u}\,du$

$= 2\pi \displaystyle\int_2^3 u^3 - 2u + \frac{4}{u}\,du$

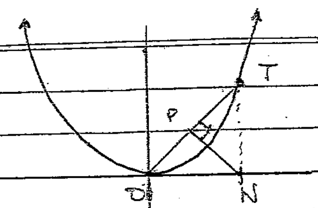
$= 2\pi \left[\dfrac{u^4}{4} - u^2 + 4\ln u\right]_2^3$

$= 2\pi \left[\dfrac{81}{4} - 9 + 4\ln 3 - 4 + 4 - 4\ln 2\right]$

$= 2\pi \left[\dfrac{45}{4} + 4\ln(3/2)\right]$

(c) $T(2t, t^2)$

$x^2 = 4y$



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$N(2t, 0)$

$m_{OT} = \dfrac{t^2}{2t}$

$= t/2.$

$\therefore m_{PN}$ must be $-2t$ Careless $-\dfrac{2}{t}$

Let $P(x, y)$

$m_{PN} = \dfrac{y}{x - 2t} = -2t.$

$y = -2tx + 4t^2$ — (1)

Similarly: $\dfrac{y}{x-2t} \times \dfrac{t}{2} = -1$

$\dfrac{ty}{2x - 4t} = -1$

$ty = -2x + 4t.$ — (2).

$t(y-4) = -2x$

$t = \dfrac{-2x}{y-4}.$

$\therefore y = -2x\left(\dfrac{-2x}{y-4}\right) + 4\left(\dfrac{-2x}{y-4}\right)^2$

$y = \dfrac{4x}{y-4} + \dfrac{16x^2}{(y-4)^2}$

$y(y-4)^2 = 4x(y-4) + 16x^2$

$y(y^2 - 8y + 16) = 4xy - 16x + 16x^2$

$y^3 - 8y^2 + 16y = 4xy - 16x + 16x^2$

Eq^n of PN

$y = -\dfrac{2}{t}(x - 2t)$

$ty = -2x + 4t$

$\therefore 2x + ty - 4t = 0$

solving with $y = \dfrac{t}{2}x$ eq^n of OT

$\therefore 2x + t\left(\dfrac{xt}{2}\right) - 4t = 0$

$\therefore x\left(2 + \dfrac{t^2}{2}\right) = 4t$

$x = \dfrac{4t}{2 + \frac{t^2}{2}}$; $y = \dfrac{t}{2} \cdot \dfrac{8t}{4+t^2}$

$= \dfrac{8t}{4+t^2}$ $= \dfrac{4t^2}{4+t^2}$

Because locus is a circle then square both x and y

$x^2 = \cancel{16t^2}\ \dfrac{64t^2}{(4+t^2)^2}$

$y^2 = \dfrac{16t^4}{(4+t^2)^2}$

Now add $x^2 + y^2 = \dfrac{16t^4 + 64t^2}{(4+t^2)^2} = \dfrac{16t^2(t^2+4)}{(4+t^2)^2}$ but $\dfrac{t^2}{4+t^2} = y$

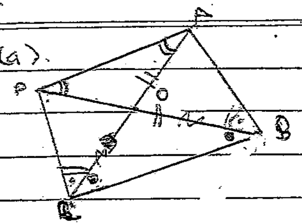
$= 16y$

$\therefore x^2 + y^2 - 16y = 0$

$x^2 + (y-4)^2 = 16$

$\therefore$ locus is a circle with centre $(0, 4)$ and radi 4.

6. (a).



(i) $D\hat{B}C = A\hat{C}B$ (given)

$\therefore$ $OC = OB$ (base $\angle$'s of isosceles Δ).

$\because$ $DB = AC$ (given)

$\therefore$ $AO = OD$ (sum of remainder).

(ii). ~~Since AO = ...~~

In Δ's ADB & DAC

AD is common

$AC = DB$ (given)

$B\hat{B}A = D\hat{A}C$ (Base Angles of ...)

$\therefore$ $\Delta ADB \equiv \Delta DAC$ (SAS)

This is right!

~~In Δ's ADB & DAC~~

~~AD is common~~

~~$AC = DB$ (given).~~

~~In Δ's AOB & POC~~

~~$AO = DO$ (given) proven above)~~

~~$OC = OB$ (" ")~~

~~$D\hat{O}C = A\hat{O}B$ (vert. opp)~~

but why ?

