SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2008
TRIAL
HIGHER SCHOOL CERTIFICATE

Mathematics Extension 1

General Instructions

- Reading Time – 5 Minutes
- Working time – 2 Hours
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators may be used.
- Start each question in a new booklet
- The questions are of equal value
- Marks may NOT be awarded for messy or badly arranged work.
- All necessary work should be shown in every question.
- Full marks will NOT be given unless the method of the solution is shown.

Total Marks – 84

- Attempt all questions

Examiner: *R. Boros*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln\left(x+\sqrt{x^2-a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

Start each question in a new answer booklet.

Question 1 (12 marks). **Marks**

a) Find the acute angle between the intersection of the curves $y = x^2 + 4$ and $y = x^2 - 2x$, correct to the nearest minute. 2

b) A is the point (-4, 2) and B is the point (3, -1). Find the coordinates of the point P which divides the interval AB externally in the ratio 2:1 2

c) Differentiate $y = \log_e\left(\sin^{-1} x\right)$ 2

d) Solve the inequality $\dfrac{x-1}{x+3} \geq -2$ 2

e) If $\cos A = \dfrac{7}{9}$ and $\sin B = \dfrac{1}{3}$ where A and B are acute angles,

Prove that $A = 2B$. 2

f) Use the substitution $u = t + 1$ to evaluate $\displaystyle\int_0^1 \frac{t}{\sqrt{t+1}}\,dt$ 2

End of Question 1.

Start a new booklet.

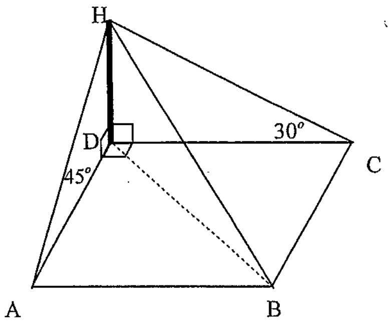
Question 2 (12 Marks). **Marks**

a) The polynomial $P(x) = ax^3 + bx^2 - 8x + 3$ has a factor of $(x-1)$ and leaves a remainder of 15 when divided by $(x+2)$. Find the values of a and b and hence fully factorise $P(x)$. 3

b) (i) Express $3\sin\theta + 2\cos\theta$ in the form $R\sin(\theta + \alpha)$ where α is an acute angle.

(ii) Hence, or otherwise solve the equation $3\sin\theta + 2\cos\theta = 2.5$ for $0^\circ \leq \theta \leq 360^\circ$. Answer correct to the nearest minute. 4

c) A post HD stands vertically at one corner of a rectangular field $ABCD$ The angle of elevation of the top of the post H from the nearest corners A and C are 45^o and 30^o respectively.



(i) If $AD = a$ units, find the length of BD in terms of a 2

(ii) Hence, find the angle of elevation of H from the corner B to the nearest minute. 1

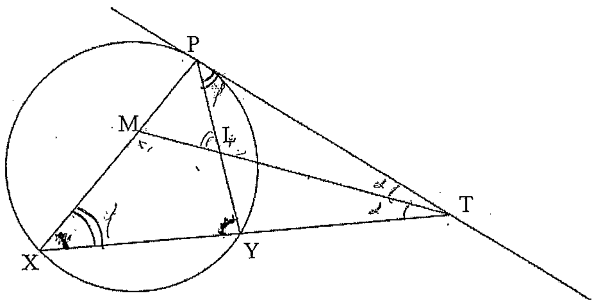
d) Taking $x = \dfrac{-\pi}{6}$ as a first approximation to the root of the equation $2x + \cos x = 0$, use Newton's method once to show that a second approximation to the root of the equation is $\dfrac{-\pi - 6\sqrt{3}}{30}$. 2

End of Question 2.

Start a new booklet.

Question 3 (12 marks). **Marks**

a) Diagram not to scale.

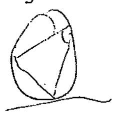


XY is any chord of a circle. XY is produced to T and TP is a tangent to the circle. The bisector of $\angle PTX$ meets XP in M and cuts PY at L. Prove that $\triangle MPL$ is isosceles. 3

b) (i) Find the domain and range of $f^{-1}(x) = \sin^{-1}(3x-1)$. 2

(ii) Sketch the graph of $y = f^{-1}(x)$. 1

(iii) Find the equation representing the inverse function $f(x)$ and state the domain and range. 3



c) Newton's Law of Cooling states that the rate of cooling of a body is proportional to the excess of the temperature of a body above the surrounding temperature. This rate can be represented by the differential equation $\frac{dT}{dt} = -k(T - T_0)$, where T is the temperature of the body, T_0 is the temperature of the surroundings, t is the time in minutes and k is a constant.

