Centre Number

Student Number

SCEGGS Darlinghurst

2009

HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Mathematics Extension 1

This is a TRIAL PAPER only and does not necessarily reflect the content or format of the Higher School Certificate Examination for this subject.

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 84

- Attempt Questions 1–7
- All questions are of equal value

STANDARD INTEGRALS

$\int x^n \, dx$	$= \frac{1}{n+1} x^{n+1}, \quad n \neq 1; \; x \neq 0, \text{ if } n < 0$
$\int \frac{1}{x} \, dx$	$= \ln x, \quad x > 0$
$\int e^{ax} \, dx$	$= \frac{1}{a} e^{ax}, \quad a \neq 0$
$\int \cos ax \, dx$	$= \frac{1}{a} \sin ax, \quad a \neq 0$
$\int \sin ax \, dx$	$= -\frac{1}{a} \cos ax, \quad a \neq 0$
$\int \sec^2 ax \, dx$	$= \frac{1}{a} \tan ax, \quad a \neq 0$
$\int \sec ax \tan ax \, dx$	$= \frac{1}{a} \sec ax, \quad a \neq 0$
$\int \frac{1}{a^2 + x^2} \, dx$	$= \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$
$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx$	$= \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$
$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx$	$= \ln \left(x + \sqrt{x^2 - a^2} \right) \quad x > a > 0$
$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx$	$= \ln \left(x + \sqrt{x^2 + a^2} \right)$

Total marks – 84
Attempt Questions 1–7
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (12 marks) Use a SEPARATE writing booklet.

(a) Find $\lim_{x \to 0} \frac{\sin 3x}{5x}$ **1**

(b) Find the co-ordinates of the point P which divides AB externally in the ratio $2:3$ where $A(1,-4)$ and $B(6,9)$. **2**

(c) Solve for x: **3**

$$\frac{4}{x-1} \geq 1$$

(d) The angle between two lines $y = mx$ and $y = \frac{1}{3}x$ is $\frac{\pi}{4}$. **2**

Find the exact values of m.

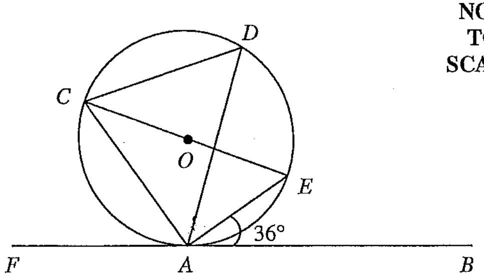
(e) If α, β and γ are the roots of $x^3 - 3x + 5 = 0$ find the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$. **2**

(f) Use the table of standard integrals to find $\int \sec 2x \tan 2x \, dx$. **2**

Marks

Question 2 (12 marks) Use a SEPARATE writing booklet.

(a)



NOT TO SCALE

FB is a tangent meeting a circle at A. CE is the diameter, O is the centre and D lies on the circumference. $\angle BAE = 36°$.

(i) Find the size of $\angle ACE$, giving reasons. **1**

(ii) Find the size of $\angle ADC$, giving reasons. **2**

(b) Find $\int \frac{dx}{\sqrt{25 - 4x^2}}$ **2**

(c) (i) If $\sin x - \sqrt{3} \cos x = R \sin(x - \alpha)$ find R and α where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. **2**

(ii) Find the general solution for $\sin x - \sqrt{3} \cos x = \sqrt{2}$ (leave your answer in exact form). **2**

(d) Colour blindness effects 5% of all men. What is the probability that any random sample of 20 men should contain:

(i) no colour blindness. **1**

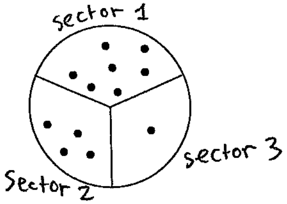
(ii) two or more colour blind men (to 3 decimal places). **2**

Marks

Question 3 (12 marks) Use a SEPARATE writing booklet.

(a) Evaluate $\int_0^{\frac{3}{2}} \sqrt{9-x^2}\, dx$ using the substitution $x = 3\sin\theta$. **3**

(b) Twelve points lie inside a circle. No three points are collinear. Seven of the points lie in sector 1, four lie in sector 2 and the other point lies in sector 3.