~~$\therefore$ $\Delta AOB \equiv \Delta POC$. (SAS)~~

~~Since $\Delta AOB \equiv \Delta POC$~~

Is this valid. $\Longleftarrow$

~~Add common Δ DOA~~

~~gives $\Delta ADB \equiv \Delta DAC$.~~

(iii) Similarly, $\Delta ACB \equiv ADB$

$\therefore$ $A\hat{C}B = A\hat{D}B$ (corresponding $\angle$'s of $\equiv \Delta$'s)

$= D\hat{A}C$ (Base $\angle$'s of iso Δ)

$\therefore$ $AC \parallel BC$ (corresponding $\angle$'s are equal)

(b). $\ddot{x} = -9x$

$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$RHS = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

let $z = v^2$

$\frac{dz}{dv} = 2v$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{1}{2} \times \frac{dz}{dv} \times \frac{dv}{dx}$

$= \frac{1}{2} \times 2v \times \frac{dv}{dx}$

$= v \frac{dv}{dx}$

$= \frac{dx}{dt} \times \frac{dv}{dx}$

$= \frac{dv}{dt}$

$= \ddot{x}$

$\ddot{x} = -9x$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -9x$

$\frac{1}{2}v^2 = -9\int x \, dx$

$\frac{v^2}{2} = \frac{-9x^2}{2} + c$ careless!

$v^2 = \frac{-18x^2}{2} + c$.

At $x = 4$, $v = 12$.

$144 = -\frac{288}{2} + c$.

$c = \cancel{432}$. 288

$\therefore v^2 = \cancel{432} - \cancel{18}x^2$ (288, 9) $\Rightarrow$ $v^2 = 9(32 - x^2)$

$= 9(48 - 2x^2)$ $\quad v = 3\sqrt{32 - x^2}$ as req^d.

$v = 3\sqrt{48 - 2x^2}$

(iii). $\frac{dx}{dt} = -3\sqrt{32 - x^2}$

Verify means that you could show that $\ddot{x} = -9x$ from $x = 4\sqrt{2}\cos\left(\frac{\pi}{4} + 3t\right)$ by differentiation.

$\frac{dt}{dx} = \frac{-1}{3\sqrt{32 - x^2}}$

$t = \frac{1}{3}\int \frac{-1}{\sqrt{32 - x^2}} \, dx$ $\Rightarrow$

$t = \frac{1}{3} \cos^{-1}\left(x/4\sqrt{2}\right) + c$

At $t = 0$, $x = 4$.

$0 = 1/3 \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) + c$.

$= \pi/12 + c$

$c = -\pi/12$.

$t = \frac{1}{3} \cos^{-1}\left(\frac{x}{4\sqrt{2}}\right) - \pi/12$.

$3t + \pi/4 = \cos^{-1}\left(\frac{x}{4\sqrt{2}}\right)$

$\cos(3t + \pi/4) = x/4\sqrt{2}$.

$x = 4\sqrt{2}\cos(3t + \pi/4)$ as req^d,

$v = \frac{dx}{dt} = -12\sqrt{2}\sin\left(3t + \frac{\pi}{4}\right)$

(iv) (α) Greatest speed $= |v| = 12\sqrt{2}$

(β) " acceleration at $x = a$ i.e. $x = 4\sqrt{2}$ m/s

$\ddot{x} = \left|-9(4\sqrt{2})\right| = 36\sqrt{2}$ m/s^2

(γ) " displacement at $x = 4\sqrt{2}$ m

(v) Period is $T = \frac{2\pi}{n} = \frac{2\pi}{3}$ sec.