(i) Show that $T = T_0 + Ae^{-kt}$, where A is a constant, is a solution to the differential equation $\frac{dT}{dt} = -k(T - T_0)$. 1

(ii) A cup of coffee cools from $85\,^\circ C$ to $80\,^\circ C$ in one minute in a room temperature of $25\,^\circ C$. Find the temperature of the cup of coffee after a further 4 minutes have elapsed. Answer to the nearest degree. 2

End of Question 3.

Start a new booklet.

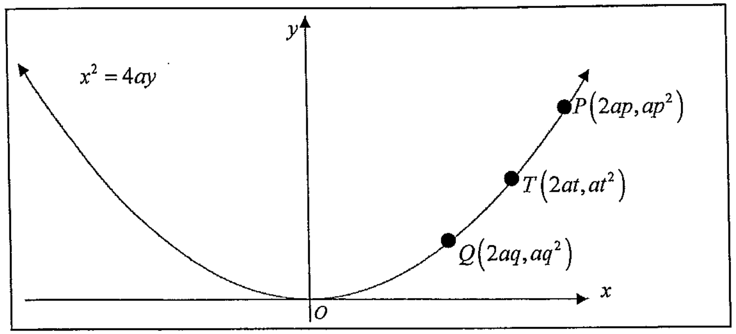
Question 4 (12 marks). **Marks**

a) Find the number of ways of seating 5 boys and 5 girls at a round table if:

(i) A particular girl wishes to sit between two particular boys. 1

(ii) Two particular persons do not wish to sit together. 1

b) $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are the points on the parabola $x^2 = 4ay$



It is given that the chord PQ has the equation $y - \frac{1}{2}(p+q)x + apq = 0$

(i) Derive the equation of the tangent to the parabola $x^2 = 4ay$ at the point $T(2at, at^2)$. 2

(ii) The tangent at T cuts the y-axis at the point R. Find the coordinates of the point R. 1

(iii) If the chord PQ passes through the point R show that p, t and q are terms of a geometric series. 2

c) A particle moves so that its distance x cm from a fixed point O at time t seconds is $x = 2\cos 3t$.

(i) Show that the particle satisfies the equation of motion $\ddot{x} = -n^2 x$ where n is a constant. 2

(ii) What is the period of the motion? 1

(iii) What is the velocity when the particle is first 1cm from O. 2

End of Question 4.

Start a new booklet.

Question 5 (12 marks). **Marks**

a) Find the general solution of the equation $\tan\theta = \sin 2\theta$ 3

b) The cubic equation $2x^3 - x^2 + x - 1 = 0$ has roots α, β and γ. Evaluate

(i) $\alpha\beta + \beta\gamma + \alpha\gamma$ 1

(ii) $\alpha\beta\gamma$ 1

The equation $2\cos^3\theta - \cos^2\theta + \cos\theta - 1 = 0$ has roots $\cos a$, $\cos b$ and $\cos c$.

Using appropriate information from parts (i) and (ii), prove that 2

$\sec a + \sec b + \sec c = 1$.

c) (i) Sketch the curve $y = 2\cos x - 1$ for $-\pi \le x \le \pi$. Mark clearly where the graph crosses each axis. 2

(ii) Find the volume generated by the rotation through a complete revolution about the x axis of the region between the x-axis and that part of the curve $y = 2\cos x - 1$ for which $|x| \le \pi$ and $y \ge 0$ 3

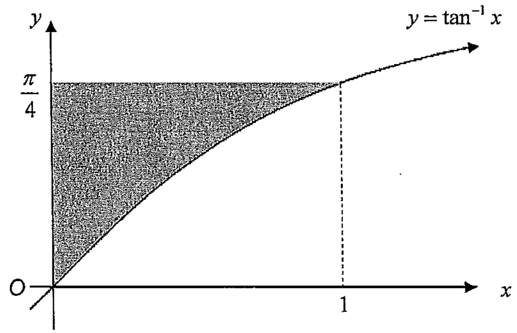
End of Question 5

Start a new booklet.

