SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2009

TRIAL HIGHER SCHOOL
CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instruction

- Reading Time – 5 Minutes
- Working time – 180 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may **NOT** be awarded for messy or badly arranged work.
- Start each question in a separate answer booklet.
- Answer in simplest exact form unless otherwise instructed.

Total Marks – 120

- Attempt questions 1-8
- All questions are of equal value

Examiner: *C. Kourtesis*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\ x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\ x > 0$

Question 1. (15 marks) Marks

(a) Find: (i) $\int \frac{1}{\sqrt{x+8}}\,dx$ 3

(ii) $\int \frac{1}{x^2+9}\,dx$

(b) Use integration by parts to find 3

$$\int x \ln x$$

(c) Use completion of squares to find 2

$$\int \frac{dx}{\sqrt{6-x-x^2}}$$

(d) i) Find real numbers a, b and c such that $\frac{1}{x^2(2-x)} = \frac{ax+b}{x^2} + \frac{c}{2-x}$ 4

ii) Hence evaluate $\int_1^{1.5} \frac{dx}{x^2(2-x)}$

(e) Use the substitution $x = \tan y$ to show that 3

$$\int_0^1 \frac{dx}{(x^2+1)^2} = \frac{\pi+2}{8}$$

Question 2. (15 marks) Marks

(a) If k is a real number and $z = k - 2i$ express $\overline{(iz)}$ in the form $x+iy$ where x and y are real numbers. 2

(b) Solve the equation 2

$$\bar{z} = 3z - 1$$

where $z = x + iy$ (x, y real)

(c) On an Argand diagram shade the region specified by both the conditions 3

$$\text{Im}(z) \le 4 \text{ and } |z-4-5i| \le 3$$

(d) If $\text{cis}\,\theta = \cos\theta + i\sin\theta$, express 2

$$(4\,\text{cis}\,\alpha)^2(2\,\text{cis}\,\beta)^3$$

in modulus-argument form.

(e) ~~i) If $w = \frac{z}{z+2}$ where $z = x+iy$ (x, y real) find the locus of w given that it is purely imaginary.~~ 4

~~ii) Sketch the locus of w on an Argand diagram.~~

? (f) If α and β are real show that $(\alpha+\beta i)^{2002} + (\beta-\alpha i)^{2002} = 0$. 2

Question 3. (15 marks) Marks

(a) Consider the function 8

$$f(x)=\frac{x^3}{(1-x)^2}$$

i) Show that $f'(x)=\dfrac{x^2[3-x]}{(1-x)^3}$

ii) Use the first derivative $f'(x)$ to determine the nature of the stationary points.

iii) Write down the equations of any asymptotes.

iv) Sketch the graph of $y=f(x)$ showing all essential features.

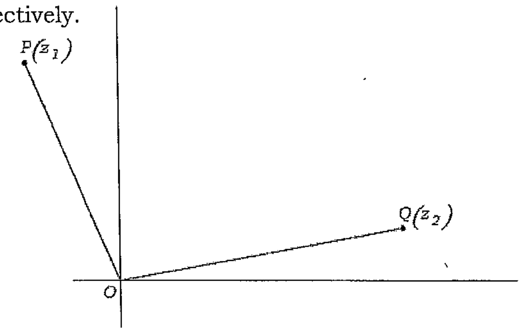
(b) 4

i) Sketch the graphs of $y=\sin x$ and $y=\sqrt{\sin x}$ for $0\le x\le\frac{\pi}{2}$ on the same diagram.

ii) Hence show that $1<\int_0^{\frac{\pi}{2}}\sqrt{\sin x}\,dx<\frac{\pi}{2}$

NOTE: You are NOT required to evaluate the integral $\int_0^{\frac{\pi}{2}}\sqrt{\sin x}\,dx$

(c) In the diagram below points P and Q represent the complex numbers z_1 and z_2 respectively. 3



i) Copy the diagram in your examination booklet and indicate the point representing the complex number z_1+z_2

ii) If the length of PQ is $|z_1-z_2|$ and $|z_1-z_2|=|z_1+z_2|$ what can be said about $\dfrac{z_2}{z_1}$

Question 4. (15 marks) Marks

(a) The real cubic polynomial $ax^3+9x^2+ax=30$ has $-3+i$ as a root. 4

i) Show that $x^2+6x+10$ is a quadratic factor of the cubic polynomial.

ii) Show that $a=2$.

iii) Write down all the roots of the polynomial.

(b) Show that the polynomial $P(x)=nx^{n+1}-(n+1)x^n+1$ is divisible by $(x-1)^2$ 2

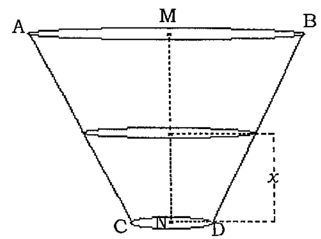
(c) 4

i) Sketch the graphs of $y=\dfrac{1}{x^2+1}$ and $y=\dfrac{1}{x^2+2}$ on the same set of axes.

ii) The area bounded by the two curves in (i) and the ordinates at $x=0$ and $x=2$ is rotated about the y-axis. Use the cylindrical shell method to show that the volume of the resulting solid is

$$\pi\ln\frac{5}{3}.$$

(d) A drinking glass is in the shape of a truncated cone, in which the internal diameter of the top and bottom are 8cm and 6cm respectively. 5



AB = 8cm
CD = 6cm

i) If the internal height of the glass, MN, is 10cm show that the area of the cross-section x cm above the base is

$$\pi\left(3+\frac{x}{10}\right)^2 \text{cm}^2.$$

ii) Hence find by integration, the volume of liquid the glass can hold (answer to the nearest mL).

