SCEGGS Darlinghurst

2004
Higher School Certificate
Trial Examination

Centre Number

9074
Student Number

Teddy

Mathematics Extension 2

This is a TRIAL PAPER only and does not necessarily reflect the content or format of the Higher School Certificate Examination for this subject.

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1–8
- All questions are of equal value

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Total marks – 120
Attempt Questions 1–8
All questions are of equal value

Answer each question on a NEW page

Marks

Question 1 (15 marks)

(a) Find:

(i) $\int x \cos\left(x^2\right) dx$. **2**

(ii) $\int \dfrac{dx}{\sqrt{x^2 - 4x + 5}}$. **2**

(b) (i) Find real numbers A, B and C such that: **2**

$$\frac{x-28}{\left(x^2+9\right)(x-2)} = \frac{Ax+B}{x^2+9} + \frac{C}{x-2} .$$

(ii) Hence find $\displaystyle\int \frac{x-28}{\left(x^2+9\right)(x-2)}\, dx$. **2**

(c) Use the substitution $t = \tan\left(\dfrac{\theta}{2}\right)$ to find **3**

$$\int \frac{1}{1+\sin\theta+\cos\theta}\, d\theta .$$

(d) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} e^x \cos x \; dx$. **4**

Marks

Question 2 (15 marks) START A NEW PAGE

(a) (i) Express $-\sqrt{27} + 3i$ in modulus-argument form. **2**

(ii) Hence find $\left(-\sqrt{27} + 3i\right)^6$. **2**

Give your answer in the form $a + ib$ where a and b are real.

(b) Solve $z^2 - (4-3i)z + (13+i) = 0$. **3**

(c) (i) Find the Cartesian equation of the locus represented by **2**

$$2|z| = 3\left(z + \bar{z}\right) .$$

(ii) Sketch this locus on the Argand Diagram. **1**

(d) (i) Find the solutions of $z^5 + 1 = 0$ and indicate the position of these solutions on the Argand Diagram. **2**

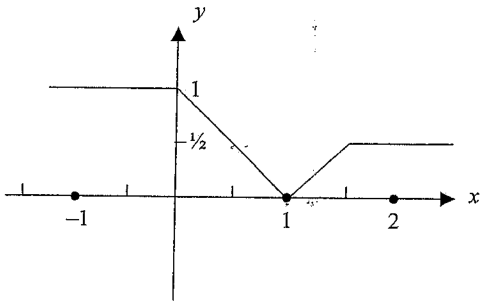
(ii) Hence show that: **3**

$$\cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \frac{1}{2} .$$

Marks

Question 3 (15 marks) START A NEW PAGE

(a) The diagram below is the function $y = f(x)$.



Draw separate sketches of the following:

(i) $y = f(-x)$ **1**

(ii) $y = f(|x|)$ **1**

(iii) $y = f(2x)$ **1**

(iv) $|y| = f(2x)$ **1**

(b) Consider the curve $x^2 + 9y^2 = 81$.

(i) Find the eccentricity. **1**

(ii) State the equation of the directrices. **1**

(iii) Show that the equation of the tangent to the curve at the point **2**

$P(x_0, y_0)$ is $\dfrac{xx_0}{81} + \dfrac{yy_0}{9} = 1$.

Question 3 continues on page 5

Marks

Question 3 (continued)

(c) Find the volume of the solid of revolution generated when the curve **3**

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is rotated about the x-axis.

(d) (i) Sketch $y = |x| - 3$ and $y = 5 + 4x - x^2$ on the same set of axes. **2**

(ii) Hence, or otherwise, solve: **2**

$$\frac{|x| - 3}{5 + 4x - x^2} \geq 0\,.$$

End of Question 3

Question 4 (15 marks) START A NEW PAGE **Marks**

(a) Using the method of cylindrical shells, find the volume generated when the area bounded by the curve $y = x(x-1)^2$, the x-axis and the lines $x = 0$ and $x = 2$ is rotated about the y axis. **4**

(b) Consider the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The equation of the tangent to the hyperbola at the point $P(a\sec\theta, b\tan\theta)$ is $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$.

(i) Show that the equation of the normal at P is **2**

$$\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2 .$$

(ii) The line through P parallel to the y axis meets the asymptote $y = \frac{bx}{a}$ at Q. The tangent at P meets the same asymptote at R. The normal at P meets the x-axis at G. Prove that $\angle RQG$ is a right angle. **3**

(iii) Hence explain why $RQPG$ is a cyclic quadrilateral. **1**

(c) The base of a solid is the circle $x^2 + y^2 = 16x$ and every cross section perpendicular to the x-axis is a rectangle whose height is twice the distance of the cross section from the origin.

(i) Show that the volume of the solid is given by: **2**

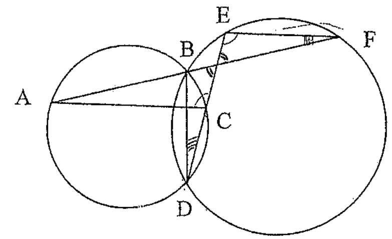
$$V = 4\int_0^{16} x\sqrt{64-(x-8)^2}\,dx$$

(ii) Using the substitution $x = 8 + 8\sin\theta$, or otherwise, show that the volume of the solid is 1024π cubic units. **3**

Question 5 (15 marks) START A NEW PAGE **Marks**

(a) The equation $x^3 - 4x^2 + 5x + 2 = 0$ has roots α, β and γ. Find the value of $\alpha^2 + \beta^2 + \gamma^2$. **2**

(b) In the diagram below, A, B, C and D lie on the circumference of the smaller circle. F, E, B and D lie on the circumference of the larger circle. ABF and DCE are straight lines. Copy the diagram and prove that AC||EF. **3**



NOT TO SCALE

(c) If $ax^4 + bx^3 + dx + e = 0$ has a non-zero triple root, show that $4a^2d + b^3 = 0$. **3**

(d) Five letters are chosen at random from the letters of the word HILARITY. These letters are placed alongside each other to form a word. Find the number of distinct arrangements possible. **4**

(e) Consider the polynomial $P(x) = x^4 - x^3 - 2x^2 + 6x - 4$. Factorise $P(x)$ completely over the complex numbers, C. **3**