Question 6 (12 marks). **Marks**

a) (i) Find $\dfrac{d}{dy}(\ln\cos y)$. 1

(ii)



Show that the shaded area is given by $A = \frac{1}{2}\ln 2$ units^2 3

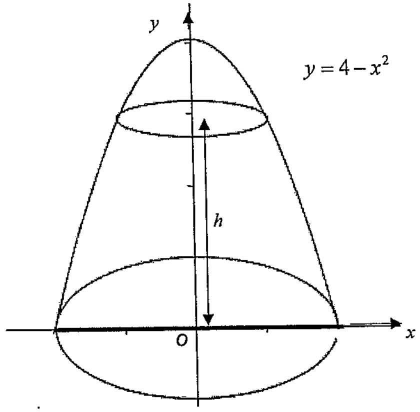
b) P, Q, R and S are four points taken in order on a circle. Prove that: 3

$$\frac{PR}{QS} = \frac{\sin P\hat{Q}R}{\sin Q\hat{P}S}$$

Question 6 continued next page.

Question 6 continued

c)



A mould for a container is made by rotating the part of the curve $y=4-x^2$ which lies in the first quadrant through one complete revolution about the y-axis. After sealing the base of the container, water is poured through a hole in the top. When the depth of water in the container is h cm, the depth is changing at a rate of $\dfrac{10}{\pi(4-h)}$ cms^{-1}.

(i) Show that when the depth is h cm, the surface area S cm^2 of the top of the water is given by $S=\pi(4-h)$. 2

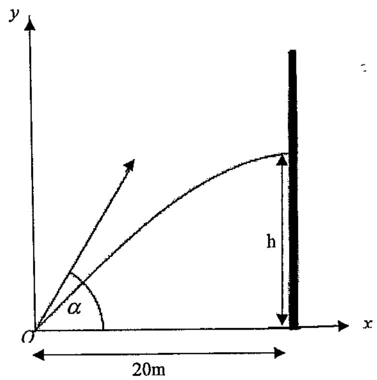
(ii) Find the rate at which the surface area of the water is changing when the depth of the water is 2cm. 3

End of Question 6.

Start a new booklet.

Question 7 (12 marks). **Marks**

a) A softball player hits the ball from ground level with a speed of 20 m/s and an angle of elevation α . It flies toward a high wall 20m away on level ground. Taking the origin at the point where the ball is hit, the derived expressions for the horizontal and vertical components of x and y of displacement at the time t seconds, taking $g=10\text{m/s}^2$, are $x=20t\cos\alpha$ and $y=-5t^2+20t\sin\alpha$



(i) Hence find the equation of the path of the ball in flight in terms of x, y and α. 1

(ii) Show that the height h at which the ball hits the wall is given by $h=20\tan\alpha-5(1+\tan^2\alpha)$ 1

(iii) Using part (ii) above, show that the maximum value of h occurs when $\tan\alpha=2$ and find this maximum height 2

Question 7 continued next page.

Question 7 continued

b) A particle of unit mass moves in a straight line. It is placed at the origin on the x-axis and is then released from rest. When at position x, its acceleration is given by:

$$-9x+\frac{5}{(2-x)^2}.$$

Prove that the particle ultimately moves between two points on the x-axis and find these points. 3

c) (i) For any angles α and β show that 1

$$\tan\alpha+\tan\beta=\tan(\alpha+\beta)[1-\tan\alpha\tan\beta]$$

(ii) Prove, by mathematical induction, that

$$\tan\theta\tan 2\theta+\tan 2\theta\tan 3\theta+\ldots+\tan n\theta\tan(n+1)\theta=\tan(n+1)\theta\cot\theta-(n+1)$$ 4

for all positive integers n

End of Question 7.

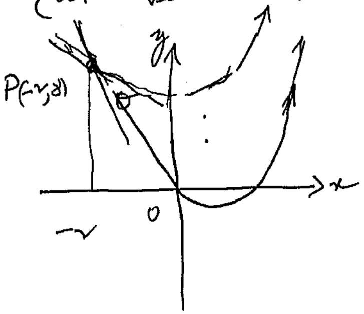
End of Examination.

QUESTION 1. (X1)

(a) Find the intersections let $x^2+4 = x^2-2x$.

$2x = -4$

$x = -2$.



now $y = x^2+4$

$y' = 2x$

$\therefore m_1 = -4$

$y = x^2-2x$

$y' = 2x-2$

$m_2 = -6$.