Question 5. (15 marks)

Marks

The equation of an ellipse E is given by $\frac{x^2}{9}+\frac{y^2}{5}=1$

i) Find the eccentricity of E — 1

ii) Write down the — 3

α) coordinates of the foci

β) equations of the directrices

γ) equation of the major auxiliary circle A.

iii) Draw a neat sketch of E showing clearly the features in part ii) — 2

iv) A line parallel to the positive y-axis meets the x-axis at N and the curves E, A at P and Q respectively. If N has coordinates $(3\cos\theta, 0)$ find the coordinates of P and Q. [P and Q are in the first quadrant] — 2

v) Show that the equations of the tangents at P and Q are $\sqrt{5}x\cos\theta+3y\sin\theta=3\sqrt{5}$ and $x\cos\theta+y\sin\theta=3$ respectively. — 4

vi) Show that the point of intersection R of these tangents lies on the major axis of E produced. — 1

vii) Prove that $ON \cdot OR$ is independent of the position of P and Q on the curves. — 2

Question 6. (15 marks)

Marks

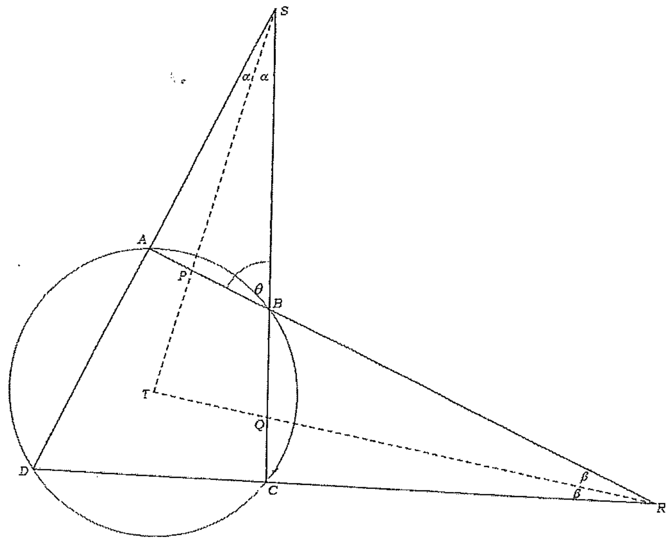
(a) i) A particle of mass m falls vertically from rest, from a point o, in a medium whose resistance is mkv, where k is a positive constant and v its velocity after t seconds. — 4

Show that $v=\frac{g}{k}\left(1-e^{-kt}\right)$

ii) An equal particle is projected vertically upwards with initial velocity U in the same medium. [The particle is released simultaneously with the first particle]. — 4

Show that the velocity of the first particle when the second particle is momentarily at rest is given by $\frac{VU}{V+U}$ where V is the terminal velocity of the first particle.

(b) — 7



ABCD is a cyclic quadrilateral.

The sides *AB* and *CD* produced intersect at *R* and the sides *CB* and *DA* produced intersect at *S*. *ST* and *RT* intersect *AR* and *CS* at *P* and *Q* respectively.

The bisectors of $C\hat{S}D$ and $A\hat{R}D$ meet at *T*.

Let $A\hat{S}T=B\hat{S}T=\alpha$ and $A\hat{R}T=D\hat{R}T=\beta$ and $A\hat{B}S=\theta$.

i) Show that $T\hat{P}B+T\hat{Q}B=\alpha+\beta+2\theta$

ii) Prove that *ST* is perpendicular to *RT*.

Question 7. (15 marks) Marks

(a) Given that $\tan 5\theta = \dfrac{t^5 - 10t^3 + 5t}{5t^4 - 10t^2 + 1}$, where $t = \tan\theta$ [Do not prove this] 5

i) Solve the equation $\tan 5\theta = 0$ for $0 \le \theta \le \pi$

ii) Hence prove that

α) $\tan\dfrac{\pi}{5}\tan\dfrac{2\pi}{5} = \sqrt{5}$

β) $\tan^2\dfrac{\pi}{5} + \tan^2\dfrac{2\pi}{5} = 10$

(b) i) Show that $\int x\tan^{-1}x\,dx = \dfrac{1}{2}(x^2+1)\tan^{-1}x - \dfrac{1}{2}x + c$ 4

ii) If $u_n = \int_0^1 x^n \tan^{-1}x\,dx$ for $n \ge 2$ show that

$$u_n = \frac{\pi}{2(n+1)} - \frac{1}{n(n+1)} - \frac{n-1}{n+1}u_{n-2}$$

(c) Show that the number of ways in which $2n$ persons may be seated at two round tables, n persons being seated at each is 2

$$\frac{(2n)!}{n^2}$$

(d) i) There are 6 persons from whom a game of tennis is to be made up, two on each side. How many different matches can be arranged if a change in either pair gives a different match? 4

ii) How many different matches are possible if two particular persons are to both play in the match?

Question 8. (15 marks) Marks

(a) Suppose a, b, c and d are positive real numbers. 5

i) Prove that $\dfrac{a}{b} + \dfrac{b}{a} \ge 2$.

ii) Deduce that $\dfrac{a+b+c}{d} + \dfrac{b+c+d}{a} + \dfrac{c+d+a}{b} + \dfrac{d+a+b}{c} \ge 12$.

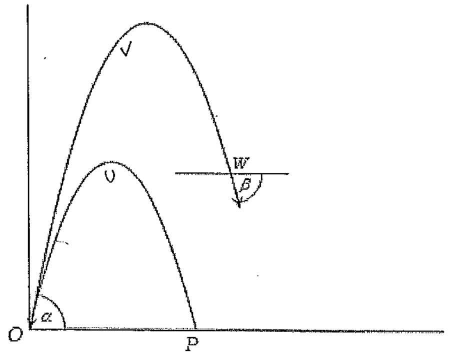
iii) Hence prove that if $a + b + c + d = 1$, then:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \ge 16.$$

(b) Two stones are thrown simultaneously from the same point O in the same direction and with the same non-zero angle of projection α, but with different velocities U and V ($U < V$). 6

The slower stone hits the ground at a point P on the same level as the point of projection.