Marks

Question 6 (15 marks) START A NEW PAGE

(a) (i) Prove that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. **2**

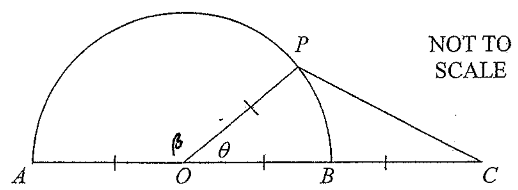
(ii) Hence show that **2**

$$\int_0^{\frac{\pi}{2}} a\cos^2 x + b\sin^2 x\,dx = \int_0^{\frac{\pi}{2}} a\sin^2 x + b\cos^2 x\,dx$$

(iii) Deduce that **2**

$$\int_0^{\frac{\pi}{2}} a\cos^2 x + b\sin^2 x\,dx = \frac{\pi}{4}(a+b)$$

(b)



In the diagram above, the fixed points A, O, B and C lie on a straight line such that $AO = OB = BC = 1$ unit. The points A and B also lie on a semi-circle centred at O. P is a variable point on this semi-circle such that $\angle POC = \theta$, $0 \le \theta \le \pi$. R is the closed region bounded by the arc AP and the straight lines PC and CA.

(i) Show that the area, S, of R is given by: $S = \frac{\pi}{2} - \frac{\theta}{2} + \sin\theta$. **1**

(ii) Find the value of θ for which S is a maximum. **2**

(iii) Show that the perimeter, L, of R is given by: **1**

$$L = 3 + \pi - \theta + \sqrt{5 - 4\cos\theta}.$$

(iv) Show that the graph of the function L would have one stationary point and that it occurs at the same value of θ for which S is a maximum. **3**

(v) Hence, find the least value of L. **2**

Marks

Question 7 (15 marks) START A NEW PAGE

(a) A sequence of integers, $u_1, u_2, u_3, \ldots$, is defined by: **5**

$$u_1 = 1$$
$$u_2 = 7$$
$$u_n = 7u_{n-1} - 12u_{n-2} \text{ for } n \ge 3$$

Use the method of mathematical induction to show that

$$u_n = 4^n - 3^n \text{ for } n \ge 1.$$

(b) Consider the function given by $f(x) = \frac{1-|x|}{|x|}$.

(i) Find whether $f(x)$ is an odd function, an even function or neither. **1**

(ii) Sketch $y = f(x)$. **1**

(iii) Hence, or otherwise, solve $f(x) \ge 1$. **2**

(iv) Sketch $y = \frac{1}{f(x)}$. **2**

(v) Hence, or otherwise, solve $\frac{1}{f(x)} \le 1$. **2**

(vi) Sketch $y = e^{f(x)}$. **2**

Question 8 (15 marks) START A NEW PAGE **Marks**

(a) $P\left(ct, \frac{c}{t}\right)$ and $Q\left(\frac{c}{t}, ct\right)$ are two distinct points on a rectangular hyperbola $xy = c^2$. R and S are two other points on the curve such that P, Q, R and S are the vertices of a rectangle.

(i) Find the co-ordinates of R and S, in terms of t. **2**

(ii) Prove that it is impossible for these 4 points to be the vertices of a square. **2**

(b) (i) Prove that $x^2 + y^2 \geq 2xy$ for all real x and y. **1**

(ii) Hence, show that $a^4 + b^4 + c^4 + d^4 \geq 4abcd$ for all real a, b, c and d. **2**

(c) (i) Sketch the curve $y = \ln x$, for all $x > 0$. **1**

(ii) Prove that the curve $y = \ln x$ is concave down, for all $x > 0$. **1**

(iii) Find an expression for the approximate area under the curve $y = \ln x$ bounded by the x-axis, the lines $x = 1$ and $x = n$ (where n is an integer) using the areas of trapezia drawn under the curve each of unit width. **1**

(iv) Show that the area under $y = \ln x$ bounded by the x-axis, the lines $x = 1$ and $x = n$ is equal to $(1 - n + n \ln n)$. **2**

(v) Hence show that **3**

$$n! < \frac{en^{n+\frac{1}{2}}}{e^n}$$

End of Paper

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EXTENSION 2 Mathematics

TRIAL EXAM, 2004.

SOLUTIONS.

QUESTION 1: (15 marks) Calc /15

(a) (i) $\int x \cdot \cos(x^2)\,dx$

$= \frac{1}{2}\int 2x \cdot \cos(x^2)\,dx$

$= \frac{1}{2}\sin(x^2) + C$ must have $+C$

(ii) $\int \frac{dx}{\sqrt{x^2-4x+5}}$

$= \int \frac{dx}{\sqrt{(x-2)^2+1}}$

$= \ln\left((x-2)+\sqrt{x^2-4x+5}\right) + C$

(b) (i) $\frac{x-28}{(x^2+9)(x-2)} = \frac{Ax+B}{x^2+9} + \frac{C}{x-2}$

$\therefore\ x-28 = (Ax+B)(x-2) + C(x^2+9)$

when $x=2$: $-26 = 13C$

$\therefore C = -2$

Equating co-eff x^2: $0 = A + C$

$\therefore A = 2$

Equating constants: $-28 = -2B + 9C$

$2B = 10$

$B = 5$

$\therefore \frac{x-28}{(x^2+9)(x-2)} = \frac{2x+5}{x^2+9} - \frac{2}{x-2}$

(ii) $\int \frac{x-28}{(x^2+9)(x-2)}\,dx$

$= \int \frac{2x}{x^2+9} + \frac{5}{x^2+9} - \frac{2}{x-2}\,dx$

$= \ln(x^2+9) - 2\ln(x-2) + \frac{5}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$

(c) $\int \frac{1}{1+\sin\theta+\cos\theta}\,d\theta$

$\sin\theta = \frac{2t}{1+t^2}$

$\cos\theta = \frac{1-t^2}{1+t^2}$

$d\theta = \frac{2}{1+t^2}\,dt$

$= \int \frac{1}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt$

$= \int \frac{2}{1+t^2+2t+1-t^2}\,dt$

$= \int \frac{2}{2t+2}\,dt$

$= \int \frac{1}{t+1}\,dt$

$= \ln(t+1) + C$

$= \ln\left(\tan\left(\frac{\theta}{2}\right)+1\right) + C$

(d) $\int_0^{\pi/2} e^x \cdot \cos x\,dx$

Integrating by parts:

