Centre Number

1	5	3	7	2	6	4	8	

Student Number

SCEGGS Darlinghurst

2005
Higher School Certificate
Trial Examination

Mathematics Extension 2

This is a TRIAL PAPER only and does not necessarily reflect the content or format of the Higher School Certificate Examination for this subject.

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1–8
- All questions are of equal value

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Total marks – 120
Attempt Questions 1–8
All questions are of equal value

SCEGGS 2005 Ext.2 trial
Darlinghurst

Answer each question in a SEPARATE writing booklet.

Question 1 (15 marks) **Marks**

(a) Integrate:

(i) $\displaystyle\int \frac{e^x}{1+e^{2x}}\,dx$ **1**

(ii) $\displaystyle\int \frac{x^2}{x^2-9}\,dx$ **3**

(iii) $\int \sin^{-1} x\,dx$ using Integration by Parts. **3**

(b) Prove that $\displaystyle\int_0^{\frac{\pi}{3}} \frac{d\theta}{1+\sin\theta} = \sqrt{3}-1$ using the substitution $t=\tan\dfrac{\theta}{2}$. **4**

(c) Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \sin^3\theta\cos^3\theta\,d\theta$. **4**

Question 2 (15 marks) BEGIN A NEW BOOKLET **Marks**

(a) Consider the complex number $u = 1-\sqrt{3}i$.

(i) Express u in mod-arg form. **1**

(ii) Evaluate u^{6n} if n is an integer. **2**

(iii) On an Argand Diagram, sketch the curve $|z-u|=2$ showing important features. **2**

(iv) Find the maximum value of $|z|$ on the curve. **1**

(b)

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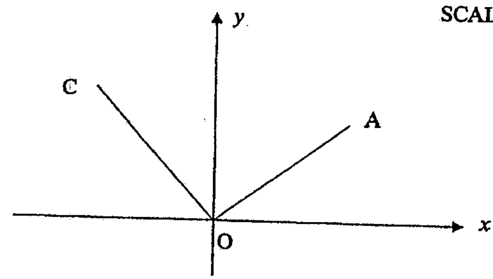

OABC is a square on an Argand Diagram where O is the origin.
The points A and C represent the complex numbers z and iz respectively.

(i) Find the complex number represented by B. **1**

(ii) The square is now rotated about O through $\dfrac{\pi}{4}$ in an anti-clockwise direction. **2**
Prove that the new position of B is given by the complex number $\sqrt{2}\,i\,z$.

Question 2 continues on page 4

Question 2 (continued) **Marks**

(c) In an Argand Diagram, an equilateral triangle has its vertices on the circle centre the origin, radius 2 units. One of the vertices is represented by the point whose argument is π.

(i) Find the 3 vertices in Cartesian form. **2**

(ii) Prove that the complex numbers represented by these vertices form the roots of the equation $z^3 + 8 = 0$. **3**

(iii) Find the area of the triangle. **1**

End of Question 2

Question 3 (15 marks) BEGIN A NEW BOOKLET **Marks**

(a) The curve $y = f(x) = \dfrac{\log_e x}{x}$ is shown below.

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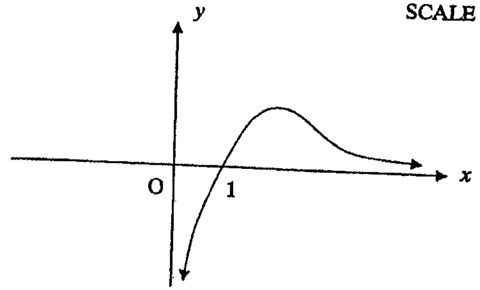


Given the maximum stationary point is $\left(e, \dfrac{1}{e}\right)$, sketch the following curves showing essential features, taking at least $\dfrac{1}{3}$ page for each.

(i) $y = |f(x)|$ **1**

(ii) $y = f(|x|)$ **1**

(iii) $y = f(x+1)$ **2**

(iv) $y = \dfrac{1}{f(x)}$ **3**

Question 3 continues on page 6

Question 3 (continued) **Marks**

(b) Consider the polynomial **3**

$$P(x)=x^4+2x^3+x^2-1$$

It is given that one zero is $\frac{-1}{2}+\frac{\sqrt{3}}{2}i$.

Find the other three zeros.

(c) Consider $f(x)=x^3-3cx$ (c is a constant).

(i) Prove that $f(x)=0$ has only one real root if $c<0$. **2**

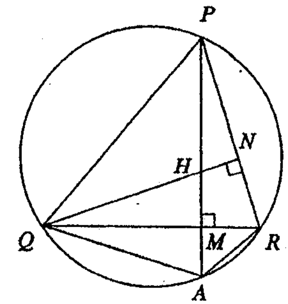
(ii) Prove that $x^3-3cx=k$ has 3 real different roots if: **3**

$$|k|<2c\sqrt{c}$$

End of Question 3

Question 4 (15 marks) BEGIN A NEW BOOKLET **Marks**

(a) **4**



NOT TO SCALE

P, Q, R and A lie on the circumference of a circle.
$PA \perp QR$ meeting QR at M.
$QN \perp PR$ meeting PA at H.
Let $\angle MQA = x°$.

Prove QR bisects HA.