(i) Show that 220 triangles can be made using these points. **1**

(ii) One triangle is chosen at random from all possible triangles. Find the probability that the triangle chosen has one vertex in each sector. **1**

(iii) Find the probability that the vertices of the triangle chosen all lie in the same sector. **1**

(c) (i) Sketch the graph of the function $f(x) = e^x - 2$. **1**

(ii) On the same diagram sketch the graph of the inverse function $f^{-1}(x)$. **1**

(iii) State the equation of the function $f^{-1}(x)$. **1**

(iv) Explain why the x co-ordinate of any point of intersection of the graphs $y = f(x)$ and $y = f^{-1}(x)$ satisfies the equation $e^x - x - 2 = 0$. **1**

(v) One root of the equation $e^x - x - 2 = 0$ lies between $x = 1$ and $x = 2$. Use one application of Newton's method, with a starting value of $x = 1.5$, to approximate the root, to 2 decimal places. **2**

Marks

Question 4 (12 marks) Use a SEPARATE writing booklet.

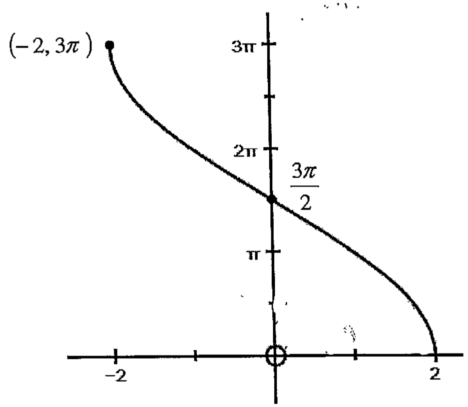
(a) $P(4p, 2p^2)$ is a variable point on the parabola $x^2 = 8y$. The normal at P cuts the y-axis at A and R is the midpoint of AP.

(i) Show that the normal at P has equation $x + py = 4p + 2p^3$. **2**

(ii) Show that R has co-ordinates $(2p, 2p^2 + 2)$. **2**

(iii) Show that the locus of R is a parabola and find its vertex and focus. **3**

(b) The graph of $y = a\cos^{-1} bx$ is drawn below.



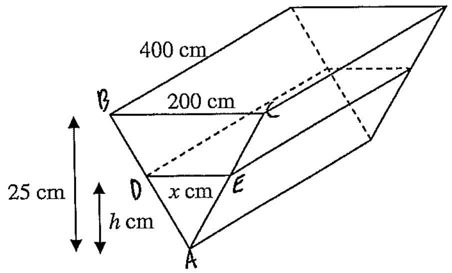
(i) Find a and b. **2**

(ii) Find the exact area bound by the curve and the y-axis for $0 \le y \le \frac{\pi}{2}$. **3**

Marks

Question 5 (12 marks) Use a SEPARATE writing booklet.

(a)



An open, flat topped water trough is in the shape of a triangular prism.

Its rectangular top measures 200 cm by 400 cm and its triangular cross-section has a vertical height of 25 cm.

When the water depth is h cm the water surface measures x cm by 400 cm.

(i) Show that when the water depth is h cm the volume V cm^3 of water in the trough is given by $V = 1600h^2$. **2**

Water is being emptied through a hole in its base at a constant rate of 16 L per second.

(ii) Find the rate at which the depth of water is changing when $h = 10$ cm. **3**

Question 5 continues on page 7

Marks

Question 5 (continued)

(b) After cooking her cheesecake, Donna puts it in the fridge. The fridge is running at a constant temperature of 8° C. At time t minutes the temperature T of the cheesecake decreases according to the equation:

$$\frac{dT}{dt} = -k\,(T-8) \text{ where } k \text{ is a positive constant.}$$

Donna puts the cheesecake in the fridge at 9.00am when its temperature is 85° C.

(i) Show that $T = 8 + 77e^{-kt}$ satisfies both this equation and the initial conditions. **2**

(ii) Donna checks the temperature of the cheesecake at 10.00am and it is 40° C. **3**

It is best served when it reaches a temperature of 10° C.

At what time (to the nearest minute) should Donna serve the cheesecake?

(c) In the expansion of $(1+ax)^{10}$, the coefficient of x^6 is twice the coefficient of x^7. **2**

Find the value of a.

End of Question 5

Marks

Question 6 (12 marks) Use a SEPARATE writing booklet.

