SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2010
HIGHER SCHOOL CERTIFICATE
TRIAL PAPER

Mathematics Extension 2

General Instructions

- Reading Time – 5 Minutes
- Working time – 3 Hours
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- Each Question is to be returned in a separate bundle.
- All necessary working should be shown in every question.
- Answer in simplest exact form unless otherwise instructed.

Total Marks – 120

- Attempt questions 1 – 8

Examiner: *A.M.Gainford*

- NOTE: This is a trial paper only and does not necessarily reflect the content or format of the final Higher School Certificate examination paper for this subject.

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ a \neq 0$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln\left(x+\sqrt{x^2-a^2}\right)\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\ x > 0$

Question 1. (15 marks) **(Start a new answer sheet.)**

			Marks
(a)	Evaluate	$\displaystyle\int_0^3 \frac{x}{\sqrt{x^2+16}}\,dx$.	2
(b)	Find	$\displaystyle\int \left(\cos^2 x - \sin^2 x\right) dx$.	1
(c)	Use integration by parts to find $\displaystyle\int xe^{-x}\,dx$.		2
(d)	(i)	Find real numbers a and b such that $\displaystyle\frac{1-3x}{x^2-3x+2} = \frac{a}{x-1} + \frac{b}{x-2}$.	2
	(ii)	Hence find $\displaystyle\int \frac{1-3x}{x^2-3x+2}\,dx$.	2
(e)	Evaluate	$\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{\cos x + 2}\,dx$.	2
(f)	(i)	If $I_n = \int_0^{\frac{\pi}{4}} \tan^n x\,dx$, $n = 1, 2, 3, \ldots$. show that $\displaystyle I_n + I_{n-2} = \frac{1}{n-1}$, $n = 2, 3, 4, \ldots$.	2
	(ii)	Hence evaluate $\int_0^{\frac{\pi}{4}} \tan^5 x\,dx$.	2

Question 2. (15 marks) **(Start a new answer sheet.)**

			Marks
(a)	If $u = 3-4i$ and $v = 2-2i$ find		4
	(i)	$u\overline{v}$	
	(ii)	$\sqrt{u}$	
	(iii)	v in modulus-argument form.	
	(iv)	v^4 using De Moivre's theorem.	
(b)	On an Argand diagram shade the region that is satisfied by both the conditions $3 \le \lvert z-4i \rvert \le 4$ and $-\frac{\pi}{4} < \arg(z-4i) < \frac{\pi}{4}$		2
(c)	Sketch, on separate Argand diagrams, the locus of the complex number z satisfying		4
	(i)	$z^2 - (\overline{z})^2 = i$	
	(ii)	$\lvert z-1 \rvert = \mathrm{Re}(z)$	
(d)	It is given that $z = \cos\theta + i\sin\theta$ where $0 < \arg z < \frac{\pi}{2}$.		5
	(i)	Show that $z+1 = 2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$ and express $z-1$ in modulus-argument form.	
	(ii)	Hence show that $\mathrm{Re}\left(\dfrac{z-1}{z+1}\right) = 0$.	

Question 3. (15 marks) **(Start a new answer sheet.)**

Marks

(a) (i) Show that $z=1+i$ is a root of $z^2-(3-2i)z+(5-i)=0$. **3**

(ii) Find the other root of the equation.

(b) If α, β and γ are roots of the equation $x^3+qx-2=0$ find, in terms of q, the monic cubic polynomial equation whose roots are α^2, β^2 and γ^2. **3**

(c) (i) Use De Moivre's theorem to find $\cos 5\theta$ in terms of powers of $\cos\theta$. **6**

(ii) Use the result in (i) to solve the equation

$$16x^4-20x^2+5=0$$

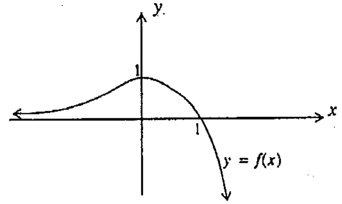
(d) If ω represents one of the complex roots of the equation $z^3-1=0$

(i) Show that $1+\omega+\omega^2=0$.

(ii) Evaluate $(1-\omega^8)(1-\omega^4)(1-\omega^2)(1-\omega)$.

Question 4 (15 marks) **(Start a new answer sheet.)**

(a) The graph of $y=f(x)$ is sketched below.
There is a stationary point at (0, 1).



Use this graph to sketch the following, on separate diagrams, showing essential features.

(i) $y=f\left(\frac{x}{2}\right)$ **2**

(ii) $y=x+f(x)$ **2**

(iii) $y=\dfrac{1}{f(x)}$ **2**

(iv) $y=f\left(\frac{1}{x}\right)$ **2**

(b) (i) Find $\displaystyle\int\frac{1}{x^2\sqrt{9-x^2}}\,dx$, using the substitution $x=3\cos\theta$. **4**

(ii) Evaluate $\int_1^e x^3\log_e x\,dx$.

(c) Given that the quartic polynomial $p(x)=x^4-5x^3-9x^2+81x-108$ has a zero of multiplicity 3, factorise $p(x)$ completely, and find all its zeroes. **3**

Question 5 (15 marks) **(Start a new answer sheet.)**

Marks

(a) A particle is moving under gravity in a fluid which exerts a resistance to its motion, per unit mass, k times its speed (k is constant).