(b) A solid has as its base the ellipse $\frac{x^2}{4}+y^2=1$ in the $x-y$ plane. Find the volume of the solid such that every cross section by a plane parallel to the y axis is a semi circle with its diameter in the $x-y$ plane. **4**

A diagram and a clear explanation should accompany your solution.

(c) Let $I_n=\int_0^{\frac{\pi}{2}} \sin^n x\,dx$ where n is a non-negative integer.

(i) Use Integration by Parts to prove that **3**

$$I_n=(n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}x\cos^2 x\,dx \quad \text{for } n\geq 2.$$

(ii) Deduce that $I_n=\frac{n-1}{n}I_{n-2}$ for $n\geq 2$. **2**

(iii) Evaluate I_4. **2**

Question 5 (15 marks) BEGIN A NEW BOOKLET **Marks**

(a) The equation $x^3 - x^2 - 3x + 5 = 0$ has roots α, β and γ.

(i) Find $\alpha + \beta + \gamma$. **1**

(ii) Find the equation whose roots are $2\alpha + \beta + \gamma$, $\alpha + 2\beta + \gamma$, $\alpha + \beta + 2\gamma$. **3**

(b) The points $P\left(2t, \frac{2}{t}\right)$ and $Q\left(2s, \frac{2}{s}\right)$ lie on the hyperbola $xy = 4$.

$$\left(t \neq 0, s \neq 0, t^2 \neq s^2\right)$$

(i) Prove that the equation of the tangent to the hyperbola at the point P is **2**

$$x + t^2 y = 4t.$$

(ii) Prove that the tangents at P and Q intersect at **2**

$$M\left(\frac{4st}{s+t}, \frac{4}{s+t}\right)$$

Suppose that $s = \frac{-1}{t}$

(iii) Prove that the locus of M is a straight line and state any conditions that may apply. **2**

(c) A tennis match between two players consists of a number of sets. The match continues until one of the players has won 3 sets.

Whenever Pat and John play, on average, for each set they play, there is a probability of $\frac{2}{3}$ that Pat wins and a probability of $\frac{1}{3}$ that John wins.

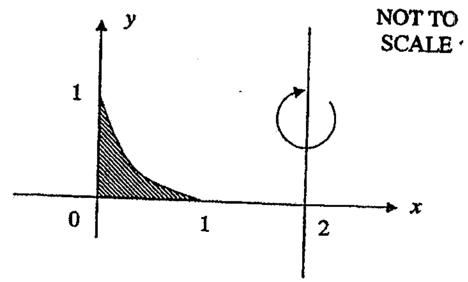
Find the probability that:

(i) Pat wins 2 of the first 3 sets. **1**

(ii) Pat wins the match. **4**

Question 6 (15 marks) BEGIN A NEW BOOKLET **Marks**

(a)



The region contained by the curve $y = (x-1)^2$ and the axes is rotated about the line $x = 2$.

(i) Taking slices perpendicular to the line of rotation prove that the volume obtained is **5**

$$\lim_{\delta y \to 0} \pi \sum_{y=0}^{1} \left(3 + 2\sqrt{y} - y\right) \delta y$$

(ii) Hence find this volume. **2**

(b) (i) Find the centre and radius of the circle $x^2 + y^2 + 4x = 0$. **2**

(ii) Prove that the line $y = mx + b$ will be a tangent to the circle if: **3**

$$4(mb + 1) = b^2$$

(iii) P is the point whose co-ordinates are $(k, 0)$. **3**
If P lies on the line $y = mx + b$ and is exterior to the circle, find possible values for k if the two tangents from P to the circle are perpendicular.

Marks

Question 7 (15 marks) BEGIN A NEW BOOKLET

(a) (i) Prove $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ using the substitution $y = a - x$. **1**

(ii) Hence evaluate $\int_0^1 x^2\sqrt{1-x}\,dx$ **3**

(b) (i) Prove $\cos(A+B) + \cos(A-B) = 2\cos A\cos B$. **1**

(ii) Hence solve: **3**

$$\cos 5x + \cos 3x - \cos x = 0 \text{ for } 0 \le x \le \frac{\pi}{2}$$

(c) An object of mass m kg is thrown vertically upwards. Air resistance is given by $R = 0.05m\,v^2$ where R is in newtons and $v\ ms^{-1}$ is the speed of the object. Take $g = 9.8\,ms^{-2}$.

(i) Explain why the equation of motion is **2**

$$\ddot{x} = -\frac{196 + v^2}{20}$$

where x is the height of the object in metres above the point from which it is thrown.

If the initial velocity was $50ms^{-1}$, find:

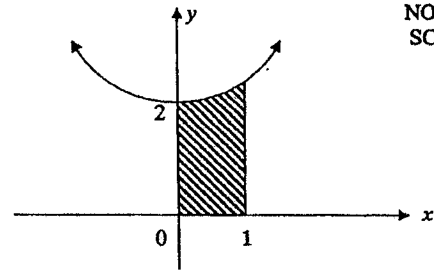
(ii) the maximum height attained. **3**

(iii) the time taken to reach this maximum height. **2**

Marks

Question 8 (15 marks) BEGIN A NEW BOOKLET

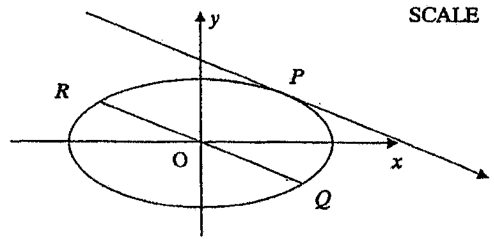
(a) **4**



NOT TO SCALE

The curve $y = e^x + e^{-x}$ is shown.
Use the method of cylindrical skills to find the volume formed when the shaded region is rotated about the y axis.