$\tan\theta = \left|\frac{m_1 - m_2}{1+m_1 m_2}\right|$

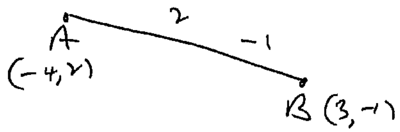
$= \left|\frac{-4--6}{1+-4\times-6}\right|$

$= \left|\frac{-4+6}{1+24}\right|$

$= \frac{2}{25}$

$\therefore \boxed{\theta = \tan^{-1}\frac{2}{25} = 4^{0}34'}$

(b)



$P = \left(\frac{2\times3+-1\times-4}{2+-1}, \frac{2\times-1+-1\times2}{2+-1}\right)$

$= \boxed{(10, -4)}$

(c) $y = \ln(\sin^{-1}x)$

$y' = \frac{\frac{1}{\sqrt{1-x^2}}}{\sin^{-1}x} = \boxed{\frac{1}{\sin^{-1}x\,\sqrt{1-x^2}}}$

(d) $\frac{x-1}{x+3} \geq -2$

$\frac{x-1}{x+3} + 2 \geq 0$

$\frac{x-1+2(x+3)}{x+3} \geq 0$

$\frac{x-1+2x+6}{x+3} \geq 0$

$\frac{3x+5}{x+3} \geq 0$

$\frac{(3x+5)}{(x+3)} \times (x+3)^2 \geq 0$

$(3x+5)(x+3) \geq 0$

$-3 \quad -\frac{5}{3}$

$\therefore \boxed{x < -3,\ x \geq -\frac{5}{3}}$

(e)

$\cos 2B = 1-2\sin^2 B$

$= 1-2\times\left(\frac{1}{3}\right)^2$

$= 1-\frac{2}{9}$

$= \frac{7}{9}$

$\therefore \cos 2B = \frac{7}{9} = \cos A$.

$\therefore \boxed{A = 2B}$

[NB doing this question on a calculator is not a proof]

(f) $u = t+1$

$du = dt$

$\therefore \int_0^1 \frac{t}{\sqrt{1+t}}\,dt = \int_1^2 \frac{u-1}{\sqrt{u}}\,du$

$= \int_1^2 \left(u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right) du$

$= \left[\frac{2}{3}u^{3/2} - 2u^{\frac{1}{2}}\right]_1^2 = \frac{2}{3}\cdot 2^{3/2} - 2\cdot 2^{\frac{1}{2}} - \left(\frac{2}{3} - 2\right)$

$= \frac{4\sqrt{2}}{3} - 2\sqrt{2} - \frac{-4}{3}$

$= \frac{4}{3} - \frac{2\sqrt{2}}{3}$

$= \boxed{\frac{4-2\sqrt{2}}{3}}$

Question 2

a) $P(x) = ax^3 + bx^2 - 8x + 3$

$P(1) = 0$

$\therefore 0 = a + b - 8 + 3$

$\boxed{a + b = 5}$ ①

$P(-2) = 15$

$15 = -8a + 4b + 16 + 3$

$\boxed{8a - 4b = 4}$ ②

① × 4

$4a + 4b = 20$ ③

② + ③

$12a = 24$

$\boxed{a = 2}$ 1 mark

sub into ①

$2 + b = 5$

$\boxed{b = 3}$ 1 mark

$\therefore P(x) = 2x^3 + 3x^2 - 8x + 3$

$$\begin{array}{r} 2x^2 + 5x - 3 \\ x - 1 \overline{)\, 2x^3 + 3x^2 - 8x + 3} \\ 2x^3 - 2x^2 \\ \hline 5x^2 - 8x + 3 \\ 5x^2 - 5x \\ \hline -3x + 3 \\ -3x + 3 \\ \hline 0 \end{array}$$

$2x^2 + 5x - 3$

$= \frac{(2x + 6)(2x - 1)}{2}$

$= \frac{2(x+3)(2x-1)}{\cancel{2}}$

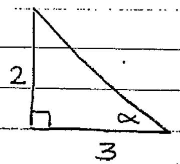
$\therefore P(x) = (x-1)(x+3)(2x-1)$ 1 mark

b) i $3\sin\theta + 2\cos\theta$

$= R\sin(\theta + \alpha)$

$R = \sqrt{3^2 + 2^2}$

$= \sqrt{13}$ 1 mark



$\tan\alpha = \frac{2}{3}$

$\alpha = \tan^{-1} 2/3$

$= 33°41'$ 1 mark

ii $\sqrt{13}\sin(\theta + 33°41') = \frac{5}{2}$

$\sin(\theta + 33°41') = \frac{5}{2\sqrt{13}}$

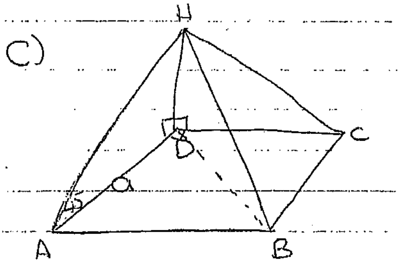
$\theta + 33°41' = \sin^{-1}\frac{5}{2\sqrt{13}}$