At that instant the faster stone is at a point W on its downward path, making an angle β with the horizontal.



i) Show that $V(\tan\alpha + \tan\beta) = 2U\tan\alpha$

ii) Deduce that if $\beta = \dfrac{1}{2}\alpha$ then $U < \dfrac{3}{4}V$

(c) i) Show by graphical means that 4

$$\ln ex > e^{-x} \text{ for } x \ge 1$$

ii) Hence, or otherwise, show that

$$\ln(n!e^n) > e^{-n}\left(\frac{e^n - 1}{e - 1}\right)$$

SBHS - 2009 TRIAL HSC SOLUTIONS

1) a) i) $\displaystyle\int \frac{1}{\sqrt{x+8}}\,dx = \int (x+8)^{-\frac{1}{2}}\,dx$

$\displaystyle = \frac{(x+8)^{\frac{1}{2}}}{\frac{1}{2}\cdot 1} + C$

$= 2\sqrt{x+8} + C$

ii) $\displaystyle\int \frac{1}{x^2+9}\,dx = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$

$a=3$

b) $\displaystyle\int x\ln x\,dx = \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx$

$u = \ln x \quad v' = x$

$u' = \frac{1}{x} \quad v = \frac{x^2}{2}$

$\displaystyle = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$

c) $\displaystyle\int \frac{dx}{\sqrt{6-x-x^2}} = \int \frac{dx}{\sqrt{-(x^2+x+\frac{1}{4})+\frac{25}{4}}}$

$\displaystyle = \int \frac{dx}{\sqrt{\frac{25}{4}-(x+\frac{1}{2})^2}}$

$a = \frac{5}{2}$

OR make a substitution

$\displaystyle = \sin^{-1}\left(\frac{(x+\frac{1}{2})}{(\frac{5}{2})}\right) + C$

$\displaystyle = \sin^{-1}\left(\frac{2x+1}{5}\right) + C$

d) i) $\displaystyle\frac{1}{x^2(2-x)} \equiv \frac{ax+b}{x^2} + \frac{c}{2-x}$

$1 \equiv (ax+b)(2-x) + cx^2$

let $x=2$

$1 = c(2)^2$

$c = \frac{1}{4}$

let $x=0$

$1 = b.2$

$b = \frac{1}{2}$

equate coefficients of x^2

$0 = -a + c$

$a = c$

$\therefore a = \frac{1}{4}$

$a = \frac{1}{4},\ b = \frac{1}{2},\ c = \frac{1}{4}$

ii) $\displaystyle\int_1^{1.5} \frac{dx}{x^2(2-x)} = \int_1^{1.5}\left(\frac{\frac{1}{4}x+\frac{1}{2}}{x^2} + \frac{\frac{1}{4}}{2-x}\right)dx$

$\displaystyle = \int_1^{1.5}\left(\frac{1}{4}\cdot\frac{1}{x} + \frac{1}{2}x^{-2} - \frac{1}{4}\cdot\frac{-1}{2-x}\right)dx$

$\displaystyle = \left[\frac{1}{4}\ln x - \frac{1}{2}x^{-1} - \frac{1}{4}\ln(2-x)\right]_1^{1.5}$

$\displaystyle = \left[-\frac{1}{2x} + \frac{1}{4}\ln\left(\frac{x}{2-x}\right)\right]_1^{1.5}$

$\displaystyle = -\frac{1}{2(1.5)} + \frac{1}{4}\ln\left(\frac{1.5}{2-1.5}\right) - \left(-\frac{1}{2(1)} + \overset{0}{\cancel{\frac{1}{4}\ln\left(\frac{1}{2-1}\right)}}\right)$

$\displaystyle = -\frac{1}{3} + \frac{1}{4}\ln 3 + \frac{1}{2}$

$\displaystyle = \frac{1}{4}\ln 3 + \frac{1}{6}$

e) $\displaystyle\int_0^1 \frac{dx}{(x^2+1)^2}$

$x = \tan y$

$\frac{dx}{dy} = \sec^2 y$

$dx = \sec^2 y.dy$

when $x=1$, $y = \frac{\pi}{4}$; $x=0$, $y=0$

$$\int_0^1 \frac{dx}{(x^2+1)^2} = \int_0^{\frac{\pi}{4}} \frac{\sec^2 y \, dy}{(\tan^2 y+1)^2}$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sec^2 y}{(\sec^2 y)^2} \, dy$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{\sec^2 y} \, dy$$

$$= \int_0^{\frac{\pi}{4}} \cos^2 y \, dy$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{1}{2} + \frac{1}{2}\cos 2y\right) dy$$

$$= \left[\frac{1}{2}y + \frac{1}{4}\sin 2y\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left(\frac{\pi}{4}\right) + \frac{1}{4}\sin 2\left(\frac{\pi}{4}\right) - \overset{0}{\cancel{\left(\frac{1}{2}(0) + \frac{1}{4}\sin 2(0)\right)}}$$

$$= \frac{\pi}{8} + \frac{1}{4}(1)$$

$$= \frac{\pi + 2}{8}$$

Question 2

(a) $z = k - 2i$

$$\overline{iz} = \overline{i(k-2i)}$$

$$= \overline{ik + 2}$$

$$= 2 - ik$$

$x = 2, \ y = -k$

(b) $\bar{z} = 3z - 1$

$$x - iy = 3(x+iy) - 1$$

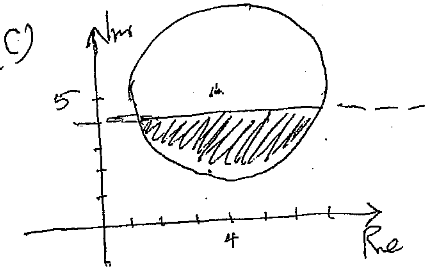
$$0 = 2x - 1 + 4iy$$

Real: $2x - 1 = 0$ Imag: $4y = 0$

$x = \frac{1}{2}$ $y = 0$

Solution $z = \frac{1}{2}$

(c)