$u = e^x \quad v = \sin x$

$u' = e^x \quad v' = \cos x$

$= \left[e^x \sin x\right]_0^{\pi/2} - \int_0^{\pi/2} e^x \sin x\,dx$

$u = e^x \quad v = -\cos x$

$u' = e^x \quad v' = \sin x$

$= \left[e^x \sin x\right]_0^{\pi/2} - \left(\left[-e^x\cos x\right]_0^{\pi/2} + \int_0^{\pi/2} e^x \cos x\,dx\right)$

$\therefore\ 2\int_0^{\pi/2} e^x \cdot \cos x\,dx = \left[e^x\sin x + e^x\cos x\right]_0^{\pi/2}$

$= e^{\pi/2} - 1.$

$\therefore \int_0^{\pi/2} e^x \cdot \cos x\,dx = \frac{1}{2}\left(e^{\pi/2} - 1\right)$

Comments:

(a). (i)

(ii) hint - use your table of standard integrals.

(b) (i)

(ii)

(c) Learn t results carefully!

$d\theta = \frac{2}{1+t^2}\,dt$

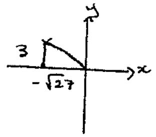
there were a few too many variations on this. Showing the derivation might improve your chances at getting it correct!

(d) Good.

Just watch careless errors in evaluation, particularly with minus signs or the calculation $0 \times 1 = 0$!!

QUESTION 2: (15 marks) Com /3 Reas /3

(a) (i) $-\sqrt{27} + 3i$



$|z| = 6$ ✓

$\arg z = \pi - \tan^{-1}\left(\frac{3}{\sqrt{27}}\right)$

$= \frac{5\pi}{6}$

$\therefore z = 6\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$ ✓

(ii) $z^6 = 6^6(\cos 5\pi + i\sin 5\pi)$ ✓ by de Moivre's Thm

$= 46656(-1 + 0)$

$= -46656$ ✓

(b) $z^2 - (4-3i)z + (13+i) = 0$

$$z = \frac{(4-3i) \pm \sqrt{(4-3i)^2 - 4.1.(13+i)}}{2}$$

$$= \frac{(4-3i) \pm \sqrt{16 - 24i - 9 - 52 - 4i}}{2}$$

$$= \frac{(4-3i) \pm \sqrt{-45-28i}}{2}$$ ✓

Let $p = \sqrt{-45-28i}$

$\therefore p^2 = -45 - 28i$

let $p = x + iy$ where $x, y \in \mathbb{R}$

$\therefore$ Equating real + imaginary coeff:

$x^2 - y^2 = -45$ ①

$2xy = -28$ ②.

Sub $y = \frac{-14}{x}$ into ①

$x^2 - \frac{14^2}{x^2} = -45$

$x^4 + 45x^2 - 14^2 = 0$

$(x^2 + 49)(x^2 - 4) = 0$

Since x is real,

$x = \pm 2$

$y = \mp 7$

$\therefore p = 2 - 7i$ or $-2 + 7i$ ✓

$\therefore z = \frac{(4-3i) \pm (2-7i)}{2}$

$= 3 - 5i$ OR $1 + 2i$. ✓

(c) (i) Let $z = x + iy$

$2|x+iy| = 3(x+iy+x-iy)$

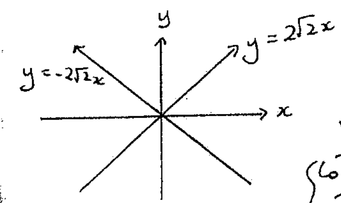
$2\sqrt{x^2+y^2} = 3(2x)$ ✓

$x^2 + y^2 = 9x^2$

$y^2 = 8x^2$

$y = \pm 2\sqrt{2}x$ ✓

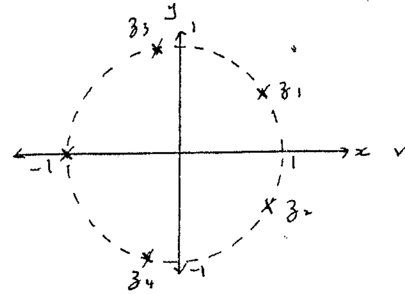
(ii)



✓

Com 1

(d) (i) $z^5 = -1$



✓

Roots are -1, $\cos\frac{\pi}{5} + i\sin\pi/5$ (z_1)

$\cos -\pi/5 + i\sin -\pi/5$ (z_2)

$\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}$ (z_3) ✓

$\cos -\frac{3\pi}{5} + i\sin -\frac{3\pi}{5}$ (z_4)

Com 2

(ii) $z^5 + 1 = 0$

$\therefore (z+1)(z-z_1)(z-z_2)(z-z_3)(z-z_4) = 0$

$\therefore$ But $z_1 + z_2 + z_3 + z_4 - 1 = -b/a$

$\therefore -1 + z_1 + z_2 + z_3 + z_4 = 0$ ✓

$\therefore (z_1 + z_2) + (z_3 + z_4) = 1$

$\cos\pi/5 + \cos -\pi/5 + \cos\frac{3\pi}{5} + \cos -\frac{3\pi}{5} = 1$ ✓

$\therefore 2\cos\pi/5 + 2\cos\frac{3\pi}{5} = 1$

$\therefore \cos\pi/5 + \cos\frac{3\pi}{5} = \frac{1}{2}$ ✓

Reas 3

Comments:

(a) (i) ✓

(ii) Simplify as much as possible ... ie -46656.

(b) NB You know how to find the $\sqrt{a+ib}$, so you must keep going beyond the initial application of the quadratic formula.

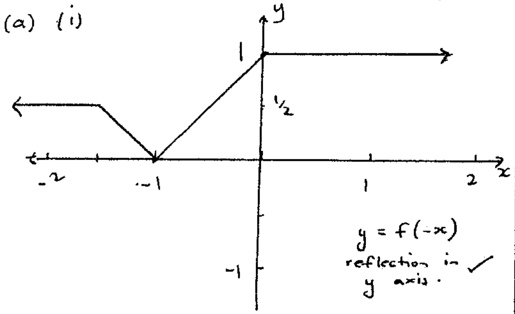
(c) (i) A few careless arithmetic errors here made (ii) much harder than it actually was.

(d) (i) List the roots and indicate clearly that they all lie on the unit circle.

