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SCEGGS Darlinghurst

Term 1, 2004
Tuesday, 24 February

Extension 2 Mathematics

Task Weighting: 25%

This is excellent work, Sisi!! You should be really pleased!!

General Instructions

- Time allowed - 70 minutes
- Write your name at the top of each page
- Start each question on a new page
- Attempt **all** questions
- Marks may be deducted for careless or badly arranged work
- Mathematical templates and geometrical equipment may be used
- Approved scientific calculators should be used
- A table of standard integrals is provided

	Com	Reas	
Question 1	2 /2	3 /3	15 /15
Question 2	6 /6	1 /2	14 /15
Question 3	2 /2	5 /7	13 /15
TOTAL	10 /10	9 /12	42 /45

Rank = 1

Question 1 (15 marks) **Marks**

(a) Find values of A, B and C such that: 2

$$\frac{x+7}{(x^2+1)(x+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1}$$

(b) The polynomial $P(z)$ is defined by: 3

$$P(z) = 6z^4 - 7z^3 + z^2 + 12z - 2$$

Given that $\underline{z-1+i}$ is a factor of $P(z)$, express $P(z)$ as a product of its complex linear factors.

(c) (i) Suppose that $x = \alpha$ is a double root of the polynomial equation 2

$P(x) = 0$. Show that $P'(\alpha) = 0$.

(ii) The polynomial $Q(x) = mx^7 + nx^6 + 1$ is divisible by $(x+1)^2$. 2
Find the values of m and n where m and n are real numbers.

a c d

(d) Let α, β and γ be the roots of $x^3 - 5x + 7 = 0$.

(i) Find the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ 2

(ii) Find $\alpha^3 + \beta^3 + \gamma^3$ 1

(iii) Find an equation whose roots are $2\alpha - 1$, $2\beta - 1$ and $2\gamma - 1$ 2

(iv) Hence, or otherwise, find the value of 1

$$(2\alpha-1)(2\beta-1) + (2\beta-1)(2\gamma-1) + (2\alpha-1)(2\gamma-1)$$

Question 2 (15 marks) **Start a new page** **Marks**

(a) (i) Express $z = -1 - \sqrt{3}\, i$ in modulus-argument form. **2**

(ii) Show that $z^7 - 64z = 0$ **3**

(b) Express $\sqrt{12 - 5i}$ in the form $a + ib$ where a and b are real numbers. **4**

(c) Let ω be a complex root of $z^3 = 1$

(i) Show that $\omega^2 + \omega + 1 = 0$ **1**

(ii) Hence simplify $(1 + \omega)^8$ **2**

(d) Prove, for all integers $n \geq 1$, that $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$ **3**

Question 3 (15 marks) **Start a new page** **Marks**

(a) (i) **On the same diagram,** draw a neat sketch of the locus represented by: **2**

1. $\left| z-(5+4i) \right| = 4$

2. $\left| z+4 \right| = \left| z-6 \right|$

(ii) Hence write down all values of z which simultaneously satisfy **1**

$$\left| z-(5+4i) \right| = 4$$

and $\left| z+4 \right| = \left| z-6 \right|$

(iii) Use your diagram in (i) to determine the values of k for which the simultaneous equations: **1**

$\left| z-(5+4i) \right| = 4$ and $\left| z-4i \right| = k$

have **exactly** one solution.

(b) Prove that:

(i) $\overline{z_1} + \overline{z_2} = \overline{z_1 + z_2}$ **1**

(ii) $\left| z_1 z_2 \right| = \left| z_1 \right| \left| z_2 \right|$ **2**

(c) Solve $x^2 - 3ix + 4 = 0$ **2**

Question 3 continues on the next page

Question 3 (continued) **Marks**

(d) On an Argard diagram, where O is the origin, and $z = a + ib$:

OP	represents	z
OQ	represents	iz
OR	represents	$i^2 z$ $-z$ $-a - ib$
OS	represents	$-iz$ $-iz$

(i) Draw a diagram showing this information. **2**

(ii) Answer TRUE or FALSE for the following statements, giving a brief reason for your answer.

1. O, P and R are collinear. **2**

2. $\overrightarrow{RS}$ must represent a real number. **2**

End of Assessment

EXT 2 MATHS

ASSESSMENT TASK 1 SOLUTIONS

February 2004.

QUESTION 1 : (15 marks)

(a) $$\frac{x+7}{(x^2+1)(x+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1}$$

$\therefore \quad x+7 = (Ax+B)(x+1) + C(x^2+1)$

$x=-1: \quad 6 = 2C \quad \Rightarrow \quad C = 3$ ✓

Equating coeff of x^2: $0 = A + C \Rightarrow A = -3$

Equating constants : $7 = B + C$

$\therefore B = 4$

✓

$$\therefore \frac{x+7}{(x^2+1)(x+1)} = \frac{-3x+4}{x^2+1} + \frac{3}{x+1}$$

2

(b) Since $1-i$ is a root, so is $1+i$

$\therefore (z-(1-i))(z-(1+i))$ is also a factor of $P(z)$

ie $z^2 - 2z + 2 \mid P(z)$ ✓

$\therefore P(z) = (z^2 - 2z + 2)(6z^2 + 5z - 1)$ ✓

$= (z-(1-i))(z-(1+i))(6z-1)(z+1)$ ✓

3

(c) (i) If $x-\alpha$ is a double root of $P(x)$,

then $P(x) = (x-\alpha)^2\, Q(x)$ for some $Q(x)$.