(i) If the particle falls vertically from rest, show that its terminal velocity is $V_T = \frac{g}{k}$, where g is acceleration due to gravity. **2**

(ii) If the particle is projected vertically upward with velocity V_T show that after time t seconds **6**

(α) its speed is $V_T\left(2e^{-kt}-1\right)$

(β) its height above the starting point is $\frac{1}{k}V_T\left(2-2e^{-kt}-kt\right)$

(iii) Hence find an expression for the greatest height reached in terms of V_T and k. **2**

(b) A box contains 6 white balls and 2 black balls. Balls are selected at random, one at a time, and not replaced. A note is kept of the number, X, of the draw which first yields a black ball. If this experiment is repeated many times, find: **5**

(i) the most probable value of X;

(ii) the probability that $X > 4$.

Question 6 (15 marks) **(Start a new answer sheet.)**

(a) A council has 14 councillors: 6 Labor, 5 Liberal and 3 Independents. Five councilors are chosen at random to form a committee. **6**

(i) (α) How many different committees can be formed?

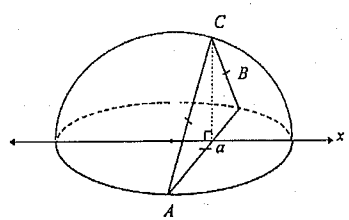
(β) Find the probability that the committee will have a majority of Labor councilors.

(ii) (α) Show that the number of different committees which can be formed with at least one councilor from each of the groups Labor, Liberal, and Independent is 1365.

(β) Given that the committee contains at least one councilor from each of the groups Labor, Liberal, and Independent, find the probability that the committee will have a majority of Labor councilors.

(b) The circle $x^2+y^2=4$ is rotated about the line $x=5$ to form a torus. Use the method of cylindrical shells to prove that the volume of the solid is $40\pi^2$ cubic units. **4**

(c) The solid drawn at right has a circular base of radius 3 units in the horizontal plane. Vertical cross-sections perpendicular to the diameter along the x-axis are equilateral triangles.



(i) A vertical slice of width Δa is positioned at the point where $x = a$. If the volume of the slice is ΔV, show that $\Delta V = \sqrt{3}\left(9-a^2\right)\Delta a$. **3**

(ii) Hence determine the volume of the solid. **2**

Question 7 (15 marks) **(Start a new answer sheet.)**

(a) On polling day in Rock Island City the ratio of electoral votes in the only four polling booths A, B, C, and D was 5:4:3:8 respectively. The percentages of votes for Mr Jones in these booths were 60%, 50%, 40%, and 70% respectively. **4**

(i) Find the probability that a voter chosen at random voted for Mr Jones.

(ii) If ten voters of this city were chosen at random, find the probability that Mr Jones gained

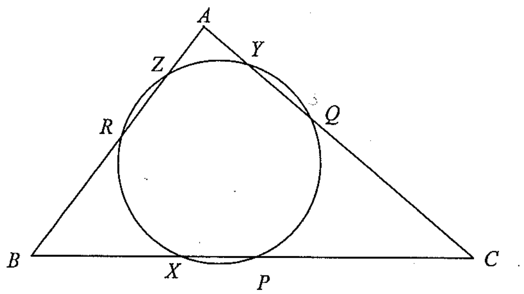
(α) at least 8 votes

(β) no more than 2 votes.

(b) The equation $e^{2x}\log_e y = 3$ implicitly defines y as a function of x. **3**

Find $\dfrac{dy}{dx}$ as a function of y.

(c) **8**



In the diagram above, P, Q, and R are the midpoints of the sides BC, CA, and AB respectively of a triangle ABC. The circle drawn through the points P, Q, and R meets the sides BC, CA, and AB again at X, Y, and Z respectively.

Copy the diagram to your answer sheet.

(i) Briefly explain why $RPCQ$ is a parallelogram.

(ii) Show that ΔXCQ is isosceles.

(iii) Show that $AX \perp BC$.

Question 8 (15 marks) **(Start a new answer sheet.)**

(a) Five women and four men are to be seated at a round table. **5**

(i) In how many ways may this be done without restrictions?

(ii) In how many ways may this be done if no two men are to be seated together?

(iii) If one man and one woman are a married couple, what is the probability that they are seated together, given the conditions of part (ii)?

(b) One root of the equation $x^3 + ax^2 + bx + c = 0$ is equal to the sum of the other two roots. **4**

Show that $a^3 - 4ab + 8c = 0$.

(c) (i) Graph, in the same xy-plane, the curves **6**

$$y = x^{-\frac{2}{3}}, x > 0 \quad \text{and} \quad y = (x-1)^{-\frac{2}{3}}, x > 1$$

(ii) Hence, or otherwise, given the sum S, where

$$S = 1 + \frac{1}{\sqrt[3]{2^2}} + \frac{1}{\sqrt[3]{3^2}} + \ldots + \frac{1}{\sqrt[3]{\left(10^9\right)^2}}$$, find the two consecutive integers between which the sum S lies.

This is the end of the paper.