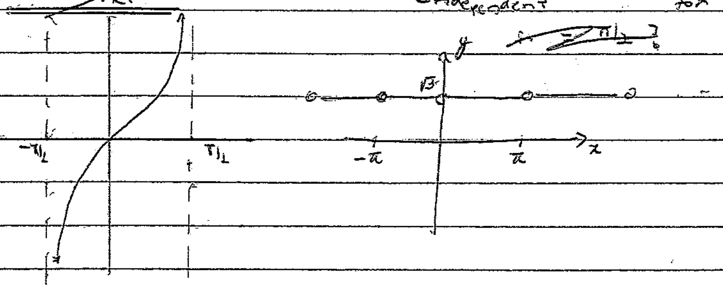
(c)(i) $\dfrac{\sin(x-\pi/6)+\sin(x+\pi/6)}{\cos(x-\pi/6)-\cos(x+\pi/6)}$

$= \dfrac{\sin x\cos\pi/6 - \cancel{\cos x\sin\pi/6} + \sin x\cos\pi/6 + \cancel{\cos x\sin\pi/6}}{\cancel{\cos x\cos\pi/6} + \sin x\sin\pi/6 - \cancel{\cos x\cos\pi/6} + \sin x\sin\pi/6}$

$= \dfrac{2\sin x\cos\pi/6}{2\cos x\cos\pi/6}$ $\quad \dfrac{2\sin x\cos\frac{\pi}{6}}{2\sin x\sin\frac{\pi}{6}} = \cot\frac{\pi}{6} = \sqrt{3}$ which is independent of x.

$= \cancel{\tan x}$ $\quad$ Independent for $x \neq n\pi$

(ii).



7.(a). $2\cos(x-\frac{7\pi}{18})+1=0.$ $\quad 0 \le x \le 2\pi$

$-\frac{7\pi}{18} \le x-\frac{7\pi}{18} \le \frac{29\pi}{18}$

$\cos(x-\frac{7\pi}{18}) = -\frac{1}{2}$

$x-7\pi/18 = 2\pi/3,\ 4\pi/3$

$x = \frac{19\pi}{18},\ \frac{31\pi}{18}.$

(b)(i). $1^2+2^2+3^2+\dots n^2 = \frac{1}{6}n(n+1)(2n+1)$

Let $n=1$

LHS $=1$, RHS $= \frac{1}{6}(2)(3)$

$=1$

$=$ LHS

Assume true for $S(k)$ $\quad \sum_{r=1}^{k} r^2 = \frac{k}{6}(k+1)(2k+1)$

Prove for $n=k+1$ $\quad = \frac{(k+1)(k+2)(2k+3)}{6}$

LHS $= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$

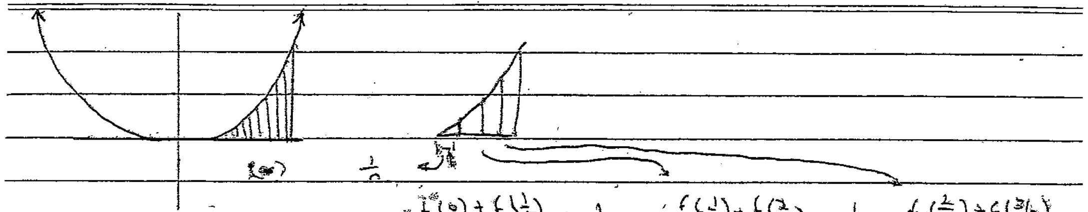
$= \frac{(k+1)[2k^2+k+6k+6]}{6}$

$= \frac{(k+1)(2k^2+7k+6)}{6}$

$= \frac{(k+1)(k+2)(2k+3)}{6}$

$=$ RHS

If true for $n=k$, then by the Principles of Maths Induction, true for $n \ge 1$.



(α) $\frac{1}{n}$

$\frac{f(0)+f(\frac{1}{n})}{2}\times\frac{1}{n}$, $\quad \frac{f(\frac{1}{n})+f(\frac{2}{n})}{2}\times\frac{1}{n}$, $\quad \frac{f(\frac{2}{n})+f(\frac{3}{n})}{2}\times\frac{1}{n}$

$\therefore S = \frac{1}{2n}\left[f(0)+f(1) + 2\left(f(\tfrac{1}{n})+f(\tfrac{2}{n})+\dots f(\tfrac{n-1}{n})\right)\right]$

$= \frac{1}{2n}\left[(0+1) + 2\left(\frac{1}{n^2}+\frac{4}{n^2}+\dots\frac{(n-1)^2}{n^2}\right)\right]$

$= \frac{1}{2n}\left[(0+1) + \frac{2}{n^2}\left(1+2^2+3^2+\dots(n-1)^2\right)\right]$

(β). $S = \frac{1}{2n}\left[(0+1)+\frac{2}{n^2}\left(\sum_{r=1}^{n-1} r^2\right)\right]$