(d) $(4\,\text{cis}\,\alpha)^2 (2\,\text{cis}\,\beta)^3$

$$= 16\,\text{cis}\,2\alpha \cdot 8\,\text{cis}\,3\beta$$

$$= 128\,\text{cis}(2\alpha + 3\beta)$$

(f) Let $z = \alpha + i\beta$

Now $-iz = \beta - i\alpha$

Thus $(\alpha + i\beta)^{2002} + (\beta - i\alpha)^{2002}$

$$= z^{2002} + (-iz)^{2002}$$

$$= z^{2002} + (-1)^{2002} \cdot i^{2002} \cdot z^{2002}$$

$$= z^{2002} - z^{2002}$$

$$= 0.$$

Q3

(a) $f(x) = \frac{x^3}{(1-x)^2}$

(i) $f'(x) = \frac{(1-x)^2 \cdot 3x^2 - x^3 \cdot 2(1-x)^1 \times -1}{(1-x)^4}$

$= \frac{3x^2(1-x) + 2x^3}{(1-x)^3}$

$= \frac{3x^2 - 3x^3 + 2x^3}{(1-x)^3}$

$= \frac{3x^2 \cancel{+2x}^3}{(1-x)^3}$

$= \frac{x^2(3-x)}{(1-x)^3}$

(ii) Let $f'(x) = 0$

i.e. $x = 0, 3$.

$\therefore\ y = 0, \frac{27}{4}$

Test $(0, 0)$.

x	-1	0	$\frac{1}{2}$
y'	$\frac{1}{2}$	0	5
	/	—	/

$\therefore$ STATIONARY INFLEXION

Test $(3, \frac{27}{4})$

x	2	3	4
y'	-4	0	$\frac{16}{27}$
	\	—	/

REL. MIN. TURNING PT.

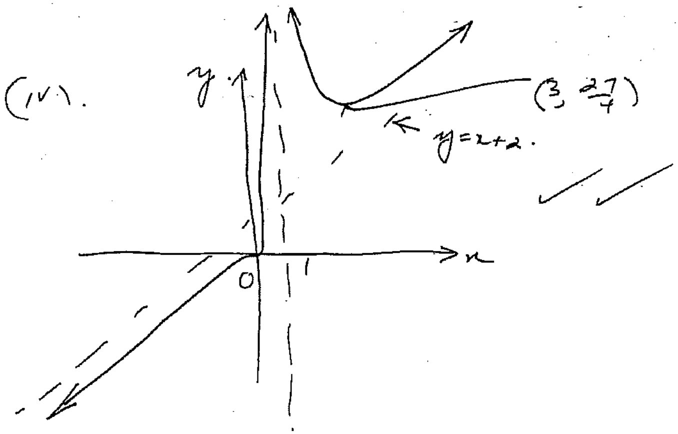
(iii) VERTICAL ASYMPTOTE at $x = 1$

now $\frac{x^3}{x^2-2x+1} = \frac{x(x^2-2x+1) + 2(x^2-2x+1) + 3x - 2}{x^2-2x+1}$

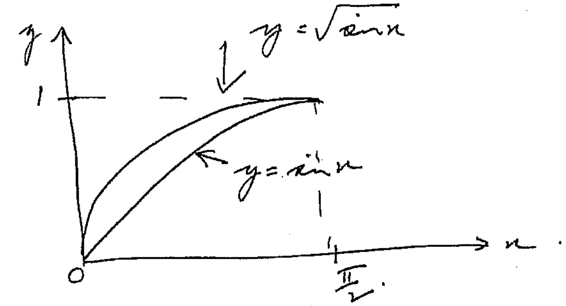
$= x + 2 + \frac{3x-2}{x^2-2x+1} \to x+2$ as $x \to \infty$.

$\therefore$ $y = x+2$ is an oblique asymptote

(iv)



(b)



now $\int_0^{\frac{\pi}{2}} \sin x\,dx < \int_0^{\frac{\pi}{2}} \sqrt{\sin x}\,dx < \frac{\pi}{2} \times 1$.

$\left[-\cos x\right]_0^{\frac{\pi}{2}} < \int_0^{\frac{\pi}{2}} \sqrt{\sin x}\,dx < \frac{\pi}{2}$.

$0 - -1 < \int_0^{\frac{\pi}{2}} \sqrt{\sin x}\,dx < \frac{\pi}{2}$

$1 < \int_0^{\frac{\pi}{2}} \sqrt{\sin x}\,dx < \frac{\pi}{2}$.

(CONTD)

(b) For $P(x)$ to be divisible by $(x-1)^2$

then $P(x)=0$ must have a multiple root of degree 2, of value 1.

now $P(1) = n - (n+1) + 1$

$= n - n - 1 + 1$

$= 0$

$P'(x) = n(n+1)x^{n} - n(n+1)x^{n-1}$

$\therefore P'(1) = n(n+1) - n(n+1)$

$= 0.$

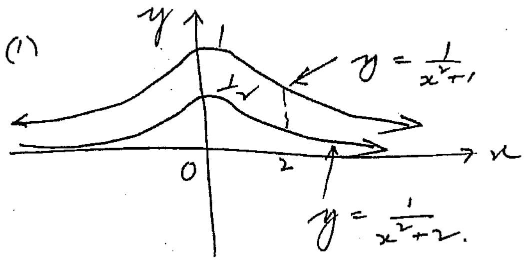
$\therefore P(1) = P'(1) = 0$

$\therefore$ by the multiple root theorem

$x = 1$ is a double root.

$\therefore (x-1)^2$ is a factor.