$\theta = \sin^{-1}\frac{5}{2\sqrt{3}} - 33°41'$

$\theta = 10°13', 102°25'$

1 mark each

c)



i) $\angle AHD = 45°$

$\therefore \triangle AHD$ is isosceles

$\therefore HD = a$

In $\triangle$'s HCD + DBA

$HD = a = DA$

$\angle HDC = 90° = \angle DAB$

(given & properties of a rectangle)

$DC = AB$ (opposite sides of a rectangle)

$\therefore \triangle HCD \equiv \triangle DBA$ (SAS)

$\therefore \angle DBA = 30°$ 1 mark

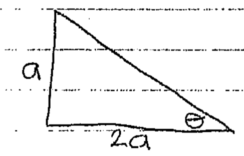
In $\triangle DAB$

$\sin 30 = \frac{a}{BD}$

$\frac{1}{2} = a/BD$

$\therefore BD = 2a$ 1 mark

ii) $\triangle HDB$



$\tan\theta = a/2a$

$\tan\theta = \frac{1}{2}$

$\theta = 26°33'54.18''$

$= 26°34'$ (nearest min) 1 mark

d) $a_1 = a - \frac{f(a)}{f'(a)}$

$f(x) = 2x + \cos x$

$f'(x) = 2 - \sin x$

$a = -\pi/6$

$f(a) = \frac{-\pi}{3} + \frac{\sqrt{3}}{2}$

$f'(a) = 2 - -\frac{1}{2}$

$= 5/2$

$a_1 = -\pi/6 - \left[\frac{-2\pi + 3\sqrt{3}}{6}\right] \div \frac{5}{2}$ 1 mark

$= \frac{-\pi}{6} - \left[\frac{-4\pi + 6\sqrt{3}}{30}\right]$

$= -\frac{5\pi}{30} + \frac{4\pi - 6\sqrt{3}}{30}$

$= \frac{-\pi - 6\sqrt{3}}{30}$ 1 mark

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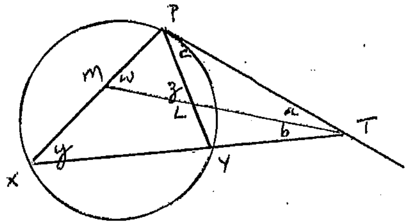
Question 3:



$a = b$ (TM bisects $\angle PTX$, given)

$c = y$ (alternate segment theorem)

$z = a + c$ (exterior $\angle$ of ΔPLT)

$\quad = b + y$

$w = b + y$ (exterior $\angle$ of Δ MTX)

$\therefore z = w$

$\therefore \Delta PLM$ is isosceles (base angles equal) (3)

(b) (i) $f^{-1}(x) = \sin^{-1}(3x-1)$

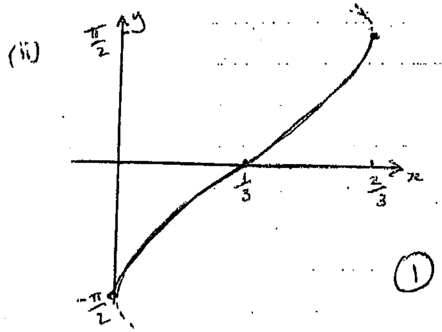
Domain: $-1 \le 3x - 1 \le 1$

$0 \le 3x \le 2$

$0 \le x \le \frac{2}{3}$ (2)

Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

(ii)



(1)

(iii) $x = \sin^{-1}(3y-1)$

$\sin x = 3y - 1$

$3y = \sin x + 1$

$y = \frac{1}{3}(\sin x + 1)$

$f(x) = \frac{1}{3}(\sin x + 1)$

Domain: $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

Range: $0 \le y \le \frac{2}{3}$ (3)

