SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

May 2011
Assessment Task 2
Year 12

Mathematics Extension 1

General Instructions

- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators
- All necessary working should be shown in every question.maybe used.
- **All answers to be given in simplified exact form unless otherwise stated.**

Total Marks – 66

- Attempt questions 1-6
- All questions are of equal value.
- Start each new question in a separate answer booklet.
- **Hand in your answers in 3 separate bundles:**

Section A (Questions 1 and 2),
Section B (Questions 3 and 4) and
Section C (Questions 5 and 6).

Examiner: *A Ward*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate

Start a new booklet.

Section A

Question 1 (11 Marks). **Marks**

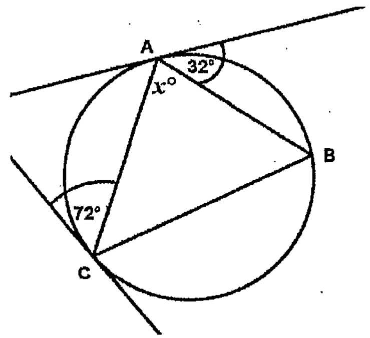
a) Three roads lead from town A to town B and five roads lead from town B to town C. How many ways are there of going from town A to town C? 1

b) Find: 2

(i) $\int (x^2+6x)dx$

(ii) $\int \left(\frac{1}{\sqrt{x}}-\sqrt{x}\right)dx$

c) Find the value of x giving reasons. 2



d) Using the letters of the word DISPLAY, how many 5 letter arrangements are possible? 1

e) Find a set of parametric equations for the curve: 2

$$x^2 = 5y$$

f) Write in factorial form $6\times5\times4$. 1

g) Evaluate the following to 3 significant figures: 2

$$\int_1^{20}\left(\sqrt[3]{x^2}-\sqrt[5]{x}\right)dx$$

End of Question 1

Question 2 (11 Marks). **Marks**

a) Find the Cartesian equation of the curve whose parametric equations are: 2

$$x = 6t, \ y = 3t^2$$

b) Using isosceles triangles, prove that the angle subtended by a diameter at the circumference of a circle is a right angle. 3

c) By differentiating $(3x+1)^4$, find the integral of $\int (3x+1)^3 \, dx$. 2

d) Find the volume of the solid generated by rotating the region bounded by the curve $y = 4 - x^2$ and $y = 0$ about the x-axis through a complete revolution. Leave your answer in terms of π. 4

End of Question 2

End of Section A

Section B – Start a new booklet.

Question 3 (11 Marks). **Marks**

a) A curve is defined by 5

$$y = \frac{2x^2 - x + 2}{x}, \ x \neq 0.$$

(i) Find the co-ordinates of any turning points.

(ii) Sketch the curve, showing these turning points and determine the equations of any asymptotes.

b) A curve has the parametric equations $x = 2t^2$ and $y = 4t$. Find the value(s) of k if $y = x + k$ is a tangent to the curve. 3

c) Using the trapezoidal rule with 2 strips, evaluate the following to 1 decimal places: 3

$$\int_2^6 \left(x - \frac{1}{x} \right) dx$$

End of Question 3

Question 4 (11 Marks). **Marks**

a) Prove by mathematical induction where n is a positive integer: 4

$$\sum_{r=1}^{n} 4^r = \frac{4}{3}\left(4^n - 1\right)$$

b) Use Simpson's Rule with 4 sub intervals to evaluate the following to 2 decimal places: 3

$$\int_0^2 \left(2^x + 1\right) dx$$

c) Find the area of the region bounded by the line $y = \frac{x}{2}$ and the parabola $y^2 = 8 - x$. 4

End of Question 4

End of Section B

Section C – Start a new booklet.

Question 5 (11 Marks). **Marks**

a) Take 0.5 as a first approximation to the root of $y = x^3 + 2x - 1$, and use Newton's Method to improve this to 2 decimal places. 3

b) (i) How many permutations of the letters of the word DEFEATED are there? 4

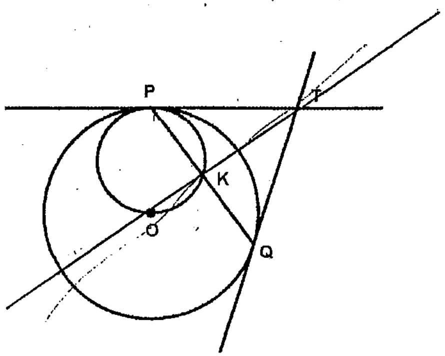
(ii) How many permutations are there in which the E's are separated from each other?

c) Prove by mathematical induction that $n! > 2^n$ for all positive integers greater than or equal to 4. 4

End of Question 5

Question 6 (11 Marks). **Marks**

a) Two circles touch internally at a point P. The smaller circle passes through the centre O, of the larger circle. PQ is any chord on the larger circle, intersecting the smaller circle at K. Tangents at P and Q of the larger circle meet at T. 5



(i) Prove that $PK=QK$.