(ii) Often, not attempted. Easiest method is using polynomial techniques of the sum of the roots. (Other, longer, methods are also possible!)

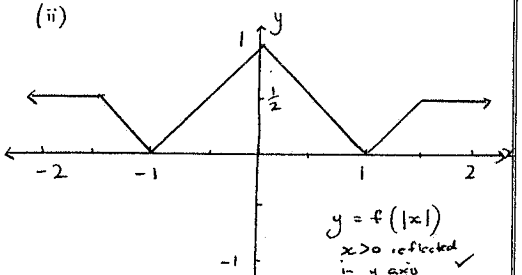
QUESTION 3 : (15 marks) Com /4 Reas /2 Calc /3

(a) (i)



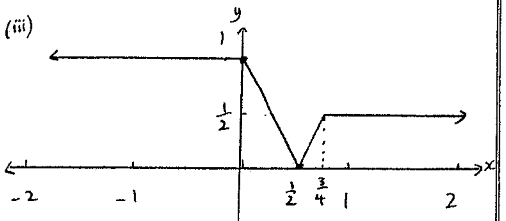
$y = f(-x)$
reflection in y axis. ✓

(ii)



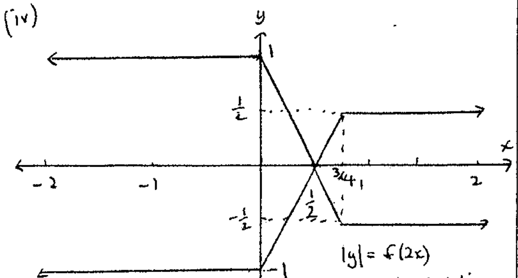
$y = f(|x|)$
$x>0$ reflected in y axis ✓

(iii)



$y = f(2x)$
horizontal scale altered. ✓

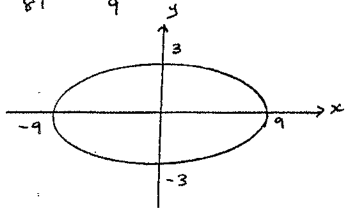
(iv)



$|y| = f(2x)$
(iii) + (iii) reflected in x-axis ✓

(b) $x^2 + 9y^2 = 81$

$\therefore \dfrac{x^2}{81} + \dfrac{y^2}{9} = 1$



(i) $9 = 81(1-e^2)$

$\dfrac{1}{9} = 1 - e^2$

$\therefore e = \dfrac{2\sqrt{2}}{3}$ ✓

(ii) $x = \pm \dfrac{27}{2\sqrt{2}}$ $\left(= \pm \dfrac{27\sqrt{8}}{8}\right)$ ✓

(iii) Differentiating,

$2x + 18y \cdot \dfrac{dy}{dx} = 0$

$\therefore \dfrac{dy}{dx} = \dfrac{-x}{9y}$

$\therefore$ Gradient of tangent at $P(x_0, y_0)$

$= \dfrac{-x_0}{9y_0}$ ✓

$\therefore$ Equation :

$y - y_0 = \dfrac{-x_0}{9y_0}(x - x_0)$

$9yy_0 - 9y_0^2 = -xx_0 + x_0^2$

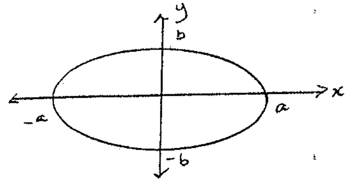
$\therefore xx_0 + 9yy_0 = x_0^2 + 9y_0^2$

$= 81$ ✓

$\therefore \dfrac{xx_0}{81} + \dfrac{yy_0}{9} = 1$

as required.

(c)



$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$\therefore y = \pm \dfrac{b}{a}\sqrt{a^2 - x^2}$

$\therefore V = \pi \int_{-a}^{a} \dfrac{b^2}{a^2}\left(a^2 - x^2\right) dx$ ✓

$= \pi \cdot \dfrac{2b^2}{a^2} \int_0^a a^2 - x^2 \, dx$ since curve is even.

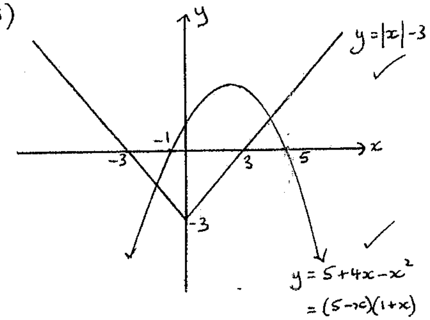
$= \dfrac{2b^2\pi}{a^2}\left[a^2x - \dfrac{x^3}{3}\right]_0^a$ ✓

$= \dfrac{2b^2\pi}{a^2}\left(a^3 - \dfrac{a^3}{3} - 0\right)$

$= \dfrac{2b^2\pi}{a^2} \cdot \dfrac{2a^3}{3}$

$= \dfrac{4}{3}\pi a b^2$ ✓ Calc 3

(d) (i)



$y = |x| - 3$ ✓

$y = 5 + 4x - x^2$
$= (5-x)(1+x)$ ✓

(ii) $\dfrac{|x| - 3}{5 + 4x - x^2} \geq 0$

$\therefore -3 \leq x < -1$ ✓ and $3 \leq x < 5$ ✓ Reas 2

Comments :

(a) (i) ✓

(ii) ✓

(iii) ✓

(iv) Not well done.
Note : $y = \pm f(2x)$ might make it easier to draw?

(b) Good.
Note : make sure that you find the gradient of the tangent at P ie $\dfrac{-x_0}{9y_0}$.

(c) Many careless errors - wrong limits, no π, no y^2, rotation around y etc.

(d) (i) Again, label diagrams well.

(ii) Always look for the connection to the part before if "hence" is used...

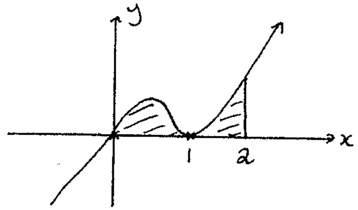
To be positive, either :
- $|x| - 3$ and $5 + 4x - x^2$ are >0

or - $|x| - 3$ and $5 + 4x - x^2$ are <0

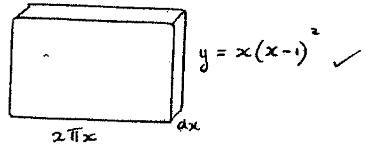
Also make sure that zero denominator solutions are excluded!

