Centre Number

Student Number

SCEGGS Darlinghurst

2003
HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Mathematics Extension 2

This is a TRIAL PAPER only and does not necessarily reflect the content or format of the Higher School Certificate Examination for this subject.

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks - 120

- Attempt Questions 1–8
- All questions are of equal value

Total marks – 120
Attempt Questions 1–8
All questions are of equal value

Answer each question on a NEW page.

Marks

Question 1 (15 marks)

(a) Find

(i) $\displaystyle\int \frac{e^x}{(1+e^x)^2}\,dx$ 1

(ii) $\displaystyle\int x\cos x\,dx$ 2

(iii) $\displaystyle\int \frac{2x-3}{x^2-4x+8}\,dx$ 3

(b) (i) Find real numbers A, B and C such that: 2

$$\frac{10}{(3+x)(1+x^2)} \equiv \frac{A}{3+x} + \frac{Bx+C}{1+x^2} \quad (x \neq -3)$$

(ii) Hence find $\displaystyle\int \frac{10}{3+\tan\theta}\,d\theta$ using the substitution $x=\tan\theta$. 3

(c) Use the substitution $x=\sin^2\theta$ to evaluate 4

$$\int_0^{\frac{1}{2}} \frac{\sqrt{x}}{(1-x)^{\frac{3}{2}}}\,dx$$

Question 2 (15 marks) Start a NEW page. **Marks**

(a) By considering the expansion of $(1+x)^{2n}$ in ascending powers of x, prove that **3**

$$\sum_{k=0}^{2n}\binom{2n}{k}=4^n$$

(b) Clearly indicate on an Argand Diagram the regions in the complex plane satisfied by:

(i) $0 \le \arg z \le \frac{\pi}{3}$ and $2 \le \operatorname{Im} z \le 3$. **2**

(ii) $|z-2i| \le 2$ and $|z-2-2i| \le 2$. **3**

(c) (i) Express $z=-1+i\sqrt{3}$ in modulus argument form. **1**

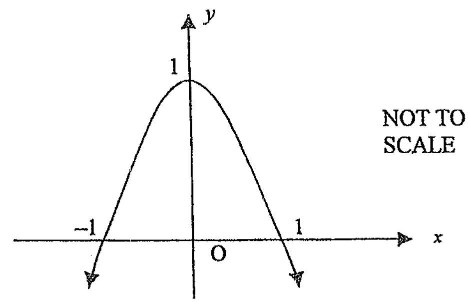
(ii) Hence or otherwise, confirm that z is a solution of the equation **3**

$$z^4-4z^2-16z-16=0$$

(iii) Find the other three solutions of the equation. **3**

Question 3 (15 marks) Start a NEW page. **Marks**

(a)



NOT TO SCALE

The graph is of the curve $y=1-x^2$ where $f(x)=1-x^2$.

Without using calculus, sketch the following showing all important features.

(i) $y=\frac{-1}{f(x)}$ **1**

(ii) $|y|=|f(x)|$ **2**

(iii) $y=f(e^x)$ **2**

(iv) $y=\log_e(f(x))$ **2**

Question 3 continues on page 5

Question 3 (continued) Marks

(b) Consider the cubic 3

$$P(x) = x^3 + Ax + B \qquad \text{(where } A \text{ and } B \text{ are real)}$$

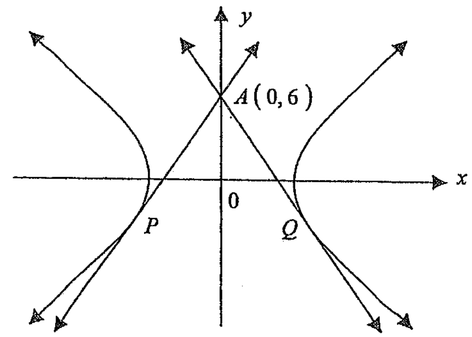
Prove that $P(x) = 0$ has exactly one real root if $A \geq 0$.