(b)



NOT TO SCALE

P is the point $(a\cos\theta, b\sin\theta)$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
RQ is the diameter of the ellipse parallel to the tangent at P.

(i) Prove that the equation of RQ is **1**

$$y = -\frac{bx\cos\theta}{a\sin\theta}$$

(ii) Hence find the co-ordinates of R and Q. **3**

Question 8 continues on the next page

Question 8 (continued) **Marks**

(b) (iii) Prove that the length of RQ is: **1**

$$2\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}$$

(iv) Explain the relationship between this result and the length of the diameter of a circle centre the origin radius a units. **2**

(c) (i) Find the sum of: **1**

$$x+x^2+x^3+\cdots+x^n$$

(ii) Hence prove: **3**

$$x+2x^2+3x^3+4x^4+\cdots+nx^n=\frac{x}{(x-1)^2}\left[nx^{n+1}-(n+1)x^n+1\right]$$

End of Paper

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Question 1.

a) (i) $\int \frac{e^x}{1+e^{2x}}\,dx = \tan^{-1}(e^x) + c$ ✓

(ii) $\int \frac{x^2}{x^2-9}\,dx = \int \frac{x^2-9+9}{x^2-9}\,dx$

$= \int 1 + \frac{9}{(x+3)(x-3)}\,dx$ ✓

Let $\frac{9}{(x+3)(x-3)} \equiv \frac{A}{x+3} + \frac{B}{x-3}$

$9 = A(x-3) + B(x+3)$

$= (A+B)x - 3A + 3B.$

$\therefore A + B = 0$ and $3B - 3A = 9$

$-A + B = 3$ $\quad B - A = 3.$

$2B = 3$

$B = \frac{3}{2}, \quad A = -\frac{3}{2}$ ✓

$\therefore \int \frac{x^2}{x^2-9}\,dx = \int 1 - \frac{3}{2(x+3)} + \frac{3}{2(x-3)}\,.\,dx$

$= x - \frac{3}{2}\log_e(x+3) + \frac{3}{2}\log_e(x-3) + c.$ ✓

(iii) $\int \sin^{-1}x\,dx$ $\quad$ Let $u = \sin^{-1}x$ $\quad v' = 1$

$u' = \frac{1}{\sqrt{1-x^2}}$ $\quad v = x.$ ✓

$\therefore \int \sin^{-1}x\,dx = x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}\,dx.$ ✓ $\qquad \frac{d}{dx}(1-x^2)^{\frac{1}{2}}$

$= x\sin^{-1}x + \sqrt{1-x^2} + c.$ ✓ $\qquad = \frac{1}{2}\times -2x \times \frac{1}{\sqrt{1-x^2}}$

b) Let $t = \tan\frac{\theta}{2}$ $\qquad$ if $\theta = \frac{\pi}{3}$, $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

$d\theta = \frac{2\,dt}{1+t^2}$ $\qquad \theta = 0 \quad \tan 0 = 0.$

$\therefore \int_0^{\frac{\pi}{3}} \frac{d\theta}{1+\sin\theta} = \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{1+\frac{2t}{1+t^2}} \times \frac{2\,dt}{1+t^2}$ ✓

$= 2\int_0^{\frac{1}{\sqrt{3}}} \frac{dt}{1+2t+t^2}$ ✓

$= 2\int_0^{\frac{1}{\sqrt{3}}} \frac{dt}{(1+t)^2}$

$= \left[\frac{-2}{1+t}\right]_0^{\frac{1}{\sqrt{3}}}$ ✓

$= \frac{-2}{1+\frac{1}{\sqrt{3}}} + \frac{2}{1+0}$

$= 2 - \frac{2\sqrt{3}}{\sqrt{3}+1}$

$= \frac{2\sqrt{3}+2-2\sqrt{3}}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1}$

$= \frac{2(\sqrt{3}-1)}{3-1}$

$= \sqrt{3} - 1.$ ✓

c) $\int_0^{\frac{\pi}{4}} \sin^3\theta\cos^3\theta\,d\theta = \int_0^{\frac{\pi}{4}} \sin\theta(1-\cos^2\theta)\cos^3\theta\,d\theta$ ✓

$= \int_0^{\frac{\pi}{4}} \sin\theta\cos^3\theta - \sin\theta\cos^5\theta\,.\,d\theta$ ✓

$= \left[-\frac{\cos^4\theta}{4} + \frac{\cos^6\theta}{6}\right]_0^{\frac{\pi}{4}}$ ✓

$= \left(-\frac{1}{4}\times\frac{1}{4} + \frac{1}{6}\times\frac{1}{8} + \frac{1}{4} - \frac{1}{6}\right)$

2) a) (i) $u = 2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$ ✓ 1

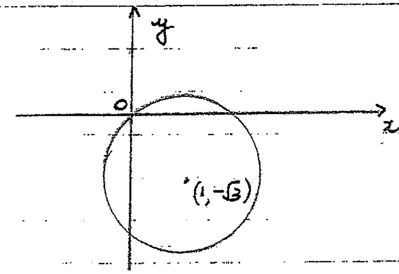
(ii) $u^{6n} = 2^{6n}\left(\cos\left(-\frac{\pi}{3}\times 6n\right) + i\sin\left(-\frac{\pi}{3}\times 6n\right)\right)$ ✓

$= 2^{6n}\left(\cos(-2\pi n) + i\sin(-2\pi n)\right)$

$= 2^{6n}(1+0)$

$= 2^{6n}$ ✓ 2.