(a) (i) In the expansion of $(a+b)^{20}$ show that $\frac{T_{n+1}}{T_n}$ is given by: **2**

$$\frac{21-n}{n}\,\frac{b}{a}$$

$\left(\text{where } T_{n+1} = {}^{20}C_n\, a^{20-n}\, b^n\right)$

(ii) In the game of craps, 2 dice are thrown and the score is recorded as the sum of the uppermost faces of the dice.

α) Find the probability that a score of 7 is recorded. **1**

β) If two dice are rolled 20 times, what is the most probable number of scores of 7 thrown? Calculate the probability that this occurs. **3**

(b) (i) Use the method of mathematical induction to show that if x is a positive integer then $(1+x)^n - 1$ is divisible by x for all positive integers $n \geq 1$. **3**

(ii) Factorise $12^n - 4^n - 3^n + 1$. **1**

(iii) Without using the method of mathematical induction, deduce that $12^n - 4^n - 3^n + 1$ is divisible by 6 for all positive integers $n \geq 1$. **2**

Marks

Question 7 (12 marks) Use a SEPARATE writing booklet.

(a) (i) Show that in the expansion of $\left(x-\frac{1}{x}\right)^{2n}$, the term independent of x is $(-1)^n \times {}^{2n}C_n$. **2**

(ii) Show that $\left(1+x\right)^{2n}\left(1-\frac{1}{x}\right)^{2n} = \left(x-\frac{1}{x}\right)^{2n}$ **1**

(iii) Deduce that:

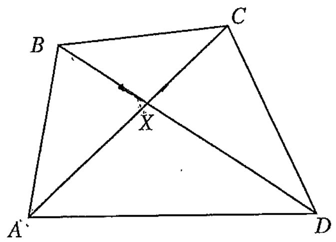
$$\left({}^{2n}C_0\right)^2 - \left({}^{2n}C_1\right)^2 + \left({}^{2n}C_2\right)^2 - \ldots + \left({}^{2n}C_{2n}\right)^2 = (-1)^n \times {}^{2n}C_n$$ **2**

Question 7 continues on page 10

Marks

Question 7 (continued)

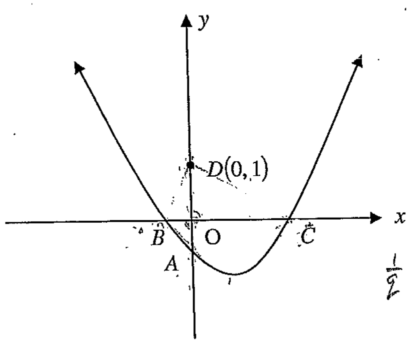
(b) (i) **4**



In the diagram above, the diagonals of quadrilateral $ABCD$ intersect at X.

Show that if $AX \,.\, XC = BX \,.\, XD$, $ABCD$ is a cyclic quadrilateral.

(ii)



Consider the parabola $y = x^2 + px - q$, where $q > 0$.
Let the parabola intercept the y-axis at A and the x-axis at the distinct points B and C.
D is the point $(0,1)$

α) Find the co-ordinates of B and C. **1**

β) Show that $ABDC$ is a cyclic quadrilateral. **2**

End of paper

Extension 1 Mathematics Trial HSC 2009 - Solutions

Q1 a) $\lim_{x\to 0}\frac{\sin 3x}{5x} = \frac{3}{5}\lim_{x\to 0}\frac{\sin 3x}{3x}$

Reas - 3
Calc - 2

$= \frac{3}{5}\times 1$ ✓

$= \frac{3}{5}$

Done well

b) $A(1,-4)$ $B(6,9)$

$-2:3$

$x = \frac{3\times 1 + -2\times 6}{-2+3}$ $y = \frac{3\times -4 + -2\times 9}{-2+3}$ ✓

$= \frac{-9}{1}$ $= \frac{-30}{1}$

$= -9$ $= -30$

$P(-9,-30)$ ✓

Those who used the 'cross' method were more successful than those who used the formula.