QUESTION 4: (15 marks) Reas $\frac{}{9}$ Calc $\frac{}{6}$

(a)



Typical shell:



$$V = 2\pi\int_0^2 x^2(x-1)^2\,dx \quad ✓$$

$$= 2\pi\int_0^2 x^4 - 2x^3 + x^2\,dx$$

$$= 2\pi\left[\frac{x^5}{5} - \frac{x^4}{2} + \frac{x^3}{3}\right]_0^2 \quad ✓$$

$$= 2\pi\left(\frac{32}{5} - \frac{16}{2} + \frac{8}{3}\right)$$

$$= \frac{32\pi}{15} \text{ units}^3 \quad ✓$$

Calc $\frac{}{4}$

(b) (i) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Differentiating,

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \frac{b^2x}{a^2y}$$

$\therefore$ Gradient of the normal at P:

$$= \frac{-a^2 \cdot b\tan\theta}{b^2 \cdot a\sec\theta}$$

$$= \frac{-a\tan\theta}{b\sec\theta} \quad ✓$$

$\therefore$ Equation of normal:

$$y - b\tan\theta = \frac{-a\tan\theta}{b\sec\theta}(x - a\sec\theta)$$

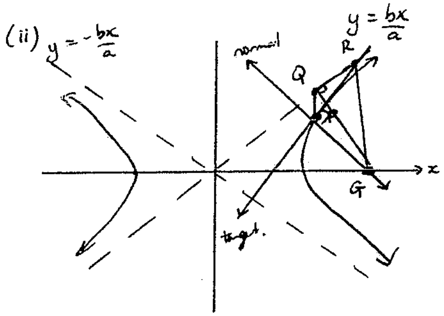
$$by\sec\theta - b^2\tan\theta\sec\theta = -ax\tan\theta + a^2\tan\theta\sec\theta \quad ✓$$

$$\therefore ax\tan\theta + by\sec\theta = (a^2+b^2)\tan\theta\sec\theta$$

$$\therefore \frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$$

Calc $\frac{}{2}$

(ii)



$P(a\sec\theta, b\tan\theta)$

$G\left(\frac{a^2+b^2}{a}\sec\theta, 0\right)$

$Q(a\sec\theta, b\sec\theta)$ ✓

$R\left(\frac{a}{\sec\theta - \tan\theta}, \frac{b}{\sec\theta - \tan\theta}\right)$

$$\therefore m_{QG} = \frac{b\sec\theta}{a\sec\theta - a\sec\theta - \frac{b^2}{a}\sec\theta}$$

$$= \frac{b\cancel{\sec\theta}}{-\frac{b^2}{a}\cancel{\sec\theta}}$$

$$= -\frac{a}{b} \quad ✓$$

$m_{QR} = \frac{b}{a}$

since both Q and R lie on the asymptote $y = \frac{bx}{a}$ which has gradient $\frac{b}{a}$.

[OR use grad. formula] ✓

$$\therefore m_{QG} \times m_{QR} = -\frac{a}{b} \times \frac{b}{a} = -1$$

Reas $\frac{}{3}$

$\therefore \angle RQG$ is a right angle.

(iii) $\angle RPG = 90°$ ($\angle$ between tangent and normal at P is 90°)

$\angle RQG = 90°$ (proved in (ii))

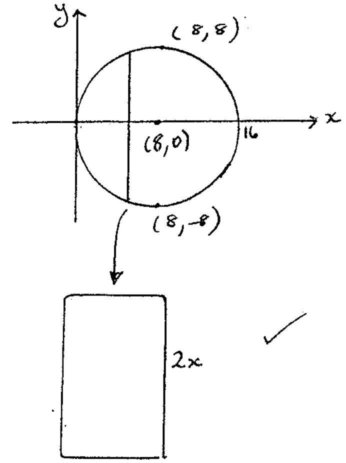
$\therefore$ RQPG is a cyclic quadrilateral since two angles ($\angle RPG + \angle RQG$) at the circumference, standing on the same arc are equal. ✓

Reas $\frac{}{1}$

(c) (i) $x^2 + y^2 = 16x$

$x^2 - 16x + y^2 = 0$

$(x-8)^2 + y^2 = 64$



$$2y = 2\sqrt{64-(x-8)^2}$$

$$\therefore \text{Volume} = \int_0^{16} 4x\sqrt{64-(x-8)^2}\,dx \quad ✓$$

$$= 4\int_0^{16} x\sqrt{64-(x-8)^2}\,dx$$

(ii) Let $x = 8 + 8\sin\theta$

$dx = 8\cos\theta \, . d\theta$

$x = 0 \rightarrow \theta = -\pi/2$

$x = 16 \rightarrow \theta = \pi/2$

$$\therefore V = 4\int_{-\pi/2}^{\pi/2}(8+8\sin\theta)\sqrt{64-64\sin^2\theta}\;8\cos\theta\,d\theta \quad ✓$$

$$= 2048\int_{-\pi/2}^{\pi/2}(1+\sin\theta)\cos^2\theta\,d\theta$$

$$= 2048\int_{-\pi/2}^{\pi/2}\cos^2\theta + \sin\theta\cos^2\theta\,d\theta \quad ✓$$

$$= 2048\int_{-\pi/2}^{\pi/2}\frac{1}{2}(\cos 2\theta + 1) + \sin\theta\cos^2\theta\,d\theta$$

$$= 2048\left[\frac{1}{4}\sin 2\theta + \frac{\theta}{2} - \frac{\cos^3\theta}{3}\right]_{-\pi/2}^{\pi/2} \quad ✓$$

$$= 2048\left[\left(0 + \frac{\pi}{4} - 0\right) - \left(0 - \frac{\pi}{4} - 0\right)\right]$$

$$= 1024\pi \text{ units}^3.$$

Reas $\frac{}{5}$

Comments:

(a) Function doesn't need to be split.

(b) (i) ✓

(ii) Draw a diagram + label everything on it. This will help with gradients etc.

(iii) usually not attempted.

(c) (i) ✓

(ii) Substitution was sloppy resulting in incorrect integrals, particularly common factors.