(c) (i) Find the points of intersection of the curve $y = x^2 - 2x$ and the straight line $y = x$. 1

(ii) Use the method of cylindrical shells to find the volume generated when the region enclosed by the parabola $y = x^2 - 2x$ and the line $y = x$ is rotated about the y axis. 4

Question 4 (15 marks) Start a NEW page. Marks

(a)



NOT TO SCALE

The diagram shows the hyperbola $9x^2 - y^2 = 36$. Tangents from the point $A(0, 6)$ touch the hyperbola at the points $P(x_1, y_1)$ and $Q(x_2, y_2)$.

(i) Prove that the equation of the tangent at P is: 2

$$y - y_1 = \frac{9x_1}{y_1}(x - x_1)$$

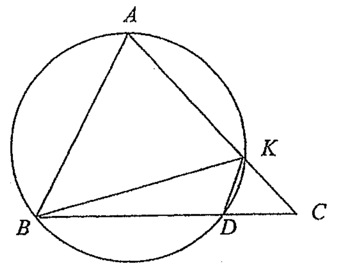
(ii) Prove that $9x_1^2 - y_1^2 = -6y_1$. 2

(iii) Hence find the co-ordinates of P and Q. 2

Question 4 continues on page 7

Question 4 (continued) **Marks**

(b)



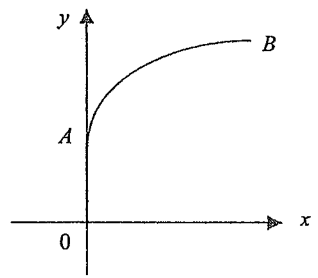
NOT TO SCALE

In the triangle ABC, $AB = AC$.
BK bisects $< ABC$.
Points A, B, D, K lie on the circumference of a circle.

(i) Assuming $< ABK = \alpha$, explain why $< DKC = 2\alpha$. **1**

(ii) Hence prove $AK = DC$. **3**

(c)



NOT TO SCALE

Points $A(0,1)$ and $B(3,2)$ lie on the curve $y^2 = x + 1$.
The region bounded by the curve, the line $y = 1$ and the line $x = 3$ is rotated about $x = 3$.

(i) By taking slices perpendicular to $x = 3$, prove that the volume formed is **3**

$$\pi \int_1^2 (4 - y^2)^2 \, dy$$

(ii) Hence find this volume. **2**

Marks

Question 5 (15 marks) Start a NEW page.

(a) Without the use of calculus, draw a sketch of $y = \frac{\sin x}{x}$ for $-3\pi \le x \le 3\pi$. **3**

(b) $P(a\cos\theta, b\sin\theta)$ is any point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

M is the midpoint of SP where S is a focus of the ellipse.

(i) Find the co-ordinates of M. **1**

(ii) Find the Cartesian equation of the locus of M. **2**

(iii) Prove that the locus is a second ellipse with centre at the midpoint of OS, where O is the origin. **2**

(c) (i) Explain the difficulty of using the formula for $\tan(A + B)$ when simplifying $\tan\left(A + \frac{\pi}{2}\right)$. **1**

(ii) Prove that $\tan\left(A + \frac{\pi}{2}\right) = -\cot A$. **2**

(iii) Hence use the method of Mathematical Induction to prove that **4**

$$\tan\left[(2n+1)\frac{\pi}{4}\right] = (-1)^n \text{ for all integers } n \ge 1.$$

Marks

Question 6 (12 marks) Start a NEW page.