c) $(x-1)^2 \frac{4}{x-1} \geq 1\,(x-1)^2$

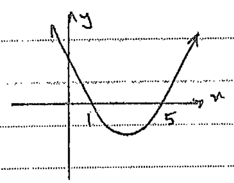
$4(x-1) \geq x^2-2x+1$

$4x-4 \geq x^2-2x+1$

$0 \geq x^2-6x+5$ ✓

Sketch $y = x^2-6x+5$

$= (x-5)(x-1)$



$1 < x \leq 5$ ✓ ✓

Reas 3

Silly algebraic mistakes were made here. Take your time with Q1

Also $x \neq 1$

d) $\left|\frac{m-\frac{1}{3}}{1+m\times\frac{1}{3}}\right| = \tan\frac{\pi}{4}$

$\frac{|m-\frac{1}{3}|}{|1+\frac{m}{3}|} = 1$ ✓

$|m-\frac{1}{3}| = |1+\frac{m}{3}|$

You need to remember exact values of trig ratios

i.e $\tan\frac{\pi}{4} = 1$ not $\frac{1}{\sqrt{2}}$

$m-\frac{1}{3} = 1+\frac{m}{3}$ or $m-\frac{1}{3} = -1-\frac{m}{3}$

$3m-1 = 3+m$ $3m-1 = -3-m$

$2m = 4$ $4m = -2$

$m = 2$ $m = -\frac{1}{2}$ ✓

e) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{\alpha\beta+\beta\gamma+\alpha\gamma}{\alpha\beta\gamma}$ ✓

$= \frac{-3}{-5}$

$= \frac{3}{5}$ ✓

Done well

f) $\int \sec 2x \tan 2x\, dx = \frac{1}{2}\sec 2x + c$ ✓ ✓

Calc - 2

Done well

Q2 a) i) $\angle ACE = 36°$ (angle between tangent and chord equals angle in alternate segment) ✓

Reas 3
Comm 3
Calc 2

ii) $\angle CAE = 90°$ (angle in a semi-circle) ✓

$\angle FAC = 180-90-36$ (angle sum of a st. line)

$= 54°$

$\angle ADC = 54°$ (angle between tangent and chord equals angle in alternate segment) ✓

Comm - 3

- Be careful of typos
- There is no such thing as "angles in alternate segments are equal".

b) $\int \frac{dx}{\sqrt{25-4x^2}} = \frac{1}{2}\int\frac{dx}{\sqrt{\frac{25}{4}-x^2}}$

$= \frac{1}{2}\sin^{-1}\left(\frac{2x}{5}\right)+c$ ✓ ✓

Calc - 2

c) i) $\sin x - \sqrt{3}\cos x = R\sin x\cos\alpha - R\cos x\sin\alpha$

$\therefore R\sin\alpha = \sqrt{3}$ ... ①

$R\cos\alpha = 1$ ... ②

① ÷ ② $\tan\alpha = \sqrt{3}$

$\alpha = \frac{\pi}{3}$ ✓

$①^2 + ②^2 = R^2\sin^2\alpha + R^2\cos^2\alpha = (\sqrt{3})^2 + 1^2$

$R^2 = 4$

$R = 2$ ✓ as $R > 0$

$\therefore \sin x - \sqrt{3}\cos x = 2\sin(x - \frac{\pi}{3})$

ii) $\sin x - \sqrt{3}\cos x = \sqrt{2}$

$2\sin(x - \frac{\pi}{3}) = \sqrt{2}$

$\sin(x - \frac{\pi}{3}) = \frac{\sqrt{2}}{2}$

$x - \frac{\pi}{3} = n\pi + (-1)^n\frac{\pi}{4}$ ✓ for $\frac{\pi}{4}$

$x = n\pi + (-1)^n\frac{\pi}{4} + \frac{\pi}{3}$ ✓

You must do the general soln part first & then add over the $\pi/3$. Those who didn't use a formula had a high success rate.

d) prob of color blindness = 0.05

prob of not color blindness = 0.95

i) $P(X=0) = {}^{20}C_0(0.05)^0(0.95)^{20}$

$= 0.358$ ✓

ii) $P(X \geq 2) = 1 - [P(X=0) + P(X=1)]$ ✓

$= 1 - ((0.95)^{20} + {}^{20}C_1(0.05)(0.95)^{19})$

$= 0.264$ ✓ Reas - 3

This is a standard binomial probability that wasn't all that successful.