✓

$\therefore P'(x) = Q(x).\,2(x-\alpha) + (x-\alpha)^2 Q'(x)$.

$= (x-\alpha)\left(2Q(x) + (x-\alpha)Q'(x)\right)$

$\therefore P'(\alpha) = 0 \times \left(2Q(x) + (x-\alpha)Q'(x)\right)$ ✓

$= 0.$

Com 2

(ii) $Q'(x) = 7mx^6 + 6nx^5$

Since $(x+1)^2$ is a factor of $Q(x)$

$x=-1$ is a double root.

$\therefore Q'(-1) = 0$

ie $0 = 7m - 6n \qquad (1)$ ✓

Also $Q(-1) = 0$

$0 = -m + n + 1 \qquad (2)$

Solving (1) and (2) simultaneously

$n + 1 = \frac{6}{7}n$

$\therefore n = -7$ ✓

and $m = -6$

2

(d) $x^3 - 5x + 7 = 0$

(i) $$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \beta\gamma + \alpha\gamma}{\alpha\beta\gamma}$$ ✓

$$= \frac{-5}{-7} = \frac{5}{7}$$ ✓

(ii) $\alpha^3 = 5\alpha - 7$

$\beta^3 = 5\beta - 7$

$\gamma^3 = 5\gamma - 7$

$\therefore \alpha^3 + \beta^3 + \gamma^3 = 5(\alpha + \beta + \gamma) - 21$ ✓

$= -21.$

(iii) Replace x with $\frac{1}{2}(y+1)$: ✓

$\frac{1}{8}(y+1)^3 - 5.\frac{1}{2}(y+1) + 7 = 0$

$(y+1)^3 - 20(y+1) + 56 = 0$

$y^3 + 3y^2 - 17y + 37 = 0$ ✓ Reas 2

(iv) $(2\alpha-1)(2\beta-1) + (2\beta-1)(2\gamma-1) + (2\alpha-1)(2\gamma-1)$ is the sum of the roots in pairs

$= -17.$ ✓ Reas 1

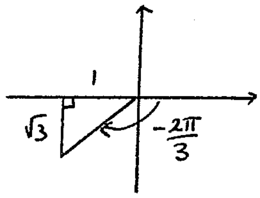
6

QUESTION 2: (15 marks)

(a)

(i) $|z| = 2$

$\arg z = -\frac{2\pi}{3}$



$\therefore z = 2\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)$

(ii) $z^7 - 64z$

$= 2^7\left[\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right]^7 - 64\left[2\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)\right]$

$= 128\left[\cos\left(-\frac{14\pi}{3}\right) + i\sin\left(-\frac{14\pi}{3}\right)\right] - 128\left[\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right]$

$= 128\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) - 128\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$

$= 0$ as required.

Com $\frac{}{3}$

5

(b) Let $z = \sqrt{12-5i}$

$\therefore z^2 = 12 - 5i$

Since $z = a + ib$ where $a, b \in \mathbb{R}$

$a^2 - b^2 + 2abi = 12 - 5i$

$\therefore$ Equating real + imaginary pts:

$a^2 - b^2 = 12$ (1)

$2ab = -5$ (2)

Solving simultaneously, $b = \frac{-5}{2a}$

$\therefore a^2 - \frac{25}{4a^2} = 12$

$4a^4 - 48a^2 - 25 = 0$

$(2a^2 - 25)(2a^2 + 1) = 0$

Since $a \in \mathbb{R}$, $a = \pm\frac{5}{\sqrt{2}}$

$\therefore b = \mp\frac{1}{\sqrt{2}}$

$\therefore$ Square roots of $12 - 5i$ are

$\frac{5}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$ and $-\frac{5}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$

4

(c) (i) $z^3 = 1$

$\therefore z^3 - 1 = 0$

$\therefore (z-1)(z^2 + z + 1) = 0$

Since ω is a complex root,

$\omega \neq 1$

$\therefore \omega$ is a solution of $z^2 + z + 1 = 0$

ie $\omega^2 + \omega + 1 = 0$.

(ii) $(1+\omega)^8$

$= (-\omega^2)^8$

$= \omega^{16}$

Reas $\frac{}{2}$

But $\omega^3 = 1$

$\therefore \omega^{16} = \omega$.