Question 1

a) $\displaystyle\int_0^3 \frac{x}{\sqrt{x^2+16}}\,dx$

let $u = x^2+16$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

$= \displaystyle\int_{16}^{25} \frac{x}{\sqrt{u}} \cdot \frac{du}{2x}$

limit change

when $x=0$ $x=3$

$u=16$ $u=25$

$= \frac{1}{2}\displaystyle\int_{16}^{25} u^{-\frac{1}{2}}\,du$

$= \frac{1}{2}\left[2u^{\frac{1}{2}}\right]_{16}^{25}$

$= \frac{1}{2}\left[2(25)^{\frac{1}{2}} - 2(16)^{\frac{1}{2}}\right]$

$= 1$

b) $\displaystyle\int (\cos^2 x - \sin^2 x)\,dx$

$= \displaystyle\int \cos 2x\,dx$

$= \frac{1}{2}\sin 2x + C$

c) $\displaystyle\int x e^{-x}\,dx$

$u = x$ $v' = e^{-x}$

$u' = 1$ $v = -e^{-x}$

$= -xe^{-x} + \displaystyle\int e^{-x}\,dx$

$= -xe^{-x} - e^{-x} + C$

d) i) $\dfrac{1-3x}{x^2-3x+2} = \dfrac{a}{x-1} + \dfrac{b}{x-2}$

$1-3x = a(x-2) + b(x-1)$

when $x=2$

$1-3(2) = b$

$b = -5$

when $x=1$

$1-3(1) = -a$

$a = 2$

ii) $\displaystyle\int \frac{1-3x}{x^2-3x+2}\,dx = 2\int\frac{dx}{x-1} - 5\int\frac{dx}{x-2}$

$= 2\ln(x-1) - 5\ln(x-2) + C$

e) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{\cos x + 2}\,dx$

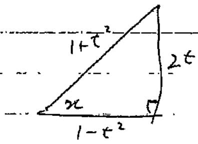
let $t = \tan\frac{x}{2}$

$\frac{x}{2} = \tan^{-1} t$

$x = 2\tan^{-1} t$

$\frac{dx}{dt} = \frac{2}{1+t^2}$

$dx = \frac{2dt}{1+t^2}$



$= \displaystyle\int_0^1 \frac{1}{\frac{1-t^2}{1+t^2}+2} \cdot \frac{2dt}{1+t^2}$

$= \displaystyle\int_0^1 \frac{2dt}{1-t^2+2(1+t^2)}$

$= \displaystyle\int_0^1 \frac{2dt}{3+t^2}$

limit change

when $x=0$ $x=\frac{\pi}{2}$

$t=0$ $t=1$

$= 2\displaystyle\int_0^1 \frac{dt}{3+t^2}$ $a=\sqrt{3}$.

$= 2\left[\frac{1}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}}\right]_0^1$

$= 2\left[\frac{1}{\sqrt{3}}\tan^{-1}\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}}\tan^{-1}\frac{0}{\sqrt{3}}\right]$

$= 2\left[\frac{1}{\sqrt{3}}\cdot\frac{\pi}{6} - \cancel{\frac{1}{\sqrt{3}}\,0}\right]$

$= \dfrac{\pi}{3\sqrt{3}}$.

f) i) $I_n = \displaystyle\int_0^{\frac{\pi}{4}} \tan^n x\,dx$, $n = 1, 2, 3, \ldots$

$I_n + I_{n-2} = \displaystyle\int_0^{\frac{\pi}{4}} \tan^n x\,dx + \int_0^{\frac{\pi}{4}} \tan^{n-2} x\,dx$

$$= \int_0^{\frac{\pi}{4}} \left(\tan^n x \, dx + \tan^{n-2} x \, dx \right) dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x \, dx \left(\tan^2 x + 1 \right) dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x \, dx . \sec^2 x \, dx$$

let $u = \tan x$

$\frac{du}{dx} = \sec^2 x$

$dx = \frac{du}{\sec^2 x}$

$$= \int_0^1 u^{n-2} \cancel{\sec^2 x} \frac{du}{\cancel{\sec^2 x}}$$

limit change

when $x = 0$ $\quad x = \frac{\pi}{4}$

$u = 0 \quad u = 1$

$$= \left[\frac{u^{n-1}}{n-1} \right]_0^1$$

$$= \frac{1^{n-1}}{n-1} - \frac{0^{n-1}}{n-1}$$

$$= \frac{1}{n-1}$$

ii) $I_5 = \int_0^{\frac{\pi}{4}} \tan^5 x \, dx$ $\qquad$ from (i) $I_n = \frac{1}{n-1} - I_{n-2}$

$$= \frac{1}{4} - I_3$$

$$= \frac{1}{4} - \left(\frac{1}{2} - I_1 \right)$$

$$= -\frac{1}{4} + \int_0^{\frac{\pi}{4}} \tan x \, dx$$

$$= -\frac{1}{4} + \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x} \, dx$$

$$= -\frac{1}{4} - \int_0^{\frac{\pi}{4}} \frac{-\sin x}{\cos x} \, dx$$

$$= -\frac{1}{4} - \left[\ln \cos x \right]_0^{\frac{\pi}{4}}$$

$$= -\frac{1}{4} - \left[\ln \cos \frac{\pi}{4} - \ln \cos 0 \right]$$

$$= -\frac{1}{4} - \left[\ln \left(\frac{1}{\sqrt{2}} \right) - \cancel{\ln (1)} \right]$$

$$= -\frac{1}{4} - \ln \left(\frac{1}{\sqrt{2}} \right)$$

$$= -\frac{1}{4} + \ln \sqrt{2}$$

$$= -\frac{1}{4} + \frac{1}{2} \ln 2$$

QUESTION 2.