Reas - 3

Comm - 3

Calc - 5

Q3 a) $\int_0^{3/2} \sqrt{9 - x^2}\,dx$ $\quad x = 3\sin\theta \quad x = \frac{3}{2} \quad \theta = \frac{\pi}{6}$

$dx = 3\cos\theta\,d\theta \quad x = 0 \quad \theta = 0$

$= \int_0^{\pi/6} \sqrt{9 - 9\sin^2\theta}\,.\,3\cos\theta\,d\theta$ ✓

$= \int_0^{\pi/6} 3\cos\theta\,.\,3\cos\theta\,d\theta$

$= 9\int_0^{\pi/6} \cos^2\theta\,d\theta$ $\quad \cos 2\theta = 2\cos^2\theta - 1$

$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$

$= \frac{9}{2}\int_0^{\pi/6} 1 + \cos 2\theta\,d\theta$

This was done poorly by many students. This is a standard integral type so you need to revise it.

Students also need to learn how to $\int \cos^2\theta\,d\theta$

$= \frac{9}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/6}$ ✓

$= \frac{9}{2}\left[\left(\frac{\pi}{6} + \frac{1}{2}\sin\frac{\pi}{3}\right) - (0 + 0)\right]$

$= \frac{9}{2}\left(\frac{\pi}{6} + \frac{\sqrt{3}}{4}\right)$

$= \frac{3\pi}{4} + \frac{9\sqrt{3}}{8}$ ✓ Calc - 3

b) i) No of triangles $= {}^{12}C_3$

$= 220$ ✓

Done well

ii) No of triangles $= {}^7C_1 \times {}^4C_1$

$= 28$

$\therefore$ Prob $= \frac{28}{220} = \frac{7}{55}$ ✓

Some students didn't read the question carefully re: finding probability.

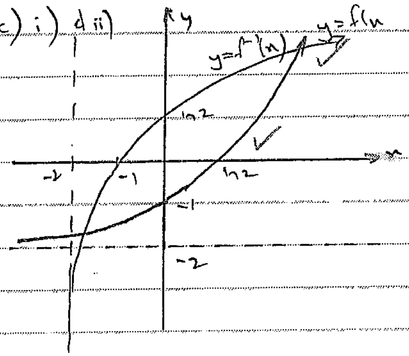
iii) No of triangles $= {}^7C_3 + {}^4C_3$

$= 35 + 4$

$= 39$

Prob $= \frac{39}{220}$ ✓ Reas - 3

c) i) & ii)



Comm - 1

Comm - 1

Done fairly well, although asymptotes were left out on many. Labelling x- and y intercepts would have been nice.

iii) $f(x): y = e^x - 2$

interchange x and y

$x = e^y - 2$

$x + 2 = e^y$

$y = \ln(x+2)$ ✓

Done well

iv) As the graph of $y=f^{-1}(x)$ is a reflection of the graph of $y=f(x)$ in the line $y=x$, points of intersection lie on $y=x$

$\therefore$ they satisfy $e^x-2=x$ ✓

$\therefore e^x-x-2=0.$ Comm - 1

The key point here is that $f(x)$ and $f^{-1}(x)$ intersect on $y=x$

v) $f(x)=e^x-x-2$

$f'(x)=e^x-1$ ✓

$\therefore x_2 = 1.5 - \dfrac{f(1.5)}{f'(1.5)}$

$= 1.5 - \dfrac{e^{1.5}-1.5-2}{e^{1.5}-1}$

$= 1.22$ (to 2 dec. pl.) ✓ Calc 2

Done very well.

Q4 a) i) $x^2=8y$

Reas - 5
Comm - 2
Calc - 3

$y=\dfrac{x^2}{8}$

$\dfrac{dy}{dx}=\dfrac{x}{4}$

$\therefore$ at P $m_{tang}=\dfrac{4p}{4}$ ✓
$=p.$

$\therefore m_{norm}=-\dfrac{1}{p}$

$\therefore y-2p^2=-\dfrac{1}{p}(x-4p)$ ✓

$py-2p^3=-x+4p$

$x+py=2p^3+4p.$ Comm - 2

ii) A: $x=0$ $\quad 0+py=2p^3+4p$

$y=2p^2+4$

$\therefore A(0, 2p^2+4)$ ✓ $\quad P(4p, 2p^2)$

$\therefore$ Midpt $\left(\dfrac{0+4p}{2}, \dfrac{2p^2+4+2p^2}{2}\right)$ ✓

$=(2p, 2p^2+2).$ Reas - 2

iii) $x=2p \qquad y=2p^2+2$

$p=\dfrac{x}{2}$ sub into y

$y=2\left(\dfrac{x}{2}\right)^2+2$ ✓

$y=\dfrac{2x^2}{4}+2$

$4y=2x^2+8$

$2y-4=x^2$

$x^2=2(y-2)$

$\therefore 4a=2 \quad a=\dfrac{1}{2} \quad$ vertex $(0,2)$ ✓

focus $(0, 2\tfrac{1}{2})$. ✓ Reas - 3

Many lost the last couple of marks because they couldn't do standard 2 unit work. Learn it!