(iii)



centre ✓

origin ✓ 2

(iv) maximum value is 4 ✓ 1

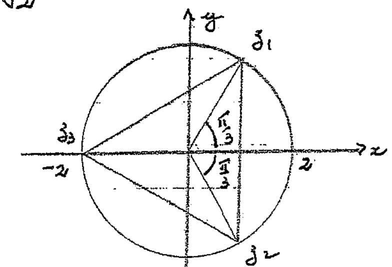
b) (i) B is $z + iz$ ✓ 1

(ii) $(z+iz)\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = (z+iz)\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$ ✓

$= \frac{z}{\sqrt{2}} + \frac{iz}{\sqrt{2}} + \frac{iz}{\sqrt{2}} - \frac{z}{\sqrt{2}}$

$= \frac{2iz}{\sqrt{2}} = \sqrt{2}iz$. is the new position of B ✓ 2

c)



(i) z_1 is point $2\left(\cos\frac{\pi}{3}, \sin\frac{\pi}{3}\right) = (1, \sqrt{3})$ ✓

z_2 is point $(1, -\sqrt{3})$ ✓ 2.

z_3 is point $(-2, 0)$

(ii) $z_1 + z_2 + z_3 = 1 + \sqrt{3}i + 1 - \sqrt{3}i - 2 = 0$. ✓

$z_1z_2 + z_1z_3 + z_2z_3 = (1+\sqrt{3}i)(1-\sqrt{3}i) - 2(1+\sqrt{3}i) - 2(1-\sqrt{3}i)$

$= 1 + 3 - 2 - 2\sqrt{3}i - 2 + 2\sqrt{3}i = 0$. ✓

$z_1z_2z_3 = (1+\sqrt{3}i)(1-\sqrt{3}i)\times -2 = (1+3)\times -2 = -8$. ✓

∴ equation where roots are z_1, z_2, z_3 is

$z^3 + 8 = 0$. $z^3 + 8 = (z^2 - 2z + 4)(z+2)$ 3

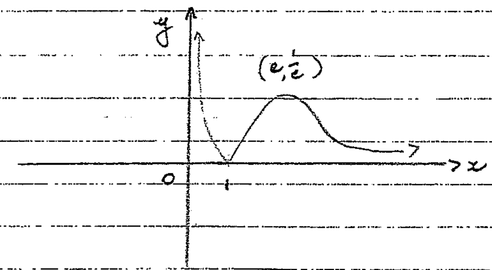
(iii) Area $= \frac{1}{2}ab\sin C$

$\sqrt{0^2 + (2\sqrt{3})^2} = 2\sqrt{3}$.

Area $= \frac{1}{2}\times 2\sqrt{3}\times 2\sqrt{3}\times\sin\frac{2\pi}{3} = 6\times\frac{\sqrt{3}}{2}$

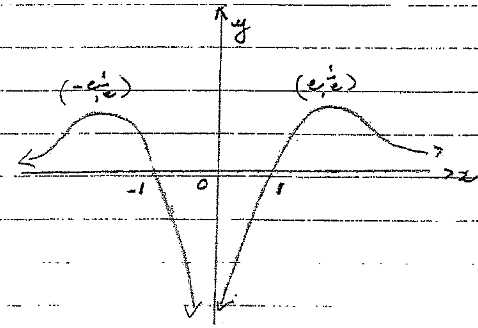
$= 3\sqrt{3}\ u^2$ ✓ 1

3) a) (i)



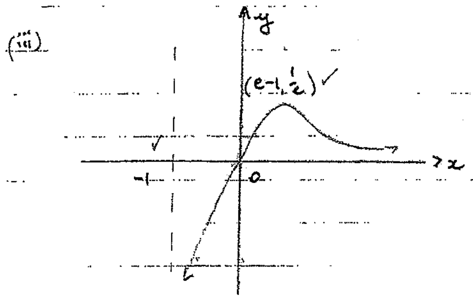
✓

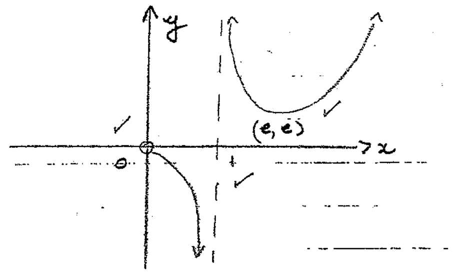
(ii)



✓

(iii)





b) one zero is $-\frac{1}{2} + \frac{\sqrt{3}}{2} i$

2nd zero is $-\frac{1}{2} - \frac{\sqrt{3}}{2} i$

$-\frac{1}{2} + \frac{\sqrt{3}}{2} i - \frac{1}{2} - \frac{\sqrt{3}}{2} i = -1$

$\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) = \frac{1}{4} + \frac{3}{4} = 1$