(a)(i) $(3-4i)(2+2i) = 6+8+6i-8i$
$= \boxed{14-2i.}$

(ii) Let $\sqrt{u} = a+ib$; $a,b \in R$; $a>0$.

ie $\sqrt{3-4i} = a+ib$

$3-4i = a^2-b^2+(2ab)i$

$\therefore 3 = a^2-b^2$ —(1)

$-4 = 2ab$ —(2)

Now $(a^2+b^2)^2 = (a^2-b^2)^2+(2ab)^2$
$= 9+16$
$= 25$

$\therefore a^2+b^2 = 5$ —(3)

Then (1) + (3)

$2a^2 = 8$

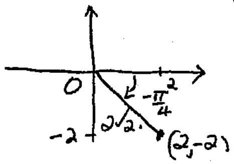
$a^2 = 4$

$a = 2$ (NB $a>0$)

$b = -1$

$\therefore \boxed{\sqrt{3-4i} = 2-i.}$

(iii)



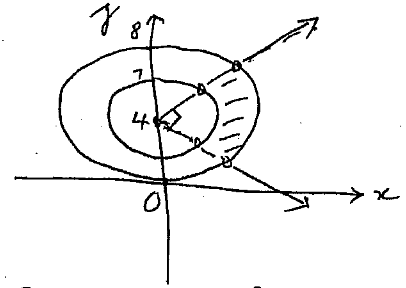
$\boxed{v = 2\sqrt{2}\ \mathrm{cis}\left(-\frac{\pi}{4}\right).}$

(iv) $v^4 = (2\sqrt{2})^4\ \mathrm{cis}(-\pi)$
$= 64 \times (\cos -\pi + i \sin -\pi)$
$= 64 \times -1$

$\boxed{v^4 = -64.}$

Q2 (CONTD)

(b)



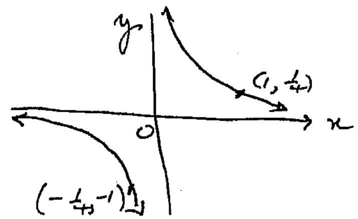
(c) (i) $z^2-(\bar{z})^2 = i$ where $z = x+iy$

$(z-\bar{z})(z+\bar{z}) = i$

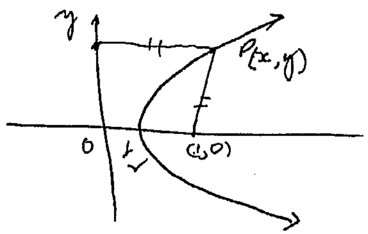
$-2iy \times 2x = i$

$-4ixy = i$

$xy = \frac{1}{4}.$



(ii)



$x = \sqrt{(x-1)^2+y^2}$

$x^2 = (x-1)^2+y^2$

$x^2 = x^2-2x+1+y^2$

$\boxed{y^2 = 2x-1}$

Q2 (CONTD)

(d) (i)

$$z = \cos\theta + i\sin\theta$$
$$= 2\cos^2\tfrac{\theta}{2} - 1 + 2i\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}$$
$$z+1 = 2\cos^2\tfrac{\theta}{2} + 2i\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}$$
$$= 2\cos\tfrac{\theta}{2}\left(\cos\tfrac{\theta}{2} + i\sin\tfrac{\theta}{2}\right)$$

now

$$z = \cos\theta + i\sin\theta.$$
$$= 1 - 2\sin^2\tfrac{\theta}{2} + 2i\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}$$
$$z-1 = -2\sin^2\tfrac{\theta}{2} + 2i\sin\tfrac{\theta}{2}\cdot\cos\tfrac{\theta}{2}.$$
$$= -2\sin\tfrac{\theta}{2}\left(\sin\tfrac{\theta}{2} - i\cos\tfrac{\theta}{2}\right)$$
$$= 2i\sin\tfrac{\theta}{2}\left(\cos\tfrac{\theta}{2} + i\sin\tfrac{\theta}{2}\right)$$
$$= 2\sin\tfrac{\theta}{2}\left(\cos\left(\tfrac{\theta}{2}+\tfrac{\pi}{2}\right) + i\sin\left(\tfrac{\theta}{2}+\tfrac{\pi}{2}\right)\right)$$

MOD-ARG FORM AS REQUIRED

(NB $i = \text{cis}\,\frac{\pi}{2}$.)

(ii)

$$\frac{z-1}{z+1} = \frac{2i\sin\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)}{2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)}$$
$$= i\tan\tfrac{\theta}{2}.$$

$$\therefore \ \text{Re}\left(\frac{z-1}{z+1}\right) = 0.$$

Extension(2) Q(3) Solutions

Question (3)

(a) (i) $z^2 - (3-2i)z + (5-i) = 0$

If $1+i$ is a root, then

$(1+i)^2 - (3-2i)(1+i) + (5-i)$

2 $= 1+2i-1-(3+3i-2i+2)+5-i$

$= 2i - \cancel{3} - i - \cancel{2} + \cancel{5} - i$ $\boxed{3}$

$= 0.$

(ii) Let w be the other root

1 $w + (1+i) = (3-2i)$

$\therefore \quad w = 2-3i$

(b) Let $y = x^2 \Rightarrow x = \sqrt{y}$. $\boxed{3}$

$\therefore (\sqrt{y})^3 + q(\sqrt{y}) = 2$

$\sqrt{y}\,(y+q) = 2 \Rightarrow y(y+q)^2 = 4$

$\therefore \ y(y^2+2qy+q^2) = 4,$

i.e. $y^3 + 2qy^2 + q^2y - 4 = 0.$

(c) Let $z = c + is$.

(i) $z^5 = \cos 5\theta + i\sin 5\theta$

$(c+is)^5 = \binom{5}{0}c^5 + i\binom{5}{1}c^4s - \binom{5}{2}c^3s^2$

$- i\binom{5}{3}c^2s^3 + \binom{5}{4}cs^4 + is^5.$ $\boxed{6}$

Equate real parts.