b) i) $a=3$ ✓

$b=\dfrac{1}{2}$ ✓

ii) $y=3\cos^{-1}\dfrac{x}{2}$

$\dfrac{y}{3}=\cos^{-1}\dfrac{x}{2}$

$\dfrac{x}{2}=\cos\dfrac{y}{3}$

$x=2\cos\dfrac{y}{3}$

$\therefore A=\int_0^{\pi/2} 2\cos\dfrac{y}{3}\,dy$ ✓

$=2\left[3\sin\dfrac{y}{3}\right]_0^{\pi/2}$ ✓

$=6\left(\sin\dfrac{\pi}{6}-\sin 0\right)$

$=6\times\dfrac{1}{2}$

$=3$ units^2. ✓ Calc - 3

Be careful making x the subject: Many did many operations in the wrong order.

Q5 a) i) $V=Ah$

Reas - 7
Comm - 2
Calc - 3

$=\dfrac{1}{2}xh\times 400$

$=200xh$

now by similar triangles

$\dfrac{x}{h}=\dfrac{200}{25}$

$x=8h$ ✓

$\therefore V=200\times 8h\times h$ ✓ Reas - 2

$=1600h^2$

Similar triangle questions are quite popular so for those who didn't get this solution you need to do further practise.

ii) $\frac{dV}{dt} = -16\ L/s \qquad 1L = 1000cm^3$

$= -16000 cm^3/s$

find $\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$

$\frac{dV}{dh} = 3200h$. ✓

$\therefore \frac{dh}{dt} = \frac{1}{3200h} \times -16000$ ✓

$= \frac{-5}{h}$

$\therefore$ when $h = 10$

$\frac{dh}{dt} = \frac{-5}{10}$

$= -\frac{1}{2}$ cm/s. ✓ Calc - 3

Most students arrived at $\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$ but made a mistake with units.

i.e. $16L = 16000cm^3$

You can't mix units in these questions.

b) $LHS = \frac{dT}{dt} \qquad RHS = -k(T-8)$

$= -k(8 + 77e^{-kt} - 8)$

$= -k \times 77e^{-kt} \qquad = -k \times 77e^{-kt}$ ✓

$\therefore LHS = RHS$

$\therefore T = 8 + 77e^{-kt}$ satisfies diff. eqn.

when $t = 0 \quad T = 8 + 77e^{-k \times 0}$

$= 8 + 77$ ✓

$= 85°C$. Comm - 2

You need to be careful with show questions. You need to have more than $\frac{dT}{dt} = -k77e^{-kt}$

$= -k(T-8)$

Initial conditions was also missed by many

Reas - 8

Comm - 2

ii) $t = 60 \quad T = 40$

$\therefore 40 = 8 + 77e^{-k \times 60}$

$32 = 77e^{-k \times 60}$

$\frac{32}{77} = e^{-k \times 60}$

$-60k = \ln \frac{32}{77}$

$k = \frac{-\ln(32/77)}{60}$ ✓

find t when $T = 10$

Done very well

$10 = 8 + 77e^{-kt}$

$2 = 77e^{-kt}$

$\frac{2}{77} = e^{-kt}$

$-kt = \ln 2/77$

$t = \frac{\ln(\frac{2}{77})}{-k}$ ✓

$= 249.45\ldots$ minutes

$= 4$hrs 9min

1:09pm. ✓ Reas - 3

c) coeff of x^6 : ${}^{10}C_6 a^6$

coeff of x^7 : ${}^{10}C_7 a^7$

$\therefore 210a^6 = 2 \times 120 a^7$ ✓

$a = \frac{210}{240}$

$= \frac{7}{8}$ ✓ Reas - 2

Most common mistake was $2 \times 210a^6 = 120a^7$

Q6 a) i) $\frac{T_{n+1}}{T_n} = \frac{{}^{20}C_n a^{20-n} b^n}{{}^{20}C_{n-1} a^{20-(n-1)} b^{n-1}}$