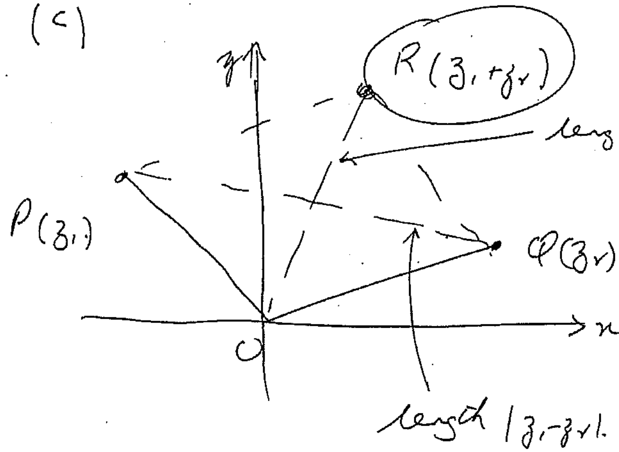
(c) (i)



(ii) $\delta V = 2\pi x\left(\frac{1}{x^2+1} - \frac{1}{x^2+2}\right)\delta x$

$V = \lim_{\delta x \to 0} \sum_{x=0}^{2} 2\pi x\left(\frac{1}{x^2+1} - \frac{1}{x^2+2}\right)\delta x$

(c)



now the diagonals of the parallelogram are equal $\therefore$ a ~~[illegible]~~ rectangle.

$\therefore z_1 = k i z_2$. or $z_2 = -k_1 i z_1$

$\frac{z_2}{z_1} = -k_1 i$

is IMAGINARY

C (CONT'D)

$$= 2\pi \int_0^2 \left(\frac{x}{x^2+1} - \frac{x}{x^2+2} \right) dx$$

$$= \pi \left[\ln(x^2+1) - \ln(x^2+2) \right]_0^2$$

$$= \pi \left[\ln 5 - \ln 6 - \ln 1 + \ln 2 \right]$$

$$= \pi \ln \frac{10}{6}$$

$$= \boxed{\pi \ln \frac{5}{3} \ u^3}$$

(d). (i) Let the radius of the cross section be r.

$\therefore$ by similarity $\frac{r-3}{x} = \frac{1}{10}$

$$r = 3 + \frac{x}{10}$$

$\therefore$ Area of the cross-section is $\pi \left(3 + \frac{x}{10}\right)^2$

$\therefore$ (ii) $V = \lim_{\delta x \to 0} \sum_{x=0}^{10} \pi \left(3 + \frac{x}{10}\right)^2 \delta x$

$$= \pi \int_0^{10} \left(3 + \frac{x}{10}\right)^2 dx$$

$$= \frac{10\pi}{3} \left[\left(3 + \frac{x}{10}\right)^3 \right]_0^{10}$$

$$= \frac{10\pi}{3} \left[4^3 - 3^3 \right]$$

$$= \frac{370\pi}{3} \text{ cc.}$$

$$\doteqdot \boxed{387 \text{ ml.}}$$

Q4. (a) (i) Given $ax^3 + 9x^2 + ax - 30 = 0$ with real co-efficients, has a root $-3+i$, it also has $-3-i$ as a root, by the conjugate root theorem.

$\therefore$ $x^2 - (-3+i+-3-i)x + (-3+i)(-3-i)$ is a factor.

i.e. $\boxed{x^2 + 6x + 10}$

(ii) now Clearly $ax^3 + 9x^2 + ax - 30 \equiv (x^2+6x+10)(ax$

now co-eff of x

LHS $= a$ RHS $= 10a - 18$.

$\therefore$ $10a - 18 = a$

$9a = 18$

$\boxed{a = 2}$

(iii) $\sum \alpha_i = -\frac{9}{2}$ $\therefore$ $-3+i+-3-i+\alpha = -\frac{9}{2}$

$$-6 + \alpha = -\frac{9}{2}$$

$$\alpha = \frac{3}{2}$$

$\therefore$ roots are $\boxed{-3 \pm i, \frac{3}{2}.}$

(b) (see next page)

[15] Solution to Question (5)

$$\frac{x^2}{9}+\frac{y^2}{5}=1$$

(i) $b^2 = a^2(1-e^2)$

[1] $5 = 9(1-e^2)$

$e^2 = 4/9 \Rightarrow e = \frac{2}{3}$

(ii) α) $(\pm ae, 0)$

$\therefore (\pm 2, 0)$.

[3] β) $x = \pm \frac{a}{e}$

$= \pm \frac{9}{2}$

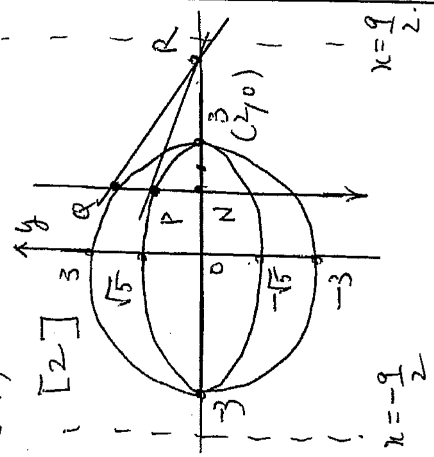
γ) $x^2+y^2=9$.

(iv) $N: (3\cos\theta, 0)$

[2] $P\ (3\cos\theta, \sqrt{5}\sin\theta)$

$Q\ (3\cos\theta, 3\sin\theta$

(iii) [2]



(v) At $P\ (3\cos\theta, \sqrt{5}\sin\theta)$

$\frac{2x}{9}+\frac{2y}{5}\frac{dy}{dx}=0$

$\therefore \frac{dy}{dx} = -\frac{5x}{9y}$

$= -\frac{15\cos\theta}{9\sqrt{5}\sin\theta}$

$= -\frac{\sqrt{5}}{3}\frac{\cos\theta}{\sin\theta}$

$\therefore y-\sqrt{5}\sin\theta = -\frac{\sqrt{5}\cos\theta}{3\sin\theta}(x-3\cos\theta)$

$(\sin\theta)\ 3y - 3\sqrt{5}\sin^2\theta = -\sqrt{5}x\cos\theta + 3\sqrt{5}\cos^2\theta$

$\therefore \sqrt{5}x\cos\theta + 3y\sin\theta = 3\sqrt{5}$. —— (1)

$2x + 2y\frac{dy}{dx} = 0$ [4]

$\therefore \frac{dy}{dx} = -\frac{\cos\theta}{\sin\theta}$

$y - 3\sin\theta = \frac{-\cos\theta}{\sin\theta}(x-3\cos\theta)$

$\therefore x\cos\theta + y\sin\theta = 3$. —— (2)

(vi) Solve (1) & (2) $y=0$

$\therefore x = \frac{3}{\cos\theta} = 3\sec\theta$.