$\therefore$ quadratic is $x^2 + x + 1$ ✓

$$\begin{array}{r|l} & x^2 + x - 1 \\ \hline x^2 + x + 1 & x^4 + 2x^3 + x^2 + 0x - 1 \\ & x^4 + x^3 + x^2 \\ \hline & x^3 + 0x^2 + 0x - 1 \\ & x^3 + x^2 + x \\ \hline & -x^2 - x + 1 \end{array}$$

$\therefore \quad P(x) = (x^2 + x - 1)(x^2 + x + 1)$ ✓

$x = \frac{-1 \pm \sqrt{1+4}}{2}$ ✓

$\therefore$ other 3 zeros are $-\frac{1}{2} - \frac{\sqrt{3}}{2} i, \; -\frac{1}{2} + \frac{\sqrt{5}}{2}, \; -\frac{1}{2} - \frac{\sqrt{5}}{2}$.

c) (i) $f(x) = x^3 - 3cx$

if $f(x) = 0, \quad x^3 - 3cx = 0$

$x(x^2 - 3c) = 0$

$x = 0, \quad x^2 = 3c$

if $c < 0, \quad x^2 < 0$ ✓ ✓

$\therefore$ if $c < 0$ there is only one real root $(x = 0)$

(ii) Consider $y = x^3 - 3cx$. for $c > 0$

$\frac{dy}{dx} = 3x^2 - 3c$

$\therefore$ stationary points occur at $x^2 = c$

$x = \pm\sqrt{c}$

if $x = \sqrt{c}, \quad y = (\sqrt{c})^3 - 3c\sqrt{c}$

$= c\sqrt{c} - 3c\sqrt{c}$

$= -2c\sqrt{c}$

if $x = -\sqrt{c}, \quad y = (-\sqrt{c})^3 + 3c\sqrt{c}$

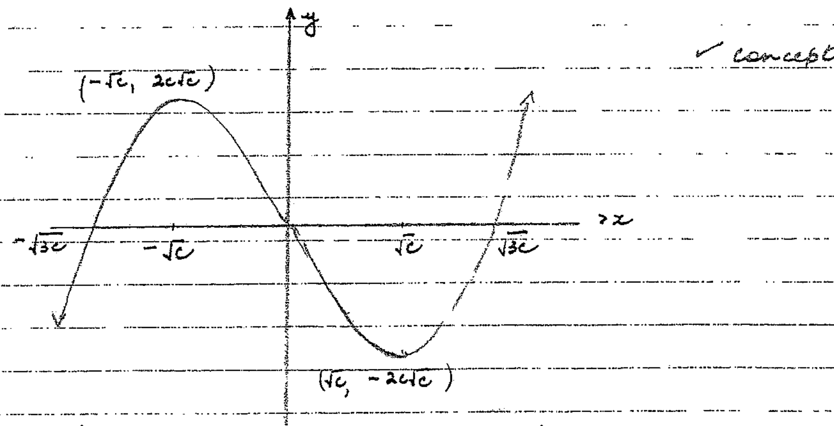
$= -c\sqrt{c} + 3c\sqrt{c}$

$= 2c\sqrt{c}$. ✓

$\therefore$ curve has a maximum stationary point at $(-\sqrt{c}, 2c\sqrt{c})$

" " " minimum " " at $(\sqrt{c}, -2c\sqrt{c})$

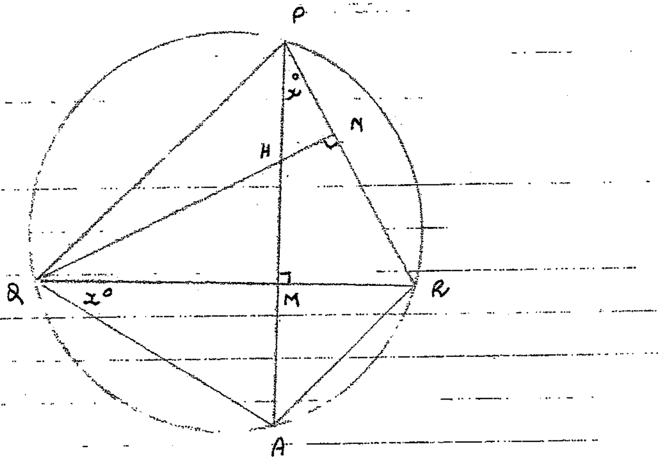
✓ concept



a horizontal line $y = k$ crosses this curve three distinct times if $-2c\sqrt{c} < y < 2c\sqrt{c}$ ✓

i.e. $|k| < 2c\sqrt{c}$

(4) a)



$\angle AQR = \angle APR = x^{\circ}$ (on same arc, equal)

In ΔPHN, $\angle PHN = 180^{\circ} - (90^{\circ} + x^{\circ})$ (angle sum of Δ is 180°)

$= 90^{\circ} - x^{\circ}$.

$\angle PHN = \angle QHM = 90^{\circ} - x^{\circ}$ (vertically opposite $\angle$'s are equal)

n ΔQAM, $\angle QAM = 180^{\circ} - (90^{\circ} - x^{\circ})$ (angle sum of ΔQAM is 180°)

$\therefore \angle QHM = \angle QAM$.