(c) (i) $\frac{dT}{dt} = \frac{d}{dt}\left(T_0 + Ae^{-kt}\right)$

$= Ae^{-kt} \times -k$

$= (T - T_0) \times -k$

$= -k(T - T_0)$ (1)

(ii) When $t = 0$: $T = 85$

$\therefore 85 = 25 + A$

$\therefore A = 60$

$\therefore T = 25 + 60e^{-kt}$

When $t = 1$: $80 = 25 + 60e^{-k}$

$\therefore 55 = 60e^{-k}$

$\therefore e^{-k} = \frac{55}{60}$

$k = -\ln\left(\frac{55}{60}\right)$

When $t = 5$: $T = 25 + 60e^{-5k}$

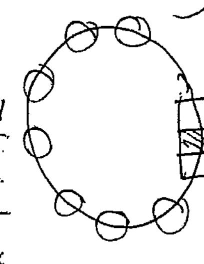
$= 63.8336\ldots$

$\approx 64°$ (2)

12

Solution to Q(4)

(a) (i)



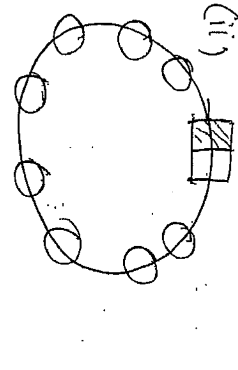
$7! \times 1 \times 2!$ [1]

the rest — particular girl. — 2 boys

$= \boxed{2 \times 7!}$

$= \boxed{10080}$

(ii)



Sitting together $= 8! \times 2!$

$\therefore$ separate to $= \boxed{9! - 8! \times 2!}$

$= \boxed{282240}$ [1]

(b) $y = x^2/4a$

(i) $\frac{dy}{dx} = \frac{x}{2a}$ [2]

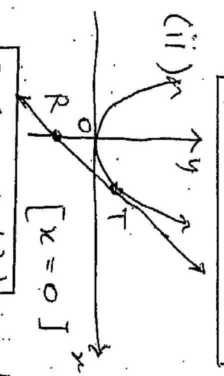
$\left.\frac{dy}{dx}\right|_{x = 2at} = \frac{2at}{2a} = t$ ✓

$\therefore$ Equ. of tgt

$y - at^2 = t(x - 2at)$

$\boxed{y = tx - at^2}$ ✓

(ii)



$[x = 0]$

$\boxed{R(0, -at^2)}$ [1]

If PQ passes through R then Coordinates of R satisfy equation of PQ

i.e. $-at^2 + apq = 0$ ✓

$t^2 = pq$

i.e. $\boxed{\frac{t}{p} = \frac{q}{t}}$ ✓ [2.]

$\therefore p, t, q$ are terms of a geometric series

(c) $x = 2\cos 3t$, $\dot{x} = -6\sin 3t$ ✓

$\ddot{x} = -9(2\cos 3t) = -9x$ ✓

(i) $\therefore n^2 = 9$.

(ii) $T = \frac{2\pi}{n} = \boxed{\frac{2\pi}{3}}$ [1]

(iii) $\cos 3t = \frac{1}{2}$ ✓, $3t = \frac{\pi}{3}, \frac{5\pi}{3}$

$\Rightarrow t = \frac{\pi}{9}, \frac{5\pi}{9}$ ✓

$\therefore \dot{x} = -6\sin\frac{\pi}{3}$, ✓ $-6\sin\frac{5\pi}{3}$.

$= \boxed{\pm 3\sqrt{3}\ \text{cm/sec}}$

$\doteqdot \pm 5.196$ cm/se [2]

QUESTION 5.

(a) $\frac{\sin\theta}{\cos\theta} = 2\sin\theta\cos\theta$

$\sin\theta = 2\sin\theta\cos^2\theta$.

$2\sin\theta(1-\sin^2\theta) - \sin\theta = 0$.

$2\sin^3\theta - \sin\theta = 0$.

$\sin\theta(2\sin^2\theta - 1) = 0$.

$\sin\theta = 0$ $\qquad$ $\sin\theta = \pm\frac{1}{\sqrt{2}}$

$\theta = \pi n \quad n \in \mathbb{Z}$ $\qquad$ $\theta = \pi n \pm \frac{\pi}{4} \quad n \in \mathbb{Z}$

(b) (i) $\alpha\beta + \beta\gamma + \alpha\gamma = \frac{1}{2}$.