(ii) Prove that O, K and T are collinear.

b) $P(2t, t^2)$ is a variable point on the parabola $x^2 = 4y$ whose focus is S. 6

$Q(x, y)$ divides the interval from P to S in the ratio $t^2 : 1$

(i) Find x and y in terms of t.

(ii) Verify that $t = \dfrac{y}{x}$.

(iii) Prove that as P moves on the parabola, Q moves on a circle, thus find the centre and radius.

End of Question 6.

End of Section C.

End of Examination.

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ a \neq 0$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln\left(x+\sqrt{x^2-a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\ x > 0$

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Q 1.

(a) $5 \times 3 = 15$ roads

(b) (i) $\int x^2 + 6x \cdot dx = \frac{x^3}{3} + 3x^2 + c$

(ii) $\int \frac{1}{\sqrt{x}} - \sqrt{x} \cdot dx = 2\sqrt{x} - \frac{2}{3} x^{3/2}$

(c) $A\hat{C}B = 32^\circ$ (Angle in Alt. Segment)

$A\hat{B}C = 72^\circ$ (" " " ")

$\therefore x = 76^\circ$ (Angle Sum in triangle).

(d) All letters are different

$\therefore$ No. of arrangements $= {}^7P_5 = 2520$

(e) $x^2 = 5y$

$\therefore 4a = 5$

$a = \frac{5}{4}$

$\therefore \quad x = 2at = \frac{5}{2} t$

$y = at^2 = \frac{5}{4} t^2$

(f) $6 \times 5 \times 4 = 120 = 5!$ or $\frac{6!}{3!}$

(g) $\int_1^{20} \sqrt[3]{x^2} - \sqrt[5]{x} \cdot dx$

$= \int_1^{20} x^{\frac{2}{3}} - x^{\frac{1}{5}} \cdot dx$

$= \left(\frac{3x^{5/3}}{5} - \frac{5x^{6/5}}{6} \right)_1^{20}$

$= 58.3$

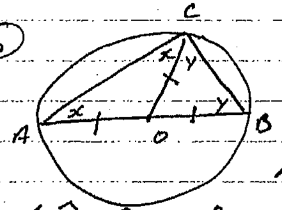
2 (a) $x = 6t \qquad y = 3t^2$

$\therefore t = \frac{x}{6}$

$\therefore y = 3\left(\frac{x}{6}\right)^2$

$12y = x^2$

(b)



$AO = CO = BO$

$\therefore O\hat{A}C = O\hat{C}A = x$ (ISOS. Δ)

and $O\hat{C}B = O\hat{B}C = y$ "

Now in Δ ACB

$2x + 2y = 180$ ($\angle$ Sum of Δ)

$\therefore x + y = 90$

Q.E.D

N.B: Use of $\hat{A} = \hat{B} = 45^\circ$ resulted in 0 marks.

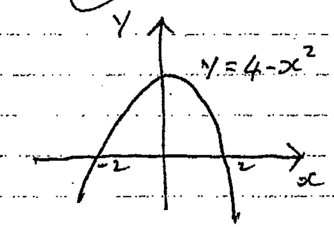
(c) $y = (3x+1)^4$

$\frac{dy}{dx} = 4(3x+1)^3 \cdot 3$ (Chain Rule)

$= 12(3x+1)^3$

$\therefore \int (3x+1)^3 \cdot dx = \frac{1}{12}(3x+1)^4 + c$

(d)



$V = \pi \int y^2 \cdot dx$

$= \pi \int_{-2}^{2} (4 - x^2)^2 \cdot dx$

$= 2\pi \left[16x - \frac{8x^3}{3} + \frac{x^5}{5} \right]_0^2$

$= 2\pi \left[32 - \frac{64}{3} + \frac{32}{5} \right]$

$= \frac{512\pi}{15} \quad u^3$

$= (34.13\pi) \quad u^3$

Question (3) [5] [redacted]