$\therefore R\ (3\sec\theta, 0)$. [1]

(vii) $|ON| = |3\cos\theta|$

$|OR| = |3\sec\theta|$ [2],

$\therefore |ON|\cdot|OR| = |3\cancel{\cos\theta}\times\frac{3}{\cancel{\cos\theta}}|$

$= 9$

2009 Mathematics Extension 2 Trial HSC: **Questions 7 & 8** solutions

7. (a) Given that $\tan 5\theta = \dfrac{t^5-10t^3+5t}{5t^4-10t^2+1}$, where $t=\tan\theta$ [Do not prove this], [5]

i) Solve the equation $\tan 5\theta = 0$ for $0 \leqslant \theta \leqslant \pi$.

Solution:
$$\begin{aligned} \tan 5\theta &= 0, \\ 5\theta &= 0+n\pi,\ n=0,1,2,3,\ldots \\ \theta &= \frac{n\pi}{5}, \\ &= 0,\ \frac{\pi}{5},\ \frac{2\pi}{5},\ \frac{3\pi}{5},\ \frac{4\pi}{5},\ \pi. \end{aligned}$$

ii) Hence prove that

α) $\tan\frac{\pi}{5}\tan\frac{2\pi}{5} = \sqrt{5}$,

Solution: Method 1—
$$\begin{aligned} t^5-10t^3+5t &= 0, \\ t(t^4-10t^2+5) &= 0, \\ t=0 \text{ or } t^2 &= \frac{10\pm\sqrt{100-20}}{2}, \\ &= 5\pm 2\sqrt{5}, \\ \text{So } t &= \pm\left(5\pm2\sqrt{5}\right). \end{aligned}$$
$$\tan\frac{\pi}{5} = \sqrt{5-2\sqrt{5}}, \qquad \tan\frac{3\pi}{5} = -\sqrt{5+2\sqrt{5}},$$
$$\tan\frac{2\pi}{5} = \sqrt{5+2\sqrt{5}}, \qquad \tan\frac{4\pi}{5} = -\sqrt{5-2\sqrt{5}},$$
$$\begin{aligned} \therefore\ \tan\frac{\pi}{5}\tan\frac{2\pi}{5} &= \sqrt{25-20}, \\ &= \sqrt{5}. \end{aligned}$$

Solution: Method 2—
$$\begin{aligned} t^5-10t^3+5t &= 0, \\ t(t^4-10t^2+5) &= 0, \\ t=0 \text{ or } t^4-10t^2+5 &= 0. \\ \textit{i.e. } \tan\frac{\pi}{5}\times\tan\frac{2\pi}{5}\times\tan\frac{3\pi}{5}\times\tan\frac{4\pi}{5} &= 5, \text{ (product of roots)} \\ \tan\frac{\pi}{5}\times\tan\frac{2\pi}{5}\times\left(-\tan\frac{2\pi}{5}\right)\times\left(-\tan\frac{\pi}{5}\right) &= 5, \\ \textit{i.e. } \tan^2\frac{\pi}{5}\times\tan^2\frac{2\pi}{5} &= 5, \\ \text{Hence } \tan\frac{\pi}{5}\tan\frac{2\pi}{5} &= \sqrt{5}. \end{aligned}$$
(Positive as both $\frac{\pi}{5}$, and $\frac{2\pi}{5}$ are in the 1st quadrant).

β) $\tan^2\frac{\pi}{5}+\tan^2\frac{2\pi}{5} = 10$.

Solution: Method 1—
$$\begin{aligned} \tan^2\frac{\pi}{5}+\tan^2\frac{2\pi}{5} &= 5-2\sqrt{5}+5+2\sqrt{5}, \\ &= 10. \end{aligned}$$

Solution: Method 2— (taking roots 2 at a time)

$$\begin{aligned}
-10 &= \tan\frac{\pi}{5}\tan\frac{2\pi}{5}+\tan\frac{\pi}{5}\tan\frac{3\pi}{5}+\tan\frac{\pi}{5}\tan\frac{4\pi}{5}+\tan\frac{2\pi}{5}\tan\frac{3\pi}{5}\\
&\quad+\tan\frac{2\pi}{5}\tan\frac{4\pi}{5}+\tan\frac{3\pi}{5}\tan\frac{4\pi}{5},\\
&= \tan\frac{\pi}{5}\tan\frac{2\pi}{5}+\tan\frac{\pi}{5}\left(-\tan\frac{2\pi}{5}\right)+\tan\frac{\pi}{5}\left(-\tan\frac{\pi}{5}\right)+\tan\frac{2\pi}{5}\left(-\tan\frac{2\pi}{5}\right)\\
&\quad+\tan\frac{2\pi}{5}\left(-\tan\frac{\pi}{5}\right)+\left(-\tan\frac{2\pi}{5}\right)\left(-\tan\frac{\pi}{5}\right),\\
&= -\tan^2\frac{\pi}{5}-\tan^2\frac{2\pi}{5},\\
10 &= \tan^2\frac{\pi}{5}+\tan^2\frac{2\pi}{5}.
\end{aligned}$$

(b) i) Show that $\displaystyle\int x\tan^{-1}x\,dx=\frac{1}{2}(x^2+1)\tan^{-1}x-\frac{1}{2}x+c.$ [4]