(a) Use the substitution $t = \tan\frac{\theta}{2}$ to find $\int \frac{d\theta}{1-\cos\theta-\sin\theta}$. **4**

(b) Given $I_n = \int_0^{\frac{\pi}{2}} \cos^{2n} x \, dx$,

(i) Prove that $I_n = \frac{2n-1}{2n} I_{n-1}$. **4**

(ii) Hence evaluate $\int_0^{\frac{\pi}{2}} \cos^6 x \, dx$. **3**

(c) (i) Prove that for $a>0$ and $n \neq 0$, $\log_{a^n} x = \frac{1}{n}\log_a x$. **2**

(ii) Hence evaluate in simplest form $\log_2 5 + \log_4 5 + \log_{16} 5 + \log_{256} 5 + \dots$. **2**

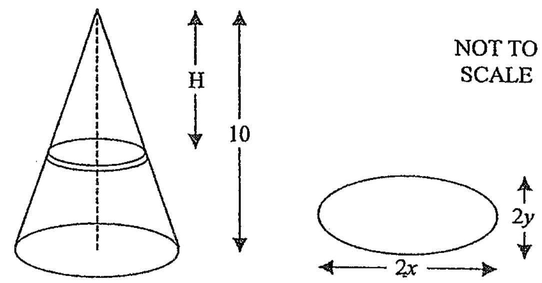
Marks

Question 7 (15 marks) Start a NEW page.

(a) (i) Evaluate $\int_{-a}^{a} \sqrt{a^2 - x^2} \, dx$ **1**

(ii) Explain how you could use the result in part (i) to prove that the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab units^2. **2**

(iii) **4**



The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ forms the base of a cone of height 10 units. A slice δH wide is taken H units from the vertex as shown. The cross section is an ellipse with major and minor axes $2x$ and $2y$ respectively.

Use the result from part (ii) to prove that the area of cross section H units from the vertex is $\frac{\pi ab H^2}{100}$ units^2

(iv) Hence find the volume of the right elliptical cone. **2**

Question 7 continues on page 11

Question 7 (continued) — Marks

(b) (i) Expand $(a+b)^3$ — 1

(ii) Use this expansion and de Moivre's Theorem to prove that — 2

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

(iii) If $\cos 3\theta = \frac{1}{2}$ and $x = \cos\theta$, prove that $8x^3 - 6x - 1 = 0$. — 2

(iv) Hence prove that $\cos\frac{\pi}{9}\cos\frac{5\pi}{9}\cos\frac{7\pi}{9} = \frac{1}{8}$. — 1

Question 8 (15 marks) Start a NEW page. — Marks

(a) From a point on the ground an object of mass m is projected vertically upwards with an initial speed of u. Air resistance is mkv^2 and g is the acceleration due to gravity.

(i) Using a diagram or otherwise explain why $\ddot{x} = -g - kv^2$. — 1

(ii) Prove that the displacement x metres above ground level is given by — 4

$$x = \frac{1}{2k}\left[\log_e\left(\frac{g+ku^2}{g+kv^2}\right)\right]$$

(iii) If the object reaches a height of 40m above the ground prove that — 3

$$u^2 = \frac{g}{k}\left(e^{80k} - 1\right)$$

(b) The equation of a curve is $x^2y^2 - x^2 + y^2 = 0$

(i) Prove $y^2 = \frac{x^2}{x^2+1}$. — 1

(ii) Explain why $-1 < y < 1$. — 1

(iii) Prove that $\frac{dy}{dx} = \left(\frac{y}{x}\right)^3$. — 3

(iv) Sketch the curve. — 2

END OF PAPER

Extension 2 Trial 2003 SCEGGS.

① a) (i) $-\dfrac{1}{1+e^{x}} + c$

(ii) $u = x \quad v' = \cos x$

$u' = 1 \quad v = \sin x$

$I = x\sin x - \int \sin x \, dx$

$= x\sin x + \cos x + c$

(iii) $\dfrac{2x-3}{x^2-4x+8} = \dfrac{2x-4+1}{x^2-4x+4+4}$

$\displaystyle\int \frac{2x-3}{x^2-4x+8}\,dx = \ln(x^2-4x+8) + \int \frac{dx}{(x-2)^2+4}$