3

(d) By induction,

$n=1$: LHS $= \cos\theta + i\sin\theta$

RHS $= \cos\theta + i\sin\theta$

$= $ LHS

Com $\frac{}{3}$

$\therefore$ True for $n = 1$.

Assume true for $n = p$

ie $(\cos\theta + i\sin\theta)^p = \cos(p\theta) + i\sin(p\theta)$

Investigate $n = p+1$:

$(\cos\theta + i\sin\theta)^{p+1} = (\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)^p$

$= (\cos\theta + i\sin\theta)(\cos(p\theta) + i\sin(p\theta))$

$= \cos\theta\cos(p\theta) - \sin\theta\sin(p\theta) + i(\sin\theta\cos(p\theta) + \cos\theta\sin(p\theta))$

$= \cos(p\theta + \theta) + i\sin(p\theta + \theta)$

$= \cos(p+1)\theta + i\sin(p+1)\theta$

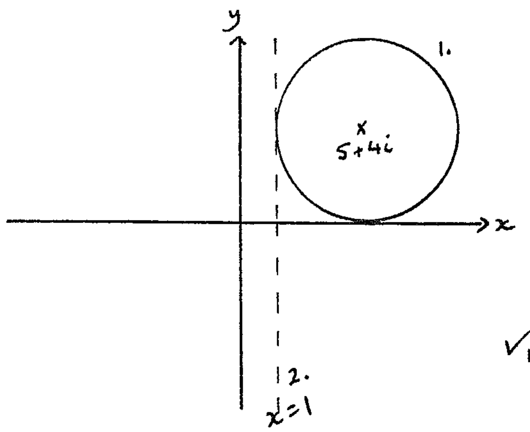
$\therefore$ If true for $n=p$, also true for $n=p+1$

Since true for $n=1$ also true for $n = 2, 3, 4, \ldots$ and hence all positive integers by PMI.

3

QUESTION 3: (15 marks)

(a) (i)



1. Circle centred $5+4i$, radius 4.

2. $|z+4| = |z-6|$

$$\sqrt{(x+4)^2+y^2} = \sqrt{(x-6)^2+y^2}$$

$$8x+16 = -12x+36$$

$$20x = 20$$

$$x = 1$$

(ii) $1+4i$

(iii) $k = 1$ OR 9

$\boxed{4}$

(b) Let $z_1 = x_1+iy_1$, $z_2 = x_2+iy_2$ $\quad x_1, x_2, y_1, y_2 \in \mathbb{R}$

(i)
$$\bar{z}_1 + \bar{z}_2 = x_1 - iy_1 + x_2 - iy_2$$
$$= (x_1+x_2) - i(y_1+y_2)$$
$$= \overline{z_1+z_2}$$

(ii)
$$|z_1 z_2| = |x_1x_2 + x_1y_2 i + x_2y_1 i - y_1y_2|$$
$$= \sqrt{(x_1x_2-y_1y_2)^2 + (x_1y_2+x_2y_1)^2}$$

$$|z_1||z_2| = \sqrt{x_1^2+y_1^2} \times \sqrt{x_2^2+y_2^2}$$
$$= \sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)}$$
$$= \sqrt{(x_1^2x_2^2 + x_1^2y_2^2 + x_2^2y_1^2 + y_1^2y_2^2)}$$
$$= \sqrt{(x_1x_2-y_1y_2)^2 + \cancel{2x_1x_2y_1y_2} + (x_1y_2+x_2y_1)^2 - \cancel{2x_1x_2y_1y_2}}$$
$$= |z_1 z_2|$$

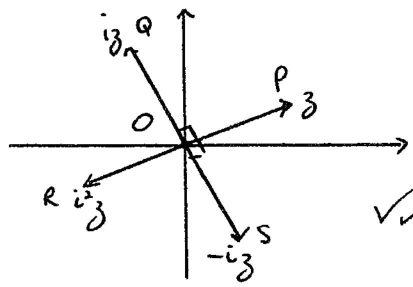
(Reas) /3

$\boxed{3}$

(c)
$$x^2 - 3ix + 4 = 0$$
$$x = \frac{3i \pm \sqrt{9i^2 - 4.1.4}}{2}$$
$$= \frac{3i \pm \sqrt{-25}}{2}$$
$$= \frac{3i \pm 5i}{2}$$
$$= 4i \text{ OR } -i$$

$\boxed{2}$

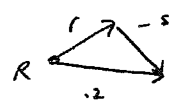
(d) (i)



(Com)

(ii) 1. TRUE.
multiplying z by i is equivalent to rotation through $90°$.
Since $R = i^2 z$, it is P rotated through $90°$ twice ie $180°$.
$\therefore$ R, P and O lie on one line.

2. FALSE



(Reas) /4

$$\vec{RS} = -iz - i^2 z$$
$$= -iz + z$$
$$= -i(a+ib) + a + ib$$
$$= (a+b) + i(b-a)$$

which is only real iff $a = b$.
which is not true for all $z \in \mathbb{C}$.

$\boxed{6}$