QUESTION 5: (15 marks) Reas 7

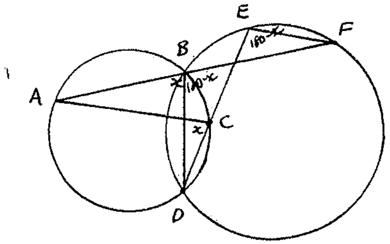
(a) $x^3 - 4x^2 + 5x + 2 = 0$

$\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\alpha\gamma)$

$= 4^2 - 2.5$

$= 6$

(b)



Let $\angle ACD = x$

$\therefore \angle ABD = x$ ($\angle$'s at the circum, standing on the same arc in the same segment are =)

$\therefore \angle DBF = 180 - x$ ($\angle \Sigma$ st. line $= 180°$)

$\therefore \angle DEF = 180 - x$ ($\angle$'s at the circum, standing on the same arc in the same segment are =)

Also $\angle ACE = 180 - x$ ($\angle \Sigma$ st. line $= 180°$)

$\therefore \angle ACE = \angle DEF$

$\therefore$ $AC \parallel EF$ since alternate angles are equal.

Reas 3

(c) Let $P(x) = ax^4 + bx^3 + dx + e = 0$

and let triple root be $x = k$.

$P'(x) = 4ax^3 + 3bx^2 + d$

$P''(x) = 12ax^2 + 6bx$

Since $x = k$ is a triple root,

$P'(k) = 0$ and $P''(k) = 0$.

$\therefore$ $12ak^2 + 6bk = 0$

$2ak + b = 0$ ① (since $k \neq 0$)

and $4ak^3 + 3bk^2 + d = 0$. ②

Solving simultaneously,

$$4a\left(\frac{-b}{2a}\right)^3 + 3b\left(\frac{-b}{2a}\right)^2 + d = 0$$

$$\frac{-b^3}{2a^2} + \frac{3b^3}{4a^2} + d = 0$$

$$\frac{b^3}{4a^2} + d = 0$$

$$\therefore 4a^2d + b^3 = 0$$

(d) H I L A R T Y
I

8 letters, 2 I's.

Choosing 5 letters, it is possible to make words with:

2 I's, 1 I or No I.

2 I's : ${}^6C_3 \times \frac{5!}{2!} = 1200$

1 I : ${}^6C_4 \times 5! = 1800$

0 I : ${}^6C_5 \times 5! = 720$

$\therefore$ Total is 3720 arrangements

Reas 4

(e) $P(x) = x^4 - x^3 - 2x^2 + 6x - 4$

$P(1) = 1 - 1 - 2 + 6 - 4 = 0$

$P(-2) = 16 - 8 - 8 - 12 - 4 = 0$

$\therefore P(x) = (x-1)(x+2)(x^2 - 2x + 2)$

$= (x-1)(x+2)(x-(1+i))(x-(1-i))$

Comments:

(a) Learn the expansion and evaluate carefully!

(b) Have a go at the Circle Geometry!! make sure ~~that~~ you write correct, logical reasons for your deductions.

(c) Poor.

(d) NB More care needed in considering possibilities of 5 letter words.

(e) Well done.

QUESTION 6: (15 marks) Reas $\frac{}{2}$ Calc $\frac{}{6}$

(a) (i) $\int_0^a f(x)\,dx$

Let $x = a - u$
$dx = -du$

$= \int_a^0 f(a-u)\,.\,-du$ ✓

$= \int_0^a f(a-u)\,du$ ✓

$= \int_0^a f(a-x)\,dx.$

(ii) $\int_0^{\frac{\pi}{2}} a\cos^2 x + b\sin^2 x\,dx$

$= \int_0^{\pi/2} a\cos^2\left(\frac{\pi}{2}-x\right) + b\sin^2\left(\frac{\pi}{2}-x\right)\,dx$ ✓ using (i).

$= \int_0^{\pi/2} a\sin^2 x + b\cos^2 x\,dx$ ✓

(iii) $\int_0^{\pi/2} a\cos^2 x + b\sin^2 x\,dx$

$= \frac{1}{2}\left[\int_0^{\pi/2} a\cos^2 x + b\sin^2 x + a\sin^2 x + b\cos^2 x\,dx\right]$

$= \frac{1}{2}\left[\int_0^{\pi/2} a(\sin^2 x + \cos^2 x) + b(\sin^2 x + \cos^2 x)\,dx\right]$

$= \frac{1}{2}\int_0^{\pi/2} (a+b)\,dx$ ✓

$= \frac{1}{2}\left[(a+b)x\right]_0^{\pi/2}$

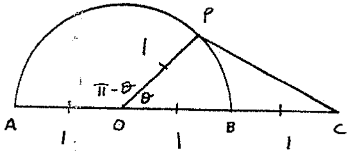
$= \frac{1}{2}\cdot(a+b).\,\pi/2$

$= \frac{\pi}{4}(a+b)$ ✓

or replace $\sin^2 x$ with $1-\cos^2 x$.

Calc $\frac{}{6}$

(b)



(i) S = sector AOP + ΔPOC

$= \frac{1}{2}.1^2.(\pi-\theta) + \frac{1}{2}.1.2.\sin\theta$ ✓

$= \frac{\pi}{2} - \frac{\theta}{2} + \sin\theta$

(ii) $\frac{dS}{d\theta} = -\frac{1}{2} + \cos\theta$

Max occurs when $\frac{dS}{d\theta} = 0$

$\therefore \cos\theta = \frac{1}{2}$

$\therefore \theta = \frac{\pi}{3}$ (since $0 \le \theta \le \pi$) ✓

$\frac{d^2S}{d\theta^2} = -\sin\theta$

when $\theta = \frac{\pi}{3}$, $\frac{d^2S}{d\theta^2} = -\frac{\sqrt{3}}{2} < 0$ ✓

$\therefore \theta = \frac{\pi}{3}$ is a maximum

(iii) L = arc AP + PC + AC

$= 1.(\pi-\theta) + PC + 3$

but $PC^2 = 1^2 + 2^2 - 2.1.2\cos\theta$

$\therefore PC = \sqrt{5-4\cos\theta}$ ✓

$\therefore L = 3 + \pi - \theta + \sqrt{5-4\cos\theta}$

(iv) $\frac{dL}{d\theta} = -1 + \frac{1}{2}(5-4\cos\theta)^{-\frac{1}{2}}.\,4\sin\theta$ ✓

Stat points occur when $\frac{dL}{d\theta} = 0$

$1 = \frac{2\sin\theta}{\sqrt{5-4\cos\theta}}$

$5 - 4\cos\theta = 4\sin^2\theta$

$5 - 4\cos\theta = 4 - 4\cos^2\theta$

$\therefore 4\cos^2\theta - 4\cos\theta + 1 = 0$ ✓

$(2\cos\theta - 1)^2 = 0$

$\therefore 2\cos\theta = 1$

$\cos\theta = \frac{1}{2}$ ✓

$\therefore \theta = \frac{\pi}{3}$ since $0 \le \theta \le \pi$

which is the same value.