(a)

(i) $y = \dfrac{2x^2 - x + 2}{x}$

$y = 2x - 1 + \dfrac{2}{x}$

$\dfrac{dy}{dx} = 2 - 2x^{-2}$

$\therefore\ 2 = \dfrac{2}{x^2}$

When $\Rightarrow x^2 = 1 \quad x = \pm 1$

$\boxed{x = 1,\ y = 3 \quad x = -1,\ y = -5}$

$\dfrac{d^2y}{dx^2} = 4x^{-3}$ (2)

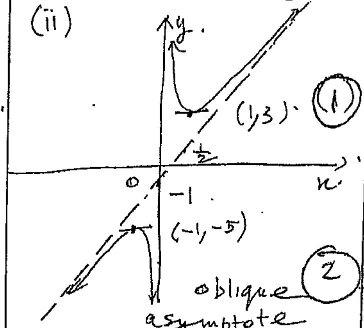
$f''(1) = 4 > 0$ (2)

$\therefore$ [redacted]

$f''(-1) = -4 < 0$

$\therefore$ [redacted]

(ii)



(1)

oblique asymptote (2)

[redacted]

vertical asymptote

[redacted] (2)

Solve 1 & 2

$x^2 + 2kx + k^2 - 8x = 0$

$x^2 + (2k - 8)x + k^2 = 0$

$\Delta = 0$ (2) if $y = x + k$ is a tgt

$\cancel{2}^2(k-4)^2 - \cancel{4}k^2 = 0$

$\therefore (2k-4)(-4) = 0$ (1)

$\Rightarrow k = 2$

(b) [3] [redacted]

$t = y/4$

$x = 2\left(y^2/16\right)$.

$\left\{\begin{array}{l}\text{[redacted]} = 0 \text{ — (1)} \\ \text{[redacted] — (2)}\end{array}\right.$

(c) [redacted]

$\int_2^6 \left(x - \frac{1}{x}\right) dx$

$A \doteq \frac{2}{2}\left[(y_1 + y_3) + 2y_2\right]$ (2)

$y_1 = f(2) = \frac{3}{2}$

$y_2 = f(4) = \frac{15}{4}$

$y_3 = f(6) = \frac{35}{6}$

$\therefore A \doteq \left(\frac{3}{2} + \frac{35}{6}\right) + \frac{15}{2}$ (1)

$= \dfrac{9 + 35 + 45}{6}$

$= 89/6 \doteq 14\,5/6$. 14.8

Question (4). [redacted]

(a) Let $S(n)$ be the propn.

$4^1 + 4^2 + \dots + 4^n = \frac{4}{3}(4^n - 1)$.

$n = 1$

L.H.S $= 4$

R.H.S $= \frac{4}{3}(4 - 1) = 4$.

$\therefore$ LHS = RHS

$\Rightarrow S(1)$ [redacted] (1)

Assume $S(k)$ is true

i.e [redacted]

Consider $n = k + 1$.

$4^1 + 4^2 + \dots + 4^k + 4^{k+1}$ (3)

$= \frac{4}{3}(4^k - 1) + 4^{k+1}$

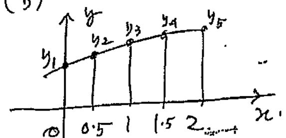
$= \frac{1}{3}4^{k+1} + 4^{k+1} - \frac{4}{3}$

$=$ [redacted]

$= \frac{4}{3}\left[4^{k+1} - 1\right]$

$\therefore$ By the principal of M.I.

(b) [redacted]



$h = \frac{1}{2} \quad \therefore h/3 = \frac{1}{6}$ (1)

$\therefore A \doteq \frac{1}{6}\left[(y_1 + y_5) + 2y_3 + 4(y_2 + y_4)\right]$ [redacted]

$\doteq \frac{1}{6}\left[7 + 6 + 4(6.242)\right]$

$\doteq$ [redacted] 6.33

$y_1 = 2$

$y_2 = \sqrt{2} + 1 = 2.414$

$y_3 = 3$

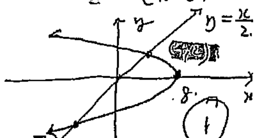
$y_4 = 2\sqrt{2} + 1$

$= 3.828$

$y_5 = 5$. (2)

(c) [redacted]

$\left\{\begin{array}{l} x = 2y \\ y^2 = -x + 8 \\ \quad = -(x - 8)\end{array}\right.$



(1)

$y^2 = 8 - 2y$.