$= \frac{\frac{20!}{(20-n)!\,n!} a^{20-n} b^n}{\frac{20!}{[20-(n-1)]!\,(n-1)!} a^{20-(n-1)} b^{n-1}}$ ✓

$= \frac{\frac{b}{n}}{\frac{a}{20-(n-1)}}$ ✓ Comm - 2

$= \frac{21-n}{n} \frac{b}{a}$

Alot of your working is in this second line, so when you are cancelling do not scribble all over it! Lots of fudging was picked up here.

ii) α) $P(7) = \frac{1}{6}$ ✓

β) $P(7) = {}^{20}C_n \left(\frac{5}{6}\right)^{20-n} \left(\frac{1}{6}\right)^n$

find n such that $\frac{T_{n+1}}{T_n} > 1$

$\therefore \frac{21-n}{n} \frac{\frac{1}{6}}{\frac{5}{6}} > 1$ from i). ✓

$\frac{21-n}{n} \times \frac{1}{5} > 1$

$\frac{21-n}{5n} > 1$

$21-n > 5n$ as $n > 0$

$6n < 21$

$n < \frac{21}{6}$

$\therefore n = 3$ ✓

$\therefore$ the most likely number of 7's thrown is 3

$P(7) = {}^{20}C_3 \left(\frac{5}{6}\right)^{17}\left(\frac{1}{6}\right)^3$ ✓ Reas - 3

$= 0.238$ (to 3 d.p.)

Reas - 7
Comm - 5

For some reason this question was skipped over by many. Those who did it, did it quite successfully.

b) i) show true for $n=1$

$(1+x)^1 - 1 = 1+x-1$

$= x$

which is divisible by x. ✓

assume true for $n=k$

$\therefore (1+x)^k - 1 = Mx$ where M is an integer

show true for $n=k+1$

i.e. $(1+x)^{k+1} - 1 = Qx$ where Q is an integer

$LHS = (1+x)^{k+1} - 1 = (1+x)(1+x)^k - 1$ ✓

$= (1+x)(Mx+1) - 1$ from above

$= Mx \cancel{+1} + Mx^2 + x \cancel{-1}$

$= x(M + Mx + 1)$ ✓

$= Qx$ as M and x are positive integers

$\therefore$ result is true for $n = k+1$ Reas - 3

$\therefore$ we have shown that if it is true for $n=k$ then the result is true for $n=k+1$. The result is true for $n=1$ $\therefore$ by the principle of Mathematical Induction the result is true for all positive n.

The induction is on n, not x.

A bit of fudging going on in this inductive step!

ii) $12^n - 4^n - 3^n + 1 = 3^n \times 4^n - 4^n - 3^n + 1$

$= 4^n(3^n - 1) - 1(3^n - 1)$

$= (3^n - 1)(4^n - 1)$ ✓

iii) $12^n - 4^n - 3^n + 1 = (3^n - 1)(4^n - 1)$

$= ((2+1)^n - 1)((3+1)^n - 1)$ ✓

now $(2+1)^n - 1$ is divisible by 2 from i)

& $(3+1)^n - 1$ is divisible by 3 from i) ✓

$\therefore (3^n - 1)(4^n - 1)$ is divisible by $2 \times 3 = 6$.

Reas - 2

DEDUCE - with a correct factorisation in part (ii) it should have been obvious what to deduce.

Q7 a) i) $\left(x - \frac{1}{x}\right)^{2n} = \sum_{r=0}^{2n} {}^{2n}C_r (x)^{2n-r} \left(-\frac{1}{x}\right)^r$

$= \sum_{r=0}^{2n} {}^{2n}C_r (x)^{2n-r} (-x)^{-r}$ ✓

$\therefore$ for the term independent of x:

$2n - r - r = 0$ ✓

$2r = 2n$

$r = n$.

$\therefore$ term is ${}^{2n}C_n (x)^{2n-n} (-x)^{-n}$

$= {}^{2n}C_n (-1)^n$ Reas - 2

Done fairly well

ii) $LHS = (1+x)^{2n}\left(1 - \frac{1}{x}\right)^{2n} = \left[(1+x)\left(1 - \frac{1}{x}\right)\right]^{2n}$

$= \left(1 - \frac{1}{x} + x - 1\right)^{2n}$ ✓

$= \left(x - \frac{1}{x}\right)^{2n}$ Comm - 1

$= RHS$.