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta)$

2 $\quad + 5\cos\theta(1-\cos^2\theta)^2$

$\therefore \cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

$= \cos\theta\,(16\cos^4\theta - 20\cos^2\theta + 5)$

(ii) $16x^5 - 20x^3 + 5x = 0$

$x(16x^4 - 20x^2 + 5) = 0.$

Let $x = \cos\theta \quad \therefore \cos 5\theta = 0$

$\therefore 5\theta = \pi/2,\ 3\pi/2,\ 5\pi/2,\ \frac{7\pi}{2},\ 9\pi/2.$

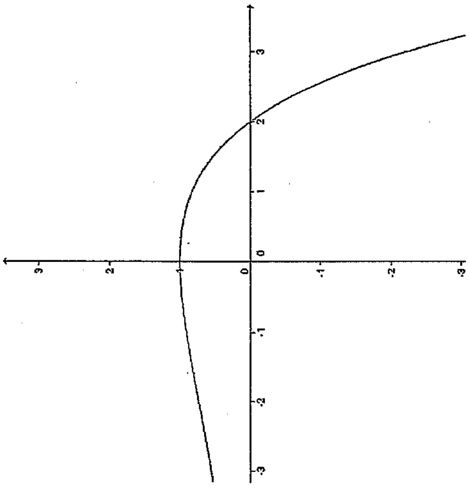
4 $\therefore x = \cos\frac{\pi}{10},\ \cos\frac{3\pi}{10},\ \cos\frac{\pi}{2},\ \cos\frac{7\pi}{10},\ \cos\frac{9\pi}{10}. \quad \therefore x \neq 0$

$\therefore$ The solutions to: $16x^4 - 20x^2 + 5 = 0$

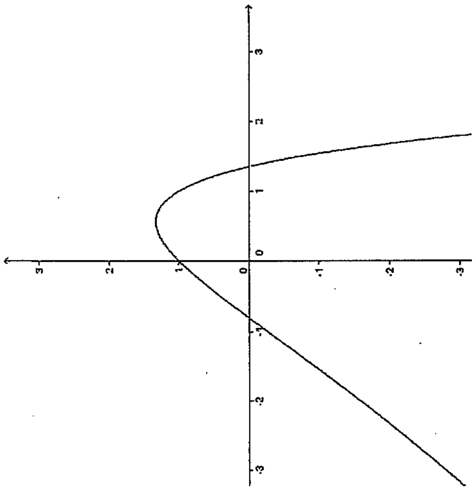
are: $\pm\cos\frac{\pi}{10},\ \pm\cos\frac{3\pi}{10}.$

Question 4

(a) (i) $y=f\left(\frac{x}{2}\right)$



(ii) $y=x+f(x)$



(d)

(i) $1+\omega+\omega^2$ is a geometric series with common ratio ω

$\therefore \quad 1+\omega+\omega^2 = \frac{\omega^3-1}{\omega-1}$

1 But $\omega^3=1$

$\therefore \quad 1+\omega+\omega^2=0$ as required.

(ii) $(1-\omega^8)(1-\omega^4)(1-\omega^2)(1-\omega)$ $\boxed{3}$

$1-\omega^8 = 1-\omega^6\cdot\omega^2 = 1-\omega^2$

$1-\omega^4 = 1-\omega^3\cdot\omega = 1-\omega$

2 $\therefore \quad (1-\omega^8)(1-\omega^4)(1-\omega^2)(1-\omega)$

$= (1-\omega^2)^2(1-\omega)^2$

$= [(1-\omega^2)(1-\omega)]^2$

$= (1-\omega-\omega^2+\omega^3)^2$

$= [2-(\omega+\omega^2)]^2$

$\because \quad 1+\omega+\omega^2=0$

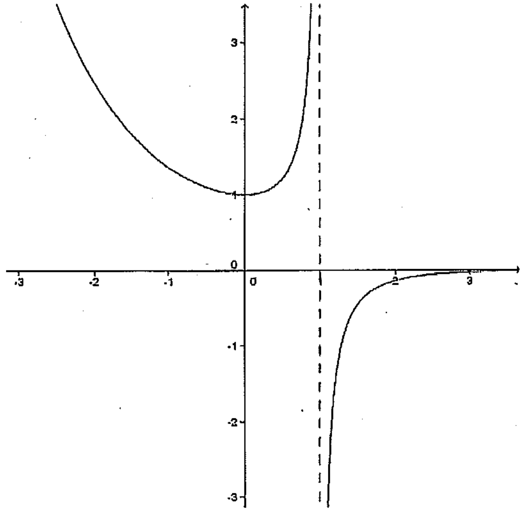
$\therefore \quad \omega+\omega^2=-1$

$\therefore \quad [2-(\omega+\omega^2)]^2$

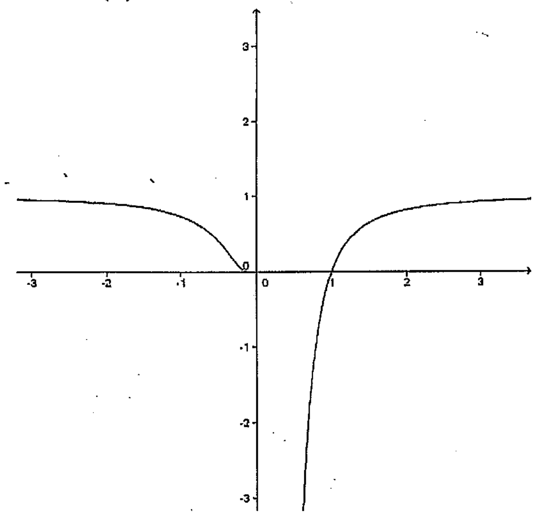
$= (2-(-1))^2$

$= 9.$

(iii) $y=\dfrac{1}{f(x)}$



(iv) $y=f\left(\dfrac{1}{x}\right)$



(b)(i)