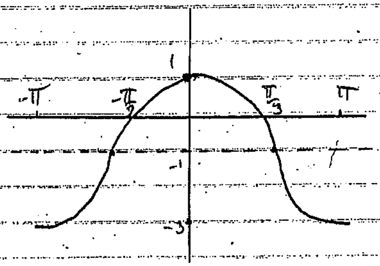
(ii) $\alpha\beta\gamma = \frac{1}{2}$.

$\frac{1}{\cos a} + \frac{1}{\cos b} + \frac{1}{\cos c} = \frac{\cos a\cos b + \cos a\cos c + \cos b\cos c}{\cos a\cos b\cos c}$.

$= \frac{\frac{1}{2}}{\frac{1}{2}}$

$= 1$

(c) (i)



$2\cos x - 1 = 0$

$\cos x = \frac{1}{2}$

$x = \pm\frac{\pi}{3}$.

(ii) $V = \pi\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 4\cos^2 x - 4\cos x + 1\, dx$.

$= \pi\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 2\cos 2x + 2 - 4\cos x + 1\, dx$.

$= \pi\left[\sin 2x - 4\sin x + 3x\right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$.

$= 2\pi\left(\sin\frac{2\pi}{3} - 4\sin\frac{\pi}{3} + \pi\right)$

$= 2\pi^2 - 3\pi\sqrt{3}$. units^3.

Question 6

a) i) $\frac{d}{dy}(\ln\cos y) = \frac{-\sin y}{\cos y}$

$= \underline{\underline{-\tan y}}$

ii) $A = \int_0^{\frac{\pi}{4}} x \, dy$

$= \int_0^{\frac{\pi}{4}} \tan y \, dy$

$= -\int_0^{\frac{\pi}{4}} -\tan y \, dy$

$= -\left[\ln(\cos y)\right]_0^{\frac{\pi}{4}}$ using (i)

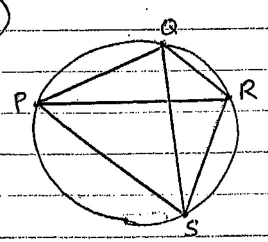
$= -\left(\ln\left(\cos\frac{\pi}{4}\right) - \ln(\cos 0)\right)$

$= -\ln\left(\frac{1}{\sqrt{2}}\right) + \cancel{\ln 1}$

$= -\ln(2)^{-\frac{1}{2}}$

$= \frac{1}{2}\ln 2$ units^2

b)



In ΔPQR

$\frac{\sin P\hat{Q}R}{PR} = \frac{\sin Q\hat{R}P}{PQ}$

In ΔQPS

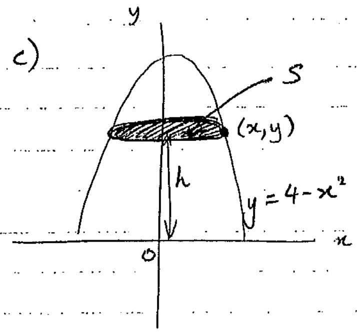
$\frac{\sin Q\hat{P}S}{QS} = \frac{\sin P\hat{S}Q}{PQ}$

$Q\hat{R}P = P\hat{S}Q$ (angles in same segment (arc PQ))

$\therefore \frac{\sin P\hat{Q}R}{PR} = \frac{\sin Q\hat{P}S}{QS}$

$\frac{\sin P\hat{Q}R}{\sin Q\hat{P}S} = \frac{PR}{QS}$

c)



i) $S = \pi r^2$

$S = \pi x^2$

when $y = h$

$h = 4 - x^2$

$x^2 = 4 - h$

$\therefore S = \pi(4-h)$

ii) $S = 4\pi - \pi h$

$\frac{dS}{dh} = -\pi$

$\frac{dS}{dt} = \frac{dS}{dh} \times \frac{dh}{dt}$

$= -\pi \times \frac{10}{\pi(4-h)}$

$= \frac{-10}{4-h}$

when $h = 2$

$\frac{dS}{dt} = \frac{-10}{(4-2)}$

$= \underline{\underline{-5}}$ cm^2/s

QUESTION 7

(a) $x = 20t\cos\alpha$

$y = -5t^2 + 20t\sin\alpha$

(i)

$\Rightarrow y = -5\left(\frac{x}{20\cos\alpha}\right)^2 + 20\left(\frac{x}{20\cos\alpha}\right)\sin\alpha$