Easy mark thrown away by many because they thought it looked hard

iii) expand $(1+x)^{2n}\left(1 - \frac{1}{x}\right)^{2n}$

$= \left({}^{2n}C_0 + {}^{2n}C_1 x + {}^{2n}C_2 x^2 + \dots + {}^{2n}C_{2n} x^{2n}\right)$

$\times \left({}^{2n}C_0 - {}^{2n}C_1\left(\frac{1}{x}\right) + {}^{2n}C_2\left(\frac{1}{x^2}\right) - \dots + {}^{2n}C_{2n}\frac{1}{x^{2n}}\right)$ ✓

$= \dots + {}^{2n}C_0 \times {}^{2n}C_0 - {}^{2n}C_1 x \, {}^{2n}C_1\left(\frac{1}{x}\right) + \dots + {}^{2n}C_{2n} x^{2n} \times {}^{2n}C_{2n}\frac{1}{x^{2n}} \dots$

$\therefore$ terms independent of x in this expansion are:

$\left({}^{2n}C_0\right)^2 - \left({}^{2n}C_1\right)^2 + \left({}^{2n}C_2\right)^2 - \dots + \left({}^{2n}C_{2n}\right)^2$

Many students didn't know which terms to multiply together.

now $(1+x)^{2n}\left(1-\frac{1}{x}\right)^{2n} = \left(x-\frac{1}{x}\right)^{2n}$

and the term independent of x in $\left(x-\frac{1}{x}\right)^{2n}$ is $(-1)^n\,{}^{2n}C_n$ ✓

Reas - 2

$\therefore$ equating terms

$$\left({}^{2n}C_0\right)^2 - \left({}^{2n}C_1\right)^2 + \left({}^{2n}C_2\right)^2 - \dots + \left({}^{2n}C_{2n}\right)^2 = (-1)^n\,{}^{2n}C_n$$

b) i) In $\triangle AXB$ and $\triangle DXC$

$\frac{AX}{DX} = \frac{XB}{XC}$ (as $AX.XC = BX.XD$) ✓

$\angle BXA = \angle CXD$ (vert. opp angles are eq.) ✓

$\therefore \triangle AXB \,|||\, \triangle DXC$ (two pairs of sides in the same ratio and the included angle is equal) ✓

$\therefore \angle ABX = \angle DCX$ (corr. angles in similar triangles)

$\therefore$ ABCD is a cyclic quadrilateral as angles in the same segment are equal. ✓

Comm - 4

Most students who attempted this question knew the concepts involved but many struggled with presenting a formal and logical proof.

ii) α) B and C are x-intercepts $\therefore y=0$

$\therefore x^2+px-q=0$

$x = \frac{-p \pm \sqrt{p^2-4.1.-q}}{2\times 1}$

$= \frac{-p \pm \sqrt{p^2+4q}}{2}$

$\therefore B\left(\frac{-p-\sqrt{p^2+4q}}{2}, 0\right)$ & $C\left(\frac{-p+\sqrt{p^2+4q}}{2}, 0\right)$ ✓

$\therefore OB = \left|\frac{-p-\sqrt{p^2+4q}}{2}\right|$ $= \frac{p+\sqrt{p^2+4q}}{2}$

$OC = \left|\frac{-p+\sqrt{p^2+4q}}{2}\right|$ $= \frac{-p+\sqrt{p^2+4q}}{2}$

$\therefore OB \times OC = \frac{p+\sqrt{p^2+4q}}{2} \times \frac{-p+\sqrt{p^2+4q}}{2}$

$= \frac{-p^2 + p\sqrt{p^2+4q} - p\sqrt{p^2+4q} + \left(\sqrt{p^2+4q}\right)^2}{4}$

$= \frac{-p^2+p^2+4q}{4}$ ✓

$= q$

$OA = |q|$ $= q$

$OD = |1|$ $= 1$

$\therefore OA \times OD = q \times 1$ $= q$ ✓

$\therefore OB \times OC = OA \times OD$

$\therefore$ ABDC is a cyclic quad.

Reas - 3

Done well for those students who knew what to do.

OB and OC are lengths so $OB = \left|\frac{-p-\sqrt{p^2+4q}}{2}\right|$.