$$\int\frac{dx}{x^2\sqrt{9-x^2}}=\int\frac{-3\sin\theta}{9\cos^2\theta\sqrt{9-9\cos^2\theta}}d\theta$$

$$=-\frac{1}{9}\int\sec^2\theta d\theta$$

$$=-\frac{1}{9}\tan\theta+C$$

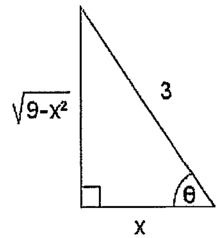
$$=-\frac{\sqrt{9-x^2}}{9x}+C$$

$x=3\cos\theta\Rightarrow\theta=\cos^{-1}\left(\dfrac{x}{3}\right)$

$\dfrac{dx}{d\theta}=-3\sin\theta$

$dx=-3\sin\theta d\theta$

2



(ii)

$$\int_1^e x^3\ln x dx=\frac{x^4}{4}\ln x\Big|_1^e-\int_1^e\frac{x^3}{4}dx$$

$$=\frac{e^4}{4}\ln e-\frac{1}{16}\left[x^4\right]_1^e$$

$$=\frac{e^4}{4}-\frac{1}{16}\left(e^4-1\right)$$

$$=\frac{4e^4}{16}-\frac{e^4}{16}+\frac{1}{16}$$

$$=\frac{3e^4}{16}+\frac{1}{16}$$

$u=\underline{\ln x}$ $\quad v=\dfrac{x^4}{4}$

$u'=\dfrac{1}{x}$ $\quad v'=\underline{x^3}$

2

(c)

$$P(x) = x^4 - 5x^3 - 9x^2 + 81x - 108$$
$$P'(x) = 4x^3 - 15x^2 - 18x + 81$$
$$P''(x) = 12x^2 - 30x - 18$$

$$2x^2 - 5x - 3 = 0$$
$$x = 3, -\frac{1}{2}$$

$$\alpha^3\beta = -108$$
$$3^3\beta = -108$$
$$\beta = -\frac{108}{3^3}$$
$$\beta = -4$$

$$P(x) = (x+4)(x-3)^3$$

Zeros $-4, 3, 3, 3$

Question 6

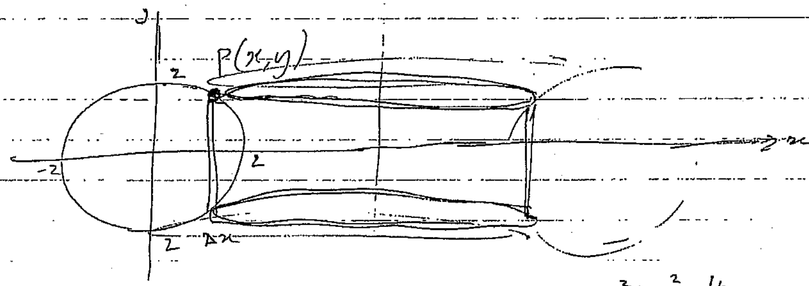
a) i) α) ${}^{14}C_5 = 2002$

β) $\dfrac{{}^6C_3 \times {}^8C_2 + {}^6C_4 \times {}^8C_1 + {}^6C_5}{2002} = \dfrac{686}{2002} = \dfrac{49}{143}$

ii) α) ${}^6C_3 \times {}^5C_1 \times {}^3C_1 + {}^6C_2 \times {}^5C_2 \times {}^3C_1 + {}^6C_2 \times {}^5C_1 \times {}^3C_2 + {}^6C_1 \times {}^5C_2 \times {}^3C_2$
$+ {}^6C_1 \times {}^5C_3 \times {}^3C_1 + {}^6C_1 \times {}^5C_1 \times {}^3C_3 = 1365.$

β) $\dfrac{{}^6C_3 \times {}^5C_1 \times {}^3C_1}{1365} = \dfrac{300}{1365} = \dfrac{20}{91}$

b)



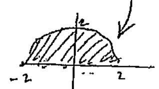
$x^2 + y^2 = 4$
$y^2 = 4 - x^2$
$y = \pm\sqrt{4-x^2}$
$y = \sqrt{4-x^2}$

$\Delta V = 2\pi r h \, \Delta x$
$\Delta V = 2\pi (5-x) 2y \, \Delta x$
$V = \lim_{\Delta x \to 0} \sum_{x=-2}^{2} 4\pi (5-x)\sqrt{4-x^2} \, \Delta x$

$V = 4\pi \int_{-2}^{2} (5-x)\sqrt{4-x^2} \, dx$

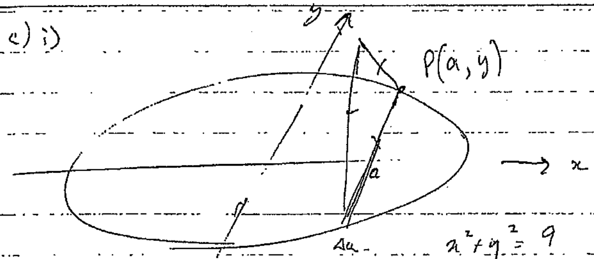
$V = 20\pi \int_{-2}^{2} \sqrt{4-x^2} \, dx - 4\pi \int_{-2}^{2} x\sqrt{4-x^2} \, dx$