$= \ln(x^2-4x+8) + \frac{1}{2}\tan^{-1}\frac{x-2}{2} + c$

b) (i) $10 \equiv A(1+x^2) + (Bx+C)(3+x)$

$10 = A + Ax^2 + 3Bx + 3C + Bx^2 + Cx$

$= (A+B)x^2 + (3B+C)x + A + 3C$

$\therefore A+B=0 \quad 3B+C=0 \quad A+3C=10$

$A=-B \quad -B+3C=10$

$-3B+9C=30$

$10C=30$

$C=3, \; B=-1, \; A=1$

(ii) $x = \tan\theta$

$dx = \sec^2\theta \, d\theta$

$= (1+\tan^2\theta)\,d\theta$

$= (1+x^2)\,d\theta$

$\displaystyle\int \frac{10}{3+\tan\theta}\,d\theta = \int \frac{10}{(3+x)(1+x^2)}\,dx$

$= \displaystyle\int \frac{dx}{3+x} + \int \frac{-x+3}{1+x^2}\,dx$

$= \ln(3+x) - \frac{1}{2}\int \frac{2x}{1+x^2}\,dx + \int \frac{3\,dx}{1+x^2}$

$= \ln(3+x) - \frac{1}{2}\ln(x^2+1) + 3\tan^{-1}x + c$

c) $x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta \, d\theta$

if $x = \frac{1}{2} \quad \theta = \frac{\pi}{4}$

$x = 0 \quad \theta = 0$

$\displaystyle\int_0^{\frac{1}{2}} \frac{\sqrt{x}}{(1-x)^{3/2}}\,dx = \int_0^{\frac{\pi}{4}} \frac{2\sin^2\theta\cos\theta\,d\theta}{(1-\sin^2\theta)^{3/2}}$

$= \displaystyle\int_0^{\frac{\pi}{4}} \frac{2\sin^2\theta\cos\theta\,d\theta}{\cos^3\theta}$

$= 2\displaystyle\int_0^{\frac{\pi}{4}} \tan^2\theta\,d\theta$

$= 2\displaystyle\int_0^{\frac{\pi}{4}} \sec^2\theta - 1 \, d\theta$

$= 2\left[\tan\theta - \theta\right]_0^{\frac{\pi}{4}}$

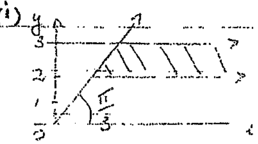
$= 2\left(1 - \frac{\pi}{4}\right)$

② a) $(1+x)^{2n} = \binom{2n}{0} + \binom{2n}{1}x + \cdots + \binom{2n}{2n}x^{2n}$

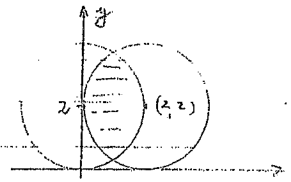
if $x=1$, $2^{2n} = \binom{2n}{0} + \binom{2n}{1} + \cdots + \binom{2n}{2n}$

$(2^2)^n = \displaystyle\sum_{k=0}^{2n} \binom{2n}{k} = 4^n$

b) (i)



(ii)



c) (i) $z = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

(ii) $z^4 = 16\left(\text{cis}\,\frac{8\pi}{3}\right) = -8 + 8\sqrt{3}i$

$-4z^2 = -16\left(\text{cis}\,\frac{4\pi}{3}\right) = 8 + 8\sqrt{3}i$

$-16z = 16 - 16\sqrt{3}i$

$z^4 - 4z^2 - 16z - 16 = 0$

$\therefore z$ is a solution

(iii) if $-1+i\sqrt{3}$ is a solution then $-1-i\sqrt{3}$ is also.