$y = -\frac{1}{80}x^2\sec^2\alpha + x\tan\alpha$

ie $y = -\frac{1}{80}(\tan^2\alpha+1)x^2 + (\tan\alpha)x$

(ii) When $x = 20$, $y = h$

$\Rightarrow h = -\frac{1}{80}(\tan^2\alpha+1)400 + 20\tan\alpha$

ie $h = -5\tan^2\alpha + 20\tan\alpha - 5$

$\boxed{h = 20\tan\alpha - 5(1+\tan^2\alpha)}$

(iii)

$h = -5\tan^2\alpha + 20\tan\alpha - 5$

Max. value of h occurs when $\tan\alpha = \frac{-20}{2(-5)} = 2$

ie $\boxed{\tan\alpha = 2}$

Max height is

$-5(2)^2 + 20(2) - 5 = \boxed{15 \text{ metres}}$

(b) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -9x + 5(x-2)^{-2}$

$\frac{1}{2}v^2 = -\frac{9x^2}{2} + \frac{5}{2-x} + C$

$\left.\begin{matrix} x=0 \\ v=0 \end{matrix}\right\} \Rightarrow C = -\frac{5}{2}$

$\therefore v^2 = -9x^2 + \frac{5}{2-x} - 5$

(b)(dd) $v^2 = -9x^2 + \frac{10}{2-x} - 5$

For motion to exist then

$v^2 \geq 0$

ie $-9x^2 + \frac{10}{2-x} - 5 \geq 0$

$-9x^2(2-x)^2 + 10(2-x) - 5(2-x)^2 \geq 0$

$(2-x)\left[-9x^2(2-x) + 10 - 5(2-x)\right] \geq 0$

ie $(2-x)(-18x^2 + 9x^3 + 5x) \geq 0$

ie $(2-x).x(9x^2 - 18x + 5) \geq 0$

$x(2-x)(3x-5)(3x-1) \geq 0$

SOLUTION

$0 \leq x \leq \frac{1}{3}$ $\qquad$ $\frac{5}{3} \leq x \leq 2$

However since particle starts at zero and changes direction at $x = \frac{1}{3}$ it can never be outside the interval $0 \leq x \leq \frac{1}{3}$.

Note For $\frac{1}{3} < x < \frac{5}{3}$ $v^2 < 0$

$\therefore$ impossible to move in this interval and therefore cannot move in $\frac{5}{3} \leq x \leq 2$.

Ultimately moves in interval

$0 \leq x \leq \frac{1}{3}$

7 (c) (i) RHS $= \tan(\alpha+\beta)\left[1 - \tan\alpha\tan\beta\right]$

$= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}\left[1 - \tan\alpha\tan\beta\right]$

$= \tan\alpha + \tan\beta$

$=$ LHS

(ii) When $n=1$ $\quad \tan\theta.\tan 2\theta = \tan 2\theta\cot\theta - 2$

RHS $= \frac{2\tan\theta}{1-\tan^2\theta}.\frac{1}{\tan\theta} - 2 = \frac{2\tan^2\theta}{1-\tan^2\theta} = \tan\theta.\frac{2\tan\theta}{1-\tan^2\theta}$

$= \tan\theta.\tan 2\theta =$ LHS

- Assume $\tan\theta\tan 2\theta + \dots + \tan k\theta\tan(k+1)\theta = \tan(k+1)\theta\cot\theta - (k+1)$

RTP $\tan\theta\tan 2\theta + \dots + \tan k\theta\tan(k+1)\theta + \tan(k+1)\theta\tan(k+2)\theta = \tan(k+2)\theta\cot\theta - (k+2)$

Now LHS $= \tan(k+1)\theta.\cot\theta - (k+1) + \tan(k+1)\theta\tan(k+2)\theta$

$= \cot\theta\left[\tan(k+1)\theta + \tan(k+1)\theta.\tan(k+2)\theta.\tan\theta\right] - (k+1)$

$= \cot\theta\left[\tan(k+1)\theta + \tan(k+2)\theta\left(1 - \frac{\tan(k+1)\theta + \tan\theta}{\tan(k+2)\theta}\right)\right] - (k+1)$

$= \cot\theta\left[\tan(k+1)\theta + \tan(k+2)\theta - \tan(k+1)\theta - \tan\theta\right] - (k+1)$ using (i)

$= \cot\theta\left[\tan(k+2)\theta - \tan\theta\right] - (k+1)$

$= \cot\theta\left[\tan(k+2)\theta\right] - 1 - (k+1)$

$= \cot\theta\left[\tan(k+2)\theta\right] - (k+2) =$ RHS