Solution:

$$\begin{aligned}
\text{I} &= \frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\int\frac{1+x^2-1}{1+x^2}\,dx,\\
&= \frac{x^2}{2}\tan^{-1}x-\frac{x}{2}+\frac{\tan^{-1}x}{2}+c,\\
&= \frac{1}{2}(x^2+1)\tan^{-1}x-\frac{x}{2}+c.
\end{aligned}
\qquad
\begin{aligned}
u &= \tan^{-1}x, & v' &= x\,dx,\\
u' &= \frac{dx}{1+x^2}, & v &= \frac{x^2}{2}.
\end{aligned}$$

ii) If $\displaystyle u_n=\int_0^1 x^n\tan^{-1}x\,dx$ for $n\geqslant 2$, show that

$$u_n=\frac{\pi}{2(n+1)}-\frac{1}{n(n+1)}-\frac{n-1}{n+1}u_{n-2}.$$

Solution: Method 1—

$$\begin{aligned}
u_n &= \int_0^1 x^n\tan^{-1}x\,dx,\\
&= \left[\frac{x^{n-1}(x^2+1)\tan^{-1}x}{2}-\frac{x^n}{2}\right]_0^1\\
&\quad-\frac{n-1}{2}\int_0^1(x^2+1)x^{n-2}\tan^{-1}x\,dx\\
&\quad+\frac{n-1}{2}\int_0^1 x^{n-1}\,dx,\\
&= \frac{\pi}{4}-\frac{1}{2}-\frac{n-1}{2}\int_0^1 x^n\tan^{-1}x\,dx\\
&\quad-\frac{n-1}{2}\int_0^1 x^{n-2}\tan^{-1}x\,dx\\
&\quad+\frac{n-1}{2}\left[\frac{x^n}{n}\right]_0^1,\\
\left(1+\frac{n-1}{2}\right)u_n &= \frac{\pi}{4}-\frac{n}{2n}-\frac{n-1}{2}u_{n-2}+\frac{n-1}{2}.\frac{1}{n},\\
\frac{n+1}{2}u_n &= \frac{\pi}{4}-\frac{n-1}{2}u_{n-2}-\frac{1}{2n},\\
u_n &= \frac{\pi}{2(n+1)}-\frac{1}{n(n+1)}-\frac{n-1}{n+1}u_{n-2}.
\end{aligned}
\qquad
\begin{aligned}
u &= x^{n-1},\\
u' &= (n-1)x^{n-2}\,dx,\\
v' &= x\tan^{-1}x\,dx,\\
v &= \frac{x^2+1}{2}\tan^{-1}x-\frac{x}{2}.
\end{aligned}$$

Solution: Method 2—

$$\begin{aligned}
u_n &= \int_0^1 x^n\tan^{-1}x\,dx,\\
&= \left[\frac{x^{n-1}\tan^{-1}x}{n+1}\right]_0^1-\frac{1}{n+1}\int_0^1\frac{x^{n+1}}{1+x^2}\,dx,\\
&= \frac{\pi}{4(n+1)}-\frac{1}{n+1}\int_0^1\frac{(x^2+1)x^{n-1}-x^{n-1}}{1+x^2}\,dx,\\
&= \frac{\pi}{4(n+1)}-\frac{1}{n+1}\int_0^1 x^{n-1}\,dx+\frac{1}{n+1}\int_0^1\frac{x^{n-1}}{1+x^2}\,dx,\\
&= \frac{\pi}{4(n+1)}-\frac{1}{n+1}\left[\frac{x^n}{n}\right]_0^1+\frac{1}{n+1}\left[x^{n-1}\tan^{-1}x\right]_0^1\\
&\quad-\frac{n-1}{n+1}\int_0^1 x^{n-2}\tan^{-1}x\,dx,\\
&= \frac{\pi}{4(n+1)}-\frac{1}{n(n+1)}+\frac{\pi}{4(n+1)}-\left(\frac{n-1}{n+1}\right)u_{n-2},\\
&= \frac{\pi}{2(n+1)}-\frac{1}{n(n+1)}-\frac{n-1}{n+1}u_{n-2}.
\end{aligned}
\qquad
\begin{aligned}
u &= \tan^{-1},\\
u' &= \frac{1}{1+x^2}\,dx,\\
v' &= x^n\,dx,\\
v &= \frac{x^{n+1}}{n+1}.\\
\hline
u &= x^{n-1},\\
u' &= (n-1)x^{n-2}\,dx,\\
v' &= \frac{1}{1+x^2}\,dx,\\
v &= \tan^{-1}.
\end{aligned}$$

(c) Show that the number of ways in which $2n$ persons may be seated at two round tables, n persons being seated at each is [2]

$$\frac{(2n)!}{n^2}.$$

Solution: Ways of choosing people for one table is ${}^{2n}C_n=\dfrac{(2n)!}{(2n-n)!n!}$.

Ways of arranging each table is $(n-1)!$

$$\begin{aligned}
\therefore\ \text{Total ways} &= \frac{(2n)!}{n!n!}.(n-1)!(n-1)!\\
&= \frac{(2n)!}{n^2}.
\end{aligned}$$

(d) i) There are 6 persons from whom a game of tennis is to be made up, two on each side. How many different matches can be arranged if a change in either pair gives a different match? [4]

Solution: Ways of choosing 1st pair $={}^6C_2$,
ways of choosing 2nd pair $={}^4C_2$.
But pair order not important,

$$\begin{aligned}
\therefore\ \text{Number of matches} &= \frac{6!}{4!2!}.\frac{4!}{2!2!}.\frac{1}{2},\\
&= 45.
\end{aligned}$$

ii) How many different matches are possible if two particular persons are to both play in the match?

Solution: If the two are on the same team,
we only need to choose the other team: ${}^4C_2=6$.
If the two are on opposing teams,
(4 ways to get one partner)$\times$(3 ways to get the other) $=12$.
$\therefore$ Number of matches is $6+12=18$ altogether.

8. (a) Suppose a, b, c and d are positive real numbers.

i) Prove that $\frac{a}{b}+\frac{b}{a} \geqslant 2$.

Solution:
$$\begin{aligned} (a-b)^2 &\geqslant 0, \\ a^2-2ab+b^2 &\geqslant 0, \\ a^2+b^2 &\geqslant 2ab, \\ \therefore\ \frac{a}{b}+\frac{b}{a} &\geqslant 2 \text{ as } a, b>0. \end{aligned}$$

ii) Deduce that $\frac{a+b+c}{d}+\frac{b+c+d}{a}+\frac{c+d+a}{b}+\frac{d+a+b}{c} \geqslant 12$.