(v) Using first derivative test,

θ	$\frac{\pi}{3}^-$	$\pi/3$	$\pi/3^+$
$\frac{dL}{d\theta}$	$-$ve	0	$-$ve

$\therefore \theta = \pi/3$ is a horizontal p.o.i

and L is decreasing over the whole domain $0 \le \theta \le \pi$. ✓

$\therefore$ Least value occurs at $x = \pi$

where $L = 3 + \pi - \pi + \sqrt{5-4\cos\pi}$
$= 6$ ✓

Reas $\frac{}{2}$

Comments:

(a) Poorly done!

(i) Need to be able to prove this.

(ii) - (iii) ✗

(b) Many easy marks in this question, which weren't always awarded.

NB At this stage of the paper, choose the questions you are going to have a real go at, and leave the others out altogether. A very half-hearted, rushed attempt at all the questions may not get you any marks at all!

QUESTION 7: (15 marks) Com /5 Reas /8

(a) $n=1$:

$u_1 = 4^1 - 3^1$

$= 1$

as defined ✓

$n=2$:

$u_2 = 4^2 - 3^2$

$= 7$

as defined ✓

Assume $u_k = 4^k - 3^k$ for all integers up to k.

Investigate u_{k+1}:

$u_{k+1} = 7u_k - 12u_{k-1}$ by the definition ✓

$= 7(4^k - 3^k) - 12(4^{k-1} - 3^{k-1})$ ✓

$= 28.4^{k-1} - 21.3^{k-1} - 12.4^{k-1} + 12.3^{k-1}$

$= 16.4^{k-1} - 9.3^{k-1}$

$= 4^2.4^{k-1} - 3^2.3^{k-1}$

$= 4^{k+1} - 3^{k+1}$

as required. ✓ Com /5

∴ If true for $n=k$, statement is also true for $n=k+1$.

Since statement true for $n=1$ and $n=2$, it is also true for $n=3, 4, 5, \ldots$ and hence all positive integers by the principle of mathematical induction.

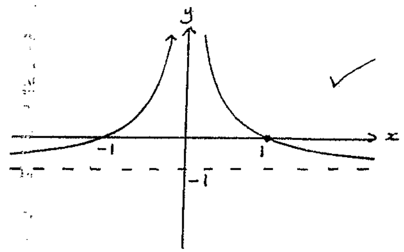
(b) (i) $f(x) = \dfrac{1-|x|}{|x|}$

$f(-x) = \dfrac{1-|-x|}{|-x|}$

$= \dfrac{1-|x|}{|x|}$ ✓

∴ $f(x)$ is even.

(ii) $f(x) = \dfrac{1}{|x|} - 1$



✓

(iii) $\dfrac{1}{|x|} - 1 = 1$

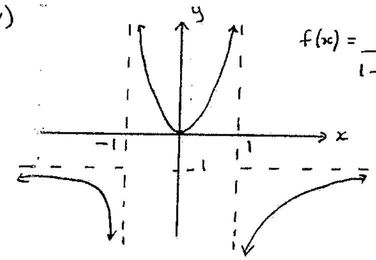
$\dfrac{1}{|x|} = 2$

$\dfrac{1}{2} = |x|$

$\therefore x = \pm\frac{1}{2}$ ✓

$\therefore f(x) \geq 1$ for $-\frac{1}{2} \leq x < 0$ and $0 < x \leq \frac{1}{2}$ ✓

(iv)



$f(x) = \dfrac{|x|}{1-|x|}$

✓✓

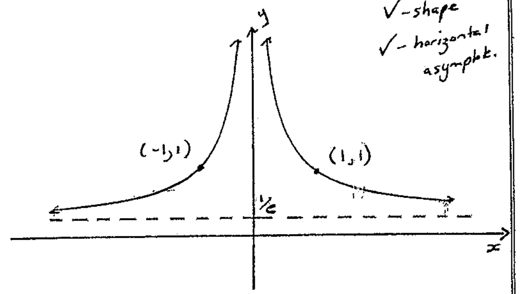
(v) $\dfrac{|x|}{1-|x|} = 1$

$2|x| = 1$

$x = \pm\frac{1}{2}$

$\therefore x < -1,\ -\frac{1}{2} \leq x \leq \frac{1}{2},\ x > 1$ ✓✓

(vi) $y = e^{f(x)}$



✓ - shape
✓ - horizontal asymptote.

Reas /8 (b)(iii) - (vi)

Comments:

(a) Need to test two values as induction step for U_n uses U_{n-1} and U_{n-2}.

Make sure you understand the notation.

(b) (i) An easy mark! ✓

(ii) ✓

(iii) NB $x \neq 0$ make sure this isn't included in any solution set.

(iv) NB $\dfrac{1}{\text{even } f^n}$ = even f^n

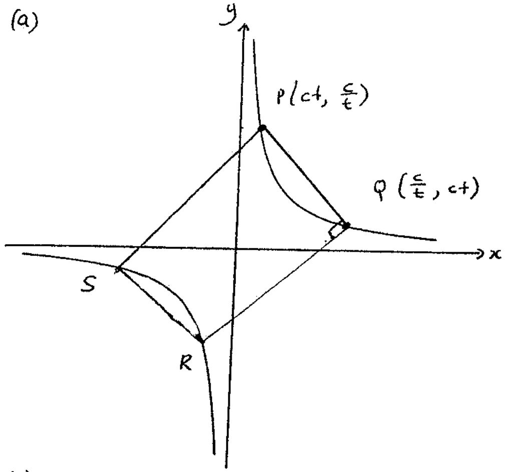
∴ Draw $x > 0$ and then reflect in y axis

(v) use your graph. Ensure solution is consistent with (iv)

(vi) Always consider horizontal assymptotes eg $x \to \infty$, $e^{f(x)} \to ?$

QUESTION 8: (15 marks) Reas $\frac{}{13}$

(a)



(i) $m_{PQ} = \dfrac{ct - \frac{c}{t}}{\frac{c}{t} - ct} = \dfrac{ct^2 - c}{c - ct^2} = -1$

$\therefore$ Gradient RQ and PS = 1

Eq^n RQ: $y - ct = x - \frac{c}{t}$

Eq^n PS: $y - \frac{c}{t} = x - ct$

and $y = \frac{c^2}{x}$ for hyperbola.