$= \cancel{\frac{1}{2n}\left(1+\frac{2}{n^2}\left(\frac{(n-1)(n)(2n-1)}{6}\right)\right)}$

$= \cancel{\frac{1}{2n}\left(1+\frac{(n-1)(2n-1)}{3n}\right)}$

$\cancel{\frac{1}{2n}(1+}$

$S = \frac{1}{2n}\left[(0+1)+\frac{2}{n^2}\left(\sum_{r=1}^{n} r^2 - n^2\right)\right]$

$= \frac{1}{2n}\left[(1)+\frac{2}{n^2}\left(\frac{1}{6}n(n+1)(2n+1)\right)\right]$

$= \frac{1}{2n}\left[1+\frac{1}{3n}(n+1)(2n+1)-2\right]$

$S = \frac{1}{2n}(0+1) + \frac{2}{n^2}\left(\sum_{r=1}^{n} -n^2\right)$

$= \frac{1}{2n}\left(1+\frac{2}{n^2}\left(\frac{n(n+1)(2n+1)}{6}\right)-\frac{2n^2}{n^2}\right)$

$= \frac{1}{2n}\left(1+\frac{(n+1)(2n+1)}{3n}-2\right)$

(δ). $A = \lim_{n\to\infty} \sum_0^1 (S)\ 2n$

$= \frac{1}{2n}\left\{1+\frac{(n+1)(2n+1)}{3n}-2\right\}$

$= \frac{\frac{1}{2n}}{1}\left\{1+\frac{2n^2+3n+1}{3n}-2\right)$

$= \frac{\frac{1}{2n}}{1}\left\{-1+2n+1+\frac{1}{3n}\right\}$

$= \frac{\frac{1}{2n}}{1}\left(2n+\frac{1}{3n}\right) = \frac{1}{2}\left(2+\frac{1}{3n^2}\right)$

As $n\to\infty$ $\quad$ As $n\to\infty$, $\frac{1}{3n^2}\to 0$

$\cancel{A \to 1/3.}$ $\quad \therefore A \to \frac{1}{2}\times 2 = 1.$

(i) $x = 50t\cos\theta$, $y = 50t\sin\theta - 5t^2$

$\frac{x}{50\cos\theta} = t$

$y = x\tan\theta - \frac{x^2\sec^2\theta}{500}$

Let $x = 200$, $y = 2$.

$2 = 200\tan\theta - 80\sec^2\theta$

$2 = 200\tan\theta - 80\tan^2\theta - 80$

$80\tan^2\theta - 200\tan\theta + 82 = 0$.

(ii). $\tan\theta = \frac{200 \pm \sqrt{200^2 - 26240}}{160}$

$= \frac{200 \pm \sqrt{13760}}{160}$

$= \frac{200 \pm 117.30}{160}$

$\tan\theta = 1.9831, \; 0.51685.$

$\theta = \tan^{-1}(1.9831), \; \tan^{-1}(0.51685)$

$= 63°14', \; 27°19$

$27°19' < \theta < 63°14'$

(iii). $x = 50t\cos 30$ $\quad y = 25t - 5t^2$

$x = \frac{50\sqrt{3}}{2}t$ $\quad = \frac{x}{\sqrt{3}} - 5\left(\frac{4x^2}{7500}\right)$

$\frac{2x}{50\sqrt{3}} = t$ $\quad y = \frac{x}{\sqrt{3}} - \frac{x^2}{375}$.

Equate t for θ & α

then x and y & hence solve for α & t

$x = -Vt\cos\alpha$ $\quad y = \frac{-gt^2}{2} + Vt\sin\alpha$

$x = -70t\cos\alpha$ $\quad y = -5t^2 + 70t\sin\alpha$

$t = \frac{-x}{70\cos\alpha}$ $\quad y = \frac{-5x^2}{4900\cos^2\alpha} - x\tan\alpha$

$\frac{x}{\sqrt{3}} - \frac{x^2}{375} = \frac{-5x^2}{4900\cos^2\alpha} - x\tan\alpha$.

$\frac{5x}{4900\cos^2\alpha} - \frac{x}{375} + \frac{1}{\sqrt{3}} - \frac{\sin\alpha}{\cos\alpha} = 0.$