$f(x) = x\sqrt{4-x^2}$ is odd



$V = 20\pi \left(\frac{1}{2}\pi(2)^2\right) - 4\pi(0)$
$= 40\pi^2$ units^3

c) i)



$x^2+y^2=9$

$y^2=9-x^2$

when $x=a$

$y^2=9-a^2$

$\Delta V=\frac{1}{2}\times 2y\times 2y\times \sin 60^\circ\,\Delta a$

$\Delta V=2y^2\cdot\frac{\sqrt{3}}{2}\cdot\Delta a$

$\Delta V=\sqrt{3}\,(9-a^2)\,\Delta a$

ii) $V=\lim\limits_{\Delta a\to 0}\sum\limits_{a=-3}^{3}\sqrt{3}\,(9-a^2)\,\Delta a$

$V=\sqrt{3}\int_{-3}^{3}(9-a^2)\,da$

$V=2\sqrt{3}\int_{0}^{3}(9-a^2)\,da$

$V=2\sqrt{3}\left[9a-\frac{a^3}{3}\right]_0^3$

$V=2\sqrt{3}\left[9(3)-\frac{(3)^3}{3}-(0)\right]$

$V=36\sqrt{3}$ units^3

QUESTION 5

(a)(i) $\downarrow g$ $\uparrow -kg$

terminal velocity $g=kv$

$V_t=\frac{g}{k}$

(ii)(α) $\frac{dv}{dt}=-g-kv$

$\frac{dt}{dv}=-\frac{1}{g+kv}$

$t=-\frac{1}{k}\log(g+kv)+c$

$t=\frac{1}{k}\log\left(\frac{g+kv_0}{g+kv}\right)$

$e^{kt}=\frac{g+kv_0}{g+kv}$

$g+kv=\frac{g+kv_0}{e^{kt}}$

$kv=\frac{g+kv_0}{e^{kt}}-g$

$V=\frac{V_t+V_t}{e^{kt}}-V_t \quad \left(V_t=\frac{g}{k}\right)$

$=V_t\left(\frac{2}{e^{kt}}-1\right)$

$V=V_t\left(2e^{-kt}-1\right)$

(ii)(β) $\frac{dh}{dt}=V_t\left(2e^{-kt}-1\right)$ from α

$h=-\frac{2}{k}e^{-kt}V_t-V_t t+C$

$h=0$ $t=0$ $C=\frac{2}{k}V_t$

$h=\frac{V_t}{k}\left(-2e^{-kt}-kt+2\right)$

(iii) greatest h velocity $=0$

$V_t\left(2e^{-kt}-1\right)=0$

$e^{-kt}=\frac{1}{2}$

$\log_e 2=kt$

$t=\frac{1}{k}\log_e 2$

$h=\frac{V_t}{k}\left(-2e^{-kt}-kt+2\right)$ from (ii)

$=\frac{V_t}{k}\left(1-\ln 2\right)$

(b)

x	Probability
1	$\frac{1}{4}$
2	$\frac{3}{14}$
3	$\frac{5}{28}$
4	$\frac{1}{7}$
5	$\frac{3}{28}$
6	$\frac{1}{14}$
7	$\frac{1}{28}$

(i) most probable $x=1$

(ii) Prob $x>4$ $=\frac{6}{28}$

$=\frac{3}{14}$

(4U)

Q7 (a)(i) $P(\text{MR. JONES}) = \frac{1}{4}\times 0.6 + \frac{1}{5}\times\frac{1}{2} + \frac{3}{20}\times 0.4 + \frac{2}{5}\times 0.7$

$= 0.15 + 0.10 + 0.06 + 0.28.$

$= \boxed{0.59.}$

(ii)(α) $P(x \geq 8) = \binom{10}{10}(0.59)^{10} + \binom{10}{9}(0.59)^9(0.41)^1 + \binom{10}{8}(0.59)^8 \times (0.41)^2$

$\doteqdot \boxed{0.152.}$

(β) $P(x \leq 2) = \binom{10}{0}(0.41)^{10} + \binom{10}{1}(0.41)^9 \times 0.59 + \binom{10}{2}(0.41)^8(0.59)^2$

$\doteqdot \boxed{0.015.}$

(b) $e^{2x}\ln y = 3$

Diff. both sides with respect to x.

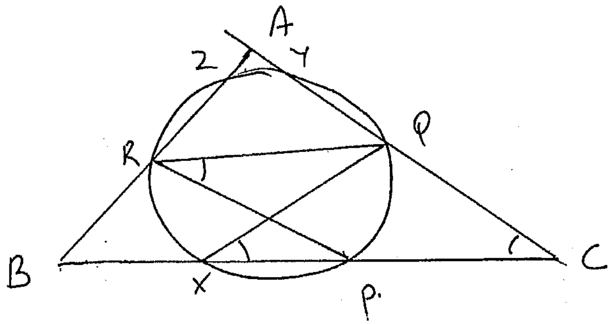
$e^{2x}\cdot\frac{1}{y}\frac{dy}{dx} + 2e^{2x}\ln y = 0$

$e^{2x}\frac{1}{y}\frac{dy}{dx} = -2e^{2x}\ln y.$

$\frac{1}{y}\frac{dy}{dx} = -2\ln y.$

(Function of y as required) $\longrightarrow$ $\boxed{\frac{dy}{dx} = -2y\ln y}$

(c)



(i) $RQ \parallel PC$
$RP \parallel QC$ (Interval joining mid-pts of two sides of a triangle is half of & parallel to the third side)

$\therefore$ $RPCQ$ is a parallelogram.