$\therefore z^2 + 2z + 4$ is a factor

$$\begin{array}{r} z^2 - 2z - 4 \\ z^2+2z+4 \,\big)\, z^4 + 0z^3 - 4z^2 - 16z - 16 \\ z^4 + 2z^3 + 4z^2 \\ \hline -2z^3 - 8z^2 - 16z \\ -2z^3 - 4z^2 - 8z \\ \hline -4z^2 - 8z - 16 \end{array}$$

if $z^2 - 2z - 4 = 0$

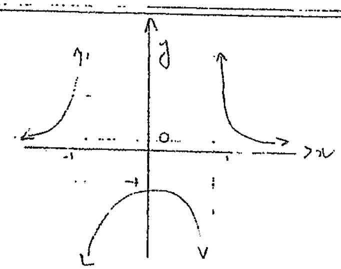
$z = \dfrac{2 \pm \sqrt{4+16}}{2}$

$= 1 \pm \sqrt{5}$

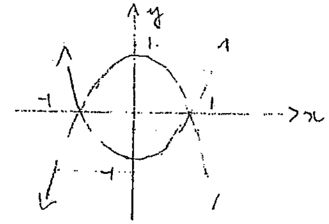
other 3 solutions are $-1-i\sqrt{3}$, $1+\sqrt{5}$ and $1-\sqrt{5}$.

Note: the technique in (iii) could have been used in (ii) as an alternative method.

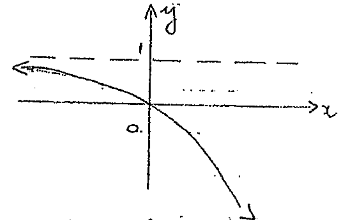
③ a) (i)



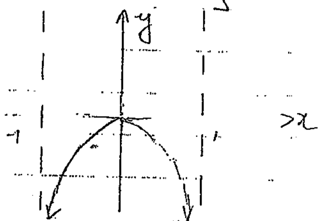
(ii)



(iii)



(iv)



b) If $A=0$, $P(x) = x^3 + B = 0$

$\therefore x = \sqrt[3]{-B}$ which has only one real root.

$P'(x) = 3x^2 + A$

if $A>0$, $P'(x) > 0$ for all real x

$\therefore$ curve monotonic increasing.

if $P'(x) = 0 \quad x = \pm\sqrt{\frac{-A}{3}}$

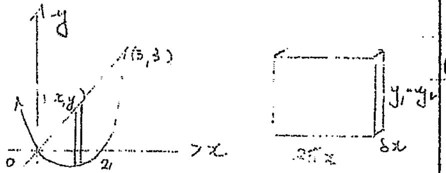
$\therefore$ stationary points do not exist if $A>0$

∴ the curve $y = P(x)$ can cross the x axis only once if $A \geq 0$. i.e. one real root.

c) (i) $x = x^2 - 2x$

$(0,0)$ and $(3,3)$

(ii)



$\delta V = 2\pi x(x - x^2 + 2x)\delta x$

$= 2\pi x(3x - x^2)\delta x$

$V = \int_0^3 2\pi x(3x - x^2)\,dx$

$= 2\pi \int_0^3 3x^2 - x^3\,dx$

$= 2\pi \left[x^3 - \frac{x^4}{4}\right]_0^3$

$= 2\pi \left[27 - \frac{81}{4}\right]$

$= \frac{27\pi}{2}\ u^3$

) a) (i) $18x - 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{9x}{y}$

gradient at P is $\frac{9x_1}{y_1}$

tangent is:

$y - y_1 = \frac{9x_1}{y_1}(x - x_1)$

(ii) Passes through $(0,6)$

∴ $6 - y_1 = \frac{9x_1}{y_1}(0 - x_1)$

∴ $6y_1 - y_1^2 = -9x_1^2$

∴ $9x_1^2 - y_1^2 = -6y_1$

(iii) But $P(x_1, y_1)$ is also on the curve

∴ $9x_1^2 - y_1^2 = 36$

∴ $-6y_1 = 36$

$y_1 = -6$

$9x_1^2 - 36 = 36$

$x_1^2 = 8$

$x_1 = \pm 2\sqrt{2}$

∴ P is $(-2\sqrt{2}, -6)$ Q is $(2\sqrt{2}, -6)$

b) (i) $\angle ABK = \angle KBC$ (BK bisects $\angle ABC$)