In Δ's, HQM and QMA,

$\angle QMH = \angle QMA$ (both 90°, given)

QM is common

$\angle QHM = \angle QAM$ (proved above)

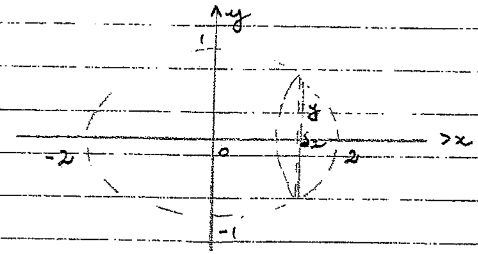
$\therefore \Delta$'s are congruent (AAS)

$\therefore HM = MA$

i.e. QR bisects HA.

These are other methods.

b)



✓

radius of cross section $= y$

Area " " " $= \frac{1}{2}\pi y^2$.

$\therefore \delta V = \frac{1}{2}\pi y^2 \delta x = \frac{1}{2}\pi\left[1 - \frac{x^2}{4}\right]\delta x$. ✓

Total Volume $= \lim_{\delta x \to 0} \sum_{x=-2}^{2} \frac{1}{2}\pi\left[1 - \frac{x^2}{4}\right]\delta x$.

$= \pi \int_0^2 1 - \frac{x^2}{4}\, dx$. ✓

$= \pi\left[x - \frac{x^3}{12}\right]_0^2$

$= \pi\left[2 - \frac{8}{12}\right]$

Volume. $= \frac{4\pi}{3} u^3$ ✓

c) (i) $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\, dx$

$= \int_0^{\frac{\pi}{2}} \sin x \sin^{n-1} x\, dx$

Let $u = \sin^{n-1} x$ $\quad v' = \sin x$.

$u' = (n-1)\sin^{n-2} x \cos x$, $v = -\cos x$ ✓

$$\therefore I_n = \left[-\cos x \sin^{n-1} x\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x \, dx.$$ ✓

$$= \left[-\cos\frac{\pi}{2} \sin^{n-1}\frac{\pi}{2} + \cos 0 \sin^{n-1} 0\right] + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x \, dx.$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x \, dx \quad \text{for } n \geq 2.$$ ✓

(ii) $$I_n = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x (1-\sin^2 x) \, dx$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \, dx - (n-1)\int_0^{\frac{\pi}{2}} \sin^n x \, dx.$$ ✓

$$I_n = (n-1) I_{n-2} - (n-1) I_n$$

$$\therefore I_n + (n-1) I_n = (n-1) I_{n-2}$$

$$\therefore n I_n = (n-1) I_{n-2}.$$

$$I_n = \frac{n-1}{n} I_{n-2}.$$ ✓

iii) $$I_4 = \frac{3}{4} I_2 = \frac{3}{4}\int_0^{\frac{\pi}{2}} \sin^2 x \, dx$$ ✓

$$= \frac{3}{4} \times \frac{1}{2}\int_0^{\frac{\pi}{2}} 1 - \cos 2x. \, dx.$$

$$= \frac{3}{8}\left[x - \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{3}{8}\left[\frac{\pi}{2} - \frac{\sin \pi}{2} - 0 + \sin 0\right]$$

$$= \frac{3\pi}{16}$$ ✓

5 a) (i) $x^3 - x^2 - 3x + 5 = 0.$

$\alpha + \beta + \gamma = 1.$ ✓

(ii) $2\alpha + \beta + \gamma = \alpha + 1$

so roots are $\alpha+1, \ \beta+1, \ \gamma+1$ ✓

Let $y = x+1$

$x = y-1$

$\therefore (y-1)^3 - (y-1)^2 - 3(y-1) + 5 = 0$ ✓

$y^3 - 3y^2 + 3y - 1 - y^2 + 2y - 1 - 3y + 3 + 5 = 0$

$y^3 - 4y^2 + 2y + 6 = 0$ ✓

equation $x^3 - 4x^3 + 2x + 6 = 0$

b) (i) $y = 4x^{-1}$

$y' = -4x^{-2} = \dfrac{-4}{x^2}$

at P, $y' = \dfrac{-4}{4t^2} = \dfrac{-1}{t^2}$ ✓

$\therefore$ tangent : $y - \dfrac{2}{t} = \dfrac{-1}{t^2}(x - 2t)$

$t^2 y - 2t = -x + 2t$

$x + t^2 y = 4t$ is tangent. ✓

(ii) tangent at Q is: $x + s^2 y = 4s$

$t^2 y - s^2 y = 4t - 4s$

$(t+s)(t-s) y = 4(t-s)$

$y = \dfrac{4}{t+s}$ ✓

$x + \dfrac{4t^2}{t+s} = 4t$

$x = 4t - \dfrac{4t^2}{}$

$$x = \frac{4t^2 + 4ts - 4t^2}{t+s} = \frac{4ts}{t+s}$$ ✓

$\therefore$ intersect at $M\left(\frac{4st}{s+t}, \frac{4}{s+t}\right)$

(iii) If $s = \frac{-1}{t}$ Let $x = \frac{4st}{s+t}$ $y = \frac{4}{s+t}$

if $st = -1$, $x = \frac{-4}{s+t} = -y$.

✓

$\therefore$ locus is the straight line $y = -x$.