$\therefore y^2 + 2y - 8 = 0$

$(y + 4)(y - 2) = 0$

$\therefore y = 2, -4$

[redacted] (1)

$\therefore A = \int_{-4}^{2}$ [redacted] dy

$\therefore A = \int_{-4}^{2}(8 - 2y - y^2)\,dy$ (2)

$= \left[8y - y^2 - y^3/3\right]_{-4}^{2}$

$= \left(16 - 4 - 8/3\right) - \left(-32 - 16 + \frac{64}{3}\right)$

$=$ [redacted]

Section C

5) a) $f(x) = x^3 + 2x - 1$

$f'(x) = 3x^2 + 2$

$x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$x_1 = 0.5 - \dfrac{(0.5)^3 + 2(0.5) - 1}{3(0.5)^2 + 2}$

$x_1 = 0.454545\ldots$

$x_1 = 0.45$ (to 2 decimal places)

b) i) D E F E A T E D

3 E's

2 D's

$\dfrac{8!}{3!\,2!} = 3360$

ii) Fix D, F, A, T, D. There are $\frac{5!}{2!}$ ways.

_ O _ O _ O _ O _ O _

since E's are all the same there are ${}^6C_3$ ways of placing them (in the gaps)

$\dfrac{5!}{2!} \times {}^6C_3 = 1200$

OR $\dfrac{5!}{2!} \times \dfrac{6 \times 5 \times 4}{3!} = 1200$.

c) Prove true for $n=4$

$LHS = 4!$ $\qquad$ $RHS = 2^4$

$= 24$ $\qquad$ $= 16$

$LHS > RHS$

$\therefore$ true for $n=4$.

Assume true for $n=k$ where $k \in N, k \geq 4$

$k! > 2^k$

Prove true for $n=k+1$

ie $(k+1)! > 2^{k+1}$

$$\begin{aligned} LHS &= (k+1)! \\ &= (k+1).k! \\ &> (k+1).2^k \quad \text{(using assumption)} \\ &\geq 5.2^k \quad (\text{since } k \geq 4) \\ &> 2.2^k \\ &= 2^{k+1} \\ &= RHS \end{aligned}$$

$\therefore$ true for $n=k+1$.

If true for $n=k$ it is true for $n=k+1$. Since true for $n=4$ by the principle of mathematical induction it is true for all positive integers greater than or equal to 4.

6) a) i) OP is the diameter of the smaller circle (when circles touch, the line of centres passes through the point of contact)

$P\hat{K}O = 90°$ (angle in a semi circle)

$PK = KQ$ (perpendicular from the centre to a chord bisects the chord)

ii) $PT = TQ$ (tangents from an external point)

$\therefore \triangle PTQ$ is isosceles

$TK \perp PQ$

(line from midpoint (K) of the base (PQ) of an isosceles triangle ($\triangle PTQ$) to opposite vertex (T) is perpendicular to the base)

Since $OK \perp PQ$ ($O\hat{K}P = 90°$) & $KT \perp PQ$ where K is common.

O, K, T are collinear

b) i) $P(2t, t^2)$ $\qquad$ $S(0,1)$

$t^2 : 1$

$$Q(x,y) = \left(\frac{1 \times 2t + t^2 \times 0}{1+t^2}, \frac{1 \times t^2 + t^2 \times 1}{1+t^2} \right)$$

$$\therefore x = \frac{2t}{1+t^2}, \quad y = \frac{2t^2}{1+t^2}$$

ii) $$\frac{y}{x} = \frac{\frac{2t^2}{1+t^2}}{\frac{2t}{1+t^2}} \times \frac{1+t^2}{1+t^2}$$

$$= \frac{2t^2}{2t}$$

$$= t$$

iii) $x = \dfrac{2t}{1+t^2}$

sub in $t = \dfrac{y}{x}$

$$x = \frac{2\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^2}$$

$$x + \frac{y^2}{x} = \frac{2y}{x}$$

$$x^2 + y^2 = 2y$$

$$x^2 + y^2 - 2y + 1 = 1$$

$$x^2 + (y-1)^2 = 1$$

which is a circle centre $(0,1)$
radius 1 unit.