(ii) $\angle PRQ = \angle PCQ$ (property of a parallelogram i.e. opposite angles are equal).

$\angle PRQ = \angle CXQ$ (angles in the same segment, subtended at the circumference are equal)

$\therefore$ $\triangle XCQ$ is isosceles (base angle equal)

(iii) $AX = QC$ (given)

$XQ = QC$ (isosceles triangle)

$\therefore$ $AX = XQ = QC.$

$\therefore$ A circle passes through A, X & C with Q the centre & AC the diameter

$\therefore$ $\angle AXC$ is 90° (angle in a semicircle)

$\therefore$ $AX \perp BC.$

8(b)

$x^3 + ax^2 + bx + c = 0$

Let the roots be $\alpha, \beta, \alpha+\beta$.

$\sum \alpha_i = -b/a$

$\therefore \alpha + \beta = -a/2$ —— (1) (2)

$\sum_{i \neq j} \alpha_i \alpha_j = b$

i.e. $\alpha\beta + (\alpha^2 + \alpha\beta) + \alpha\beta + \beta^2 = b$,

$\therefore (\alpha+\beta)^2 + \alpha\beta = b$

$\Longrightarrow \alpha\beta = b - \frac{a^2}{4}$ —— (2)

$\alpha\beta(\alpha+\beta) = -c$ (2) —— (3)

$\therefore \left(b - \frac{a^2}{4}\right)\left(-\frac{a}{2}\right) = -c$. $\boxed{4}$

simplify $a^3 - 4ab + 8c = 0$.

8(c). $y = x^{-2/3}$, $\frac{dy}{dx} = -\frac{2}{3} \cdot \frac{1}{x^{5/3}}$

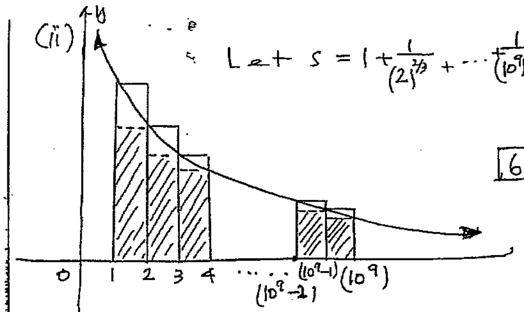
for $x > 0$ $\frac{dy}{dx} < 0$ (1)

$\frac{d^2y}{dx^2} = \frac{10}{9} x^{-8/3}$, $x > 0$ $f''(x) > 0$.

$y \longrightarrow \infty$, As $x \longrightarrow 0$

$y = (x-1)^{-2/3}$, $x > 1$ shift 1 unit to the right

(ii)



Let $S = 1 + \frac{1}{(2)^{2/3}} + \cdots + \frac{1}{(10^9)^{2/3}}$ $\boxed{6}$

Lower sum of rectangles.

$\frac{1}{2^{2/3}} + \frac{1}{3^{2/3}} + \cdots + \frac{1}{(10^9)^{2/3}} < \int_1^{10^9} x^{-2/3}\,dx$ (3)

Upper sum of rectangles.

$1 + \frac{1}{(2)^{2/3}} + \cdots + \frac{1}{(10^9-1)^{2/3}} > \int_1^{10^9} x^{-2/3}\,dx$

Now $\int_1^{10^9} x^{-2/3}\,dx = 2997$. (2)

$\therefore$ $S - 1 < 2997$, $\Rightarrow S < 2998$

$S - \frac{1}{(10^9)^{2/3}} > 2997$, $\Rightarrow S > 2997 + \frac{1}{(10^9)^{2/3}}$

$\therefore$ S is between 2997 and 2998.

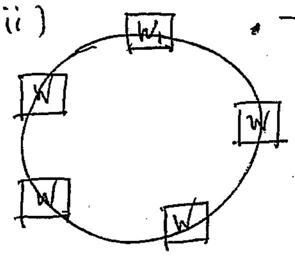
Extension (2) Q (8) Solutions

(a) Five women 4 men (1)

5 W 4 m

(i) $9!/9 = 8!$ (40320)

(ii)



• The men can seat themselves $5 \times 4 \times 3 \times 2$ ways.

$= 5!$ (2)

• 1 choice for W_1

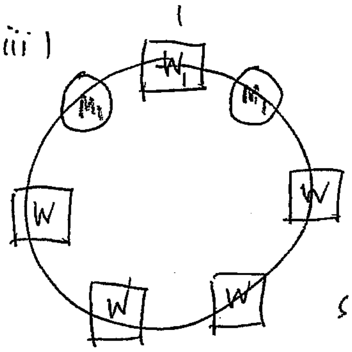
• and the women can arrange themselves $4 \times 3 \times 2 \times 1$ ways

$\therefore$ The no. of ways that no two men are to be seated together

$= 5! \times 4! = 5 \times 24^2$

$= [2880]$

(iii)



• Fix W_1 as reference, 1 way.

• Her husband M_1 can be seated 2 ways (on either side of W_1)

• The rest of 3 men can be seated $4 \times 3 \times 2$ ways!

i.e married couple are together with the condition that no two men are to be seated together

$= 1 \times 2 \times 4! \times 4!$ ways.

$= 1152$

$\therefore$ probability $= \frac{1152}{2880} = \frac{2}{5}$. (2)

$\boxed{5}$