✓

It does not have the point $(0,0)$ included.

c) (i) Pat : $P(WLW) + P(LWW) + P(WWL)$

$= 3 \times \left(\frac{2}{3}\right)^2 \times \left(\frac{1}{3}\right) = \frac{12}{27}$

$= \frac{4}{9}$. ✓

(ii) Possibilities :

Pat wins in 3 : $P(WWW) = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$

Pat wins in 4 : $P(WWLW + LWWW + WLWW$

$= 3 \times \left(\frac{2}{3}\right)^3 \times \frac{1}{3} = \frac{8}{27}$

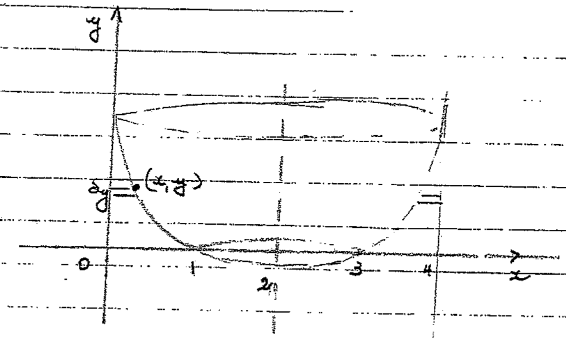
Pat wins in 5 : $P(LLWWW + LWLWW + LWWLW$
$+ WWLLW + WLLWW + WLWLW)$

$= 6 \times \left(\frac{2}{3}\right)^3 \times \left(\frac{1}{3}\right)^2 = \frac{16}{81}$

Total : $\frac{8}{27} + \frac{8}{27} + \frac{16}{81}$

$= \frac{64}{81}$

6 a) (i)



Area of cross section is an annulus
outer radius = 2 units
inner radius = $2-x$ units. ✓

Area of cross section $= \pi\left[4 - (2-x)^2\right]$

$= \pi\left[4 - 4 + 4x - x^2\right]$

$= \pi\left[4x - x^2\right]$. ✓

$y = (x-1)^2$.

$\therefore x - 1 = \pm\sqrt{y}$

in this case $x = \sqrt{y} + 1$

$x^2 = y + 2\sqrt{y} + 1$ ✓

$\therefore$ Area of cross section $= \pi\left[4(\sqrt{y}+1) - y - 2\sqrt{y} - 1\right]$

$= \pi\left[4\sqrt{y} + 4 - y - 2\sqrt{y} - 1\right]$

$= \pi\left[3 + 2\sqrt{y} - y\right]$ ✓

$\therefore$ $\delta V = \pi\,(3 + 2\sqrt{y} - y)\,\delta y$

$\therefore$ Total volume $= \lim_{\delta y \to 0} \pi \sum_{y=0}^{1} (3 + 2\sqrt{y} - y)\,\delta y$ ✓

(ii) $V = \pi \int_0^1 3 + 2y^{\frac{1}{2}} - y \, dy$

$= \pi \left[3y - \frac{y^2}{2} + \frac{2 \cdot 2y^{3/2}}{3} \right]_0^1$ ✓

$= \pi \left[3 - \frac{1}{2} + \frac{4}{3} \right]$

$= \frac{23\pi}{6} u^3$ ✓

b) (i) $x^2 + 4x + 4 + y^2 = 4$

$(x+2)^2 + y^2 = 4$

∴ centre is $(-2, 0)$ radius is 2 units. ✓✓

(ii) tangent if perpendicular distance from centre to straight line equals the radius.

Line is $mx - y + b = 0$.

could solve simultaneously and $\Delta = 0$ for equal roots.

Distance $= \left| \frac{-2m - 0 + b}{\sqrt{m^2+1}} \right| = 2$. ✓

∴ $|-2m + b| = 2\sqrt{m^2+1}$ ✓

$(-2m+b)^2 = 4(m^2+1)$

$4m^2 - 4mb + b^2 = 4m^2 + 4$

$4mb + 4 = b^2$

$4(mb+1) = b^2$. is the condition. ✓

(iii) $(k, 0)$ lies on the line

∴ $mk + b = 0$.

$-mk = b$.

∴ $4(-m^2k + 1) = m^2k^2$

∴ $m^2k^2 + 4m^2k - 4 = 0$.

$(k^2 + 4k)m^2 - 4 = 0$. ✓

there are 2 possible values of m, if the tangents are to be perpendicular the gradients must fulfil $m_1 m_2 = -1$

ie product of the roots is -1

∴ $\frac{-4}{k^2+4k} = -1$ ✓

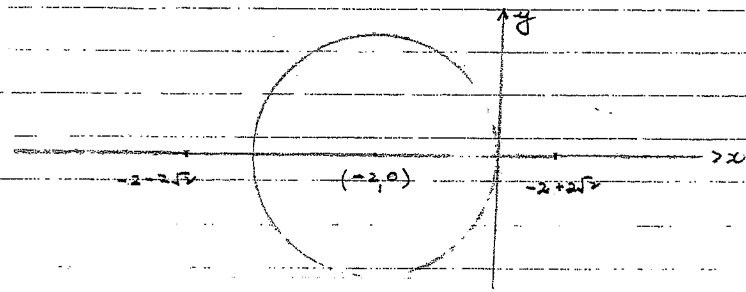
∴ $k^2 + 4k = 4$

$k^2 + 4k - 4 = 0$

$k = \frac{-4 \pm \sqrt{16+16}}{2}$

$= \frac{-4 \pm 4\sqrt{2}}{2}$

∴ $k = -2 + 2\sqrt{2}$ or $-2 - 2\sqrt{2}$ ✓ (both outside the circle)



7 a) (i) Let $y = a - x$ $\quad dy = -dx \quad x = a - y$

if $x = a$, $y = 0$

if $x = 0$, $y = a$.