∴ $\angle ABD = 2\alpha$

$\angle ABD = \angle DKC$ (ext ∠ cyclic quad = int. opp ∠)

∴ $\angle DKC = 2\alpha$

(ii) $AK = KD$ (subtend equal angles in circle)

$AB = AC$ (given)

∴ $\angle ABC = \angle ACB$ (opp equal S's in ΔABC)

∴ $\angle ACB = 2\alpha$

i.e. $\angle ACB = \angle DKC$

∴ $DK = DC$ (opp equal ∠'s in ΔDKC)

∴ $AK = DC$

c) (i)

Area of cross section is a circle radius $3 - x$ units

∴ Area $= \pi(3-x)^2$

$\delta V = \pi(3-x)^2\,\delta y$

$= \pi(3 - y^2 + 1)^2\,\delta y$

$= \pi(4 - y^2)^2\,\delta y$

∴ $V = \lim_{\delta y \to 0} \sum_{y=1}^{2} \pi(4 - y^2)^2\,\delta y$

$= \pi \int_1^2 (4 - y^2)^2\,dy$

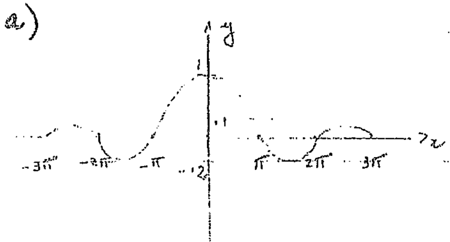
(ii) $V = \pi \int_1^2 16 - 8y^2 + y^4\,dy$

$= \pi\left[16y - \frac{8y^3}{3} + \frac{y^5}{5}\right]_1^2$

$= \pi\left[32 - \frac{64}{3} + \frac{32}{5} - 16 + \frac{8}{3} - \frac{1}{5}\right]$

$= \frac{53\pi}{15}\ u^3$

5) a)



b) (i) S is $(ae, 0)$

M is $\left(\frac{a\cos\theta + ae}{2}, \frac{b\sin\theta}{2}\right)$

(ii) Let $x = \frac{a\cos\theta + ae}{2}$ and $y = \frac{b\sin\theta}{2}$

∴ $2x = a\cos\theta + ae$ and $2y = b\sin\theta$

$\cos\theta = \frac{2x - ae}{a}$ and $\sin\theta = \frac{2y}{b}$

but $\cos^2\theta + \sin^2\theta = 1$

∴ $\left(\frac{2x - ae}{a}\right)^2 + \left(\frac{2y}{b}\right)^2 = 1$

which is the Cartesian equation

(iii) $\left(\frac{x - \frac{ae}{2}}{\frac{a}{2}}\right)^2 + \left(\frac{y}{\frac{b}{2}}\right)^2 = 1$

is of the form

$\frac{(x - x_1)^2}{a^2} + \frac{(y - y_1)^2}{b^2} = 1$

which is an ellipse.

centre is $\left(\frac{ae}{2}, 0\right)$ which is the midpoint of OS.

c) (i) using the formula for tan(A…

$\tan\left(A + \frac{\pi}{2}\right) = \frac{\tan A + \tan\frac{\pi}{2}}{1 - \tan A \tan\frac{\pi}{2}}$

but $\tan\frac{\pi}{2}$ is undefined and thus this formula will not work.

(ii) $\tan\left(A + \frac{\pi}{2}\right) = \frac{\sin\left(A + \frac{\pi}{2}\right)}{\cos\left(A + \frac{\pi}{2}\right)}$

$= \frac{\sin A\cos\frac{\pi}{2} + \cos A\sin\frac{\pi}{2}}{\cos A\cos\frac{\pi}{2} - \sin A\sin\frac{\pi}{2}}$