Solution: Similarly
$$\begin{aligned} \frac{a}{c}+\frac{c}{a} &\geqslant 2, \\ \frac{a}{d}+\frac{d}{a} &\geqslant 2, \\ \frac{b}{c}+\frac{c}{b} &\geqslant 2, \\ \frac{b}{d}+\frac{d}{b} &\geqslant 2, \\ \frac{c}{d}+\frac{d}{c} &\geqslant 2. \end{aligned}$$

Adding, $\frac{b}{a}+\frac{c}{a}+\frac{d}{a}+\frac{a}{b}+\frac{c}{b}+\frac{d}{b}+\frac{a}{c}+\frac{b}{c}+\frac{d}{c}+\frac{a}{d}+\frac{b}{d}+\frac{c}{d} \geqslant 12$,

i.e. $\frac{b+c+d}{a}+\frac{a+c+d}{b}+\frac{a+b+d}{c}+\frac{a+b+c}{d} \geqslant 12$.

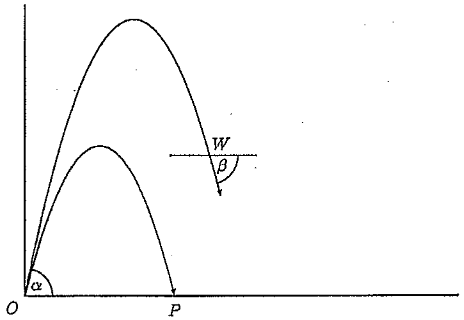
iii) Hence prove that if $a+b+c+d=1$, then:
$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} \geqslant 16.$$

Solution: Now
$$\begin{aligned} a+b+c &= 1-d, \\ a+b+d &= 1-c, \\ a+c+d &= 1-b, \\ b+c+d &= 1-a, \end{aligned}$$
$$\therefore\ \frac{1-a}{a}+\frac{1-b}{b}+\frac{1-c}{c}+\frac{1-d}{d} \geqslant 12,$$
$$\frac{1}{a}-1+\frac{1}{b}-1+\frac{1}{c}-1+\frac{1}{d}-1 \geqslant 12,$$
$$\text{So } \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} \geqslant 16.$$

(b) Two stones are thrown simultaneously from the same point O in the same direction and with the same non-zero angle of projection α, but with different velocities U and V $(U<V)$.

The slower stone hits the ground at a point P on the same level as the point of projection.

At that instant the faster stone is at a point W on its downward path, making an angle β with the horizontal.



i) Show that $V(\tan\alpha+\tan\beta)=2U\tan\alpha$.

Solution: For the OP path,
$$\ddot{x}=0,\quad \ddot{y}=-g,$$
$$\dot{x}=U\cos\alpha,\quad \dot{y}=U\sin\alpha-gt,$$
$$x=Ut\cos\alpha.\quad y=Ut\sin\alpha-g\tfrac{t^2}{2}.$$

For the OW path,
$$\ddot{x}=0,\quad \ddot{y}=-g,$$
$$\dot{x}=V\cos\alpha,\quad \dot{y}=V\sin\alpha-gt,$$
$$x=Vt\cos\alpha.\quad y=Vt\sin\alpha-g\tfrac{t^2}{2}.$$

At P, $t=\frac{2U\sin\alpha}{g}$.

So at W, $\dot{x}=V\cos\alpha$, $\dot{y}=V\sin\alpha-2U\sin\alpha$.
$$\begin{aligned} -\tan\beta=\frac{\dot{y}}{\dot{x}} &= \frac{\sin\alpha}{\cos\alpha}-\frac{2U\sin\alpha}{V\cos\alpha}, \\ \textit{i.e.}\ -V\tan\beta &= V\tan\alpha-2U\tan\alpha, \\ V(\tan\alpha+\tan\beta) &= 2U\tan\alpha. \end{aligned}$$

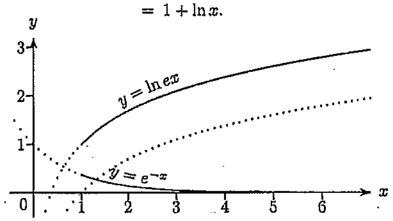
ii) Deduce that if $\beta=\frac{1}{2}\alpha$, then $U<\frac{3}{4}V$.

Solution:
$$\begin{aligned} V\left(\frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}}+\tan\frac{\alpha}{2}\right) &= \frac{2U\times 2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}}, \\ V\left(2\tan\frac{\alpha}{2}+\tan\frac{\alpha}{2}-\tan^3\frac{\alpha}{2}\right) &= 4U\tan\frac{\alpha}{2}, \\ V\left(3-\tan^2\frac{\alpha}{2}\right) &= 4U,\ (\text{as } \tan\tfrac{\alpha}{2}\neq 0) \\ U &= \frac{3V}{4}-\frac{\tan^2\frac{\alpha}{2}}{4}, \\ \textit{i.e. } U &< \frac{3V}{4}\ (\text{as } \tan^2\tfrac{\alpha}{2}>0). \end{aligned}$$

(c) i) Show by graphical means that $\boxed{4}$

$$\ln ex > e^{-x} \text{ for } x \geqslant 1.$$

Solution: $\ln ex = \ln e + \ln x,$
$\qquad\qquad\quad = 1 + \ln x.$



ii) Hence, or otherwise, show that

$$\ln\left(n!e^{n}\right) > e^{-n}\left(\frac{e^{n}-1}{e-1}\right).$$

Solution:
$$\begin{aligned}
\ln n!e^{n} &= \ln ne.(n-1)e.(n-2)e.(n-3)e\ldots(1)e,\\
&= \ln ne + \ln(n-1)e + \ln(n-2)e + \cdots + \ln e,\\
&> e^{-n} + e^{1-n} + e^{2-n} + \cdots + e^{-1},\\
&> e^{-n}(1 + e + e^{2} + \cdots + e^{n-1}),\\
&> e^{-n}\left(\frac{e^{n}-1}{e-1}\right).
\end{aligned}$$