∴ $\int_0^a f(x)\,dx = \int_a^0 f(a-y) \times -dy$

$= \int_0^a f(a-y)\,dy$

$= \int_0^a f(a-x)\,dx$. ✓ 1 com.

(ii) $\int_0^1 x^2 \sqrt{1-x}\,dx = \int_0^1 (1-x)^2 \sqrt{x}\,dx$. ✓ $a=1$

$= \int_0^1 (1 - 2x + x^2)\sqrt{x}\,dx$.

$$\int_0^1 x^{\frac{1}{2}} - 2x^{3/2} + x^{5/2}\, dx = \left[\frac{2}{3} x^{3/2} - \frac{4}{5} x^{5/2} + \frac{2}{7} x^{\frac{7}{2}} \right]_0^1 \checkmark$$

$$= \frac{2}{3} - \frac{4}{5} + \frac{2}{7}$$

$$= \frac{16}{105}$$ 3 Calc. ✓

b) (i) $\cos(A+B) + \cos(A-B) = \cos A \cos B - \sin A \sin B$

$+ \cos A \cos B + \sin A \sin B$

$= 2\cos A \cos B$. ✓ 1

(ii) consider $\cos 5x + \cos 3x$

Let $5x = A+B$

$3x = A - B$

$\therefore 8x = 2A$

$A = 4x$ and $B = x$.

$\therefore \cos 5x + \cos 3x = 2\cos 4x \cos x$. ✓

$\therefore \cos 5x + \cos 3x - \cos x = 2\cos 4x \cos x - \cos x$. Reas.

$\therefore \cos x \left[2\cos 4x - 1\right] = 0$

$\therefore$

$\therefore \cos x = 0$ or $\cos 4x = \frac{1}{2}$. ✓ 3

Acute angle $= \frac{\pi}{2}$.

$x = \frac{\pi}{2}$

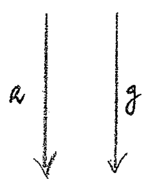
Acute angle $= \frac{\pi}{3}$.

1st 4th quadrants.

$4x = \frac{\pi}{3}, \frac{5\pi}{3}$

$x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{\pi}{2}$ for given domain. ✓

c) (i)



✓

Both gravity and air resistance are acting against the upward motion

$\therefore m\ddot{x} = -9.8m - 0.05mv^2$

$\therefore \ddot{x} = -9.8 - \frac{5v^2}{100}$

b) (i) $x = a\cos\theta$ $\quad$ $y = b\sin\theta$

$\frac{dx}{d\theta} = -a\sin\theta$ $\quad$ $\frac{dy}{d\theta} = b\cos\theta$.

$\frac{dy}{dx} = -\frac{b\cos\theta}{a\sin\theta}$ which is the gradient of tangent at and also of the line RQ.

RQ passes through O

$\therefore$ equation $y = \frac{-b\cos\theta}{a\sin\theta} x = -\frac{bx\cos\theta}{a\sin\theta}$ ✓

(ii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$\frac{x^2}{a^2} + \frac{1}{b^2} \times \frac{b^2 x^2 \cos^2\theta}{a^2 \sin^2\theta} = 1$.

$\frac{x^2}{a^2}\left[1 + \frac{\cos^2\theta}{\sin^2\theta}\right] = 1$

$\frac{x^2}{a^2}\left[\frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta}\right] = 1$ ✓

$\frac{x^2}{a^2} \times \frac{1}{\sin^2\theta} = 1$

$x^2 = a^2\sin^2\theta$

$x = \pm a\sin\theta$.

$y = -\frac{b\cos\theta}{a\sin\theta} \times \pm a\sin\theta$

$= \mp b\cos\theta$ ✓

$\therefore$ R is $(-a\sin\theta, b\cos\theta)$

Q is $(a\sin\theta, -b\cos\theta)$ ✓

(iii) $RQ^2 = (-a\sin\theta - a\sin\theta)^2 + (b\cos\theta + b\cos\theta)^2$

$= 4a^2\sin^2\theta + 4b^2\cos^2\theta$

$\therefore RQ = 2\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$ ✓

(iv) for circle centre (0,0) radius "a" units, this is a particular case of the ellipse where $a = b$

$\therefore$ length of diameter $= 2\sqrt{a^2\sin^2\theta + a^2\cos^2\theta}$

$= 2a$

i.e. diameter = twice the radius.

c) (i) G.P. $a = x$, $r = x$

$$S_n = \frac{a(r^n - 1)}{r-1} = x\left(\frac{x^n - 1}{x-1}\right)$$

$$\therefore x + x^2 + x^3 + \cdots + x^n = \frac{x(x^n - 1)}{x-1}$$

$$= \frac{x^{n+1} - x}{x-1}$$

Differentiating both sides,

$$1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} = (x-1)\left[(n+1)x\right.$$

$$= \frac{(x-1)\left[(n+1)x^n - 1\right] - (x^{n+1} - x)}{(x-1)^2}$$

multiplying by x

$$x + 2x^2 + 3x^3 + \cdots + nx^n = \frac{x}{(x-1)^2}\left[(n+1)x^{n+1} - (n+1)x^n - x + 1 - x^{n+1} + x\right]$$

$$= \frac{x}{(x-1)^2}\left[nx^{n+1} - (n+1)x^n + 1\right]$$