$\therefore$ Coordinates of R:

$\frac{c^2}{x} - ct = x - \frac{c}{t}$

$tc^2 - cxt^2 = x^2 t - cx$

$\therefore tx^2 + (t^2 - 1)cx - tc^2 = 0$

$(tx - c)(x + tc) = 0$

$\therefore x = \frac{c}{t}$ (which is Q)

or $x = -ct$ which is R

$\therefore y = \dfrac{c^2}{-ct} = -\frac{c}{t}$

$\therefore R\left(-ct, -\frac{c}{t}\right)$ ✓

Coordinates of S:

$\frac{c^2}{x} - \frac{c}{t} = x - ct$

$c^2 t - cx = x^2 t - cxt^2$

$\therefore tx^2 + (-ct^2 + c)x - c^2 t = 0$

$(tx + c)(x - ct) = 0$

$\therefore x = ct$ (which is P)

or $x = -\frac{c}{t}$ (which is S)

$\therefore y = \dfrac{c^2}{-\frac{c}{t}} = -ct$

$\therefore S\left(-\frac{c}{t}, -ct\right)$ ✓

(ii) If PQRS is a square,

then PQ = RQ

$\therefore (ct - \frac{c}{t})^2 + (\frac{c}{t} - ct)^2$

$= (\frac{c}{t} + ct)^2 + (ct + \frac{c}{t})^2$ ✓

$\therefore (t - \frac{1}{t})^2 + (\frac{1}{t} - t)^2 = (\frac{1}{t} + t)^2 + (t + \frac{1}{t})^2$

$\therefore$ Since $(t - \frac{1}{t})^2 = (\frac{1}{t} - t)^2$,

$\cancel{2}(t - \frac{1}{t})^2 = \cancel{2}(t + \frac{1}{t})^2$

$\therefore \cancel{t^2} - 2 + \cancel{\frac{1}{t^2}} = \cancel{t^2} + 2 + \cancel{\frac{1}{t^2}}$

$\therefore -2 = 2$ ✓

which is a contradiction,

$\therefore$ It is impossible for PQRS to be a square.

Reas $\frac{}{4}$

(b) (i) $(x - y)^2 \geq 0$

$\therefore x^2 - 2xy + y^2 \geq 0$ ✓

$\therefore x^2 + y^2 \geq 2xy$ for all $x, y \in \mathbb{R}$

(ii) $a^4 + b^4 + c^4 + d^4$

$= \underbrace{(a^2)^2 + (b^2)^2} + \underbrace{(c^2)^2 + (d^2)^2}$

$\geq 2a^2b^2 + 2c^2d^2$ ✓ by (i)

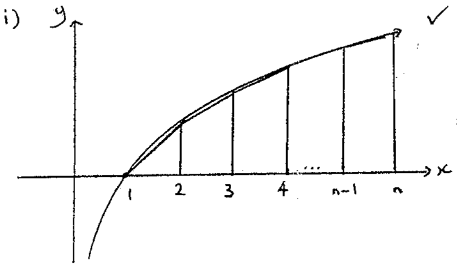
$\geq 2(a^2b^2 + c^2d^2)$

$\geq 2((ab)^2 + (cd)^2)$

$\geq 2(2abcd)$ ✓ by (i) again

$\geq 4abcd$ as required. Reas $\frac{}{3}$

(c) (i)



✓

(ii) $y = \ln x$

$\frac{dy}{dx} = \frac{1}{x}$

$\frac{d^2y}{dx^2} = -\frac{1}{x^2}$

which is < 0 for all x. ✓

$\therefore y = \ln x$ is concave down.

(iii) Area $\doteq \frac{1}{2}(\ln 1 + \ln 2) + \frac{1}{2}(\ln 2 + \ln 3) +$

$\frac{1}{2}(\ln 3 + \ln 4) + \ldots + \frac{1}{2}(\ln(n-1) + \ln n)$ ✓

$= \ln 2 + \ln 3 + \ldots + \ln(n-1) + \frac{1}{2}\ln n.$

(iv) Area $= \int_1^n \ln x \, dx$

Integrating by parts.

$u = \ln x \quad v = x$

$u' = \frac{1}{x} \quad v' = 1$ ✓

$= [x \ln x]_1^n - \int_1^n \frac{1}{x} \cdot x \, dx$

$= n \ln n - [x]_1^n$

$= n \ln n - n + 1$ ✓

(v) Since $y = \ln x$ is concave down for all x, the approximate area in (iii) is less than the exact area in (iv) for all n.

$\therefore \ln 2 + \ln 3 + \ldots + \ln(n-1) + \frac{1}{2}\ln n$

$< n \ln n - n + 1$ ✓

$\therefore \ln 2 + \ln 3 + \ldots + \ln n - \frac{1}{2}\ln n$

$< n \ln n - n + 1$

$\therefore \ln n! - \frac{1}{2}\ln n < n \ln n - n + 1$ ✓

$\therefore \ln n! - \ln\sqrt{n} < \ln n^n - n + 1$

$\therefore \ln n! < \ln n^n + \ln\sqrt{n} - n + 1$

$\therefore n! < e^{\ln n^n + \ln\sqrt{n} - n + 1}$

$< e^{\ln n^n} \cdot e^{\ln\sqrt{n}} \cdot e^{-n} \cdot e^1$

$< \dfrac{n^n \cdot \sqrt{n} \cdot e}{e^n}$

$< \dfrac{e \cdot n^{n + \frac{1}{2}}}{e^n}$ ✓

as required.

Reas $\frac{}{6}$
(c) (iii) – (v)