SCEGGS Darlinghurst

2008

HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Mathematics Extension 2

This is a TRIAL PAPER only and does not necessarily reflect the content or format of the Higher School Certificate Examination for this subject.

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1–8
- All questions are of equal value

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Total marks – 120
Attempt Questions 1–8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (15 marks)

(a) Find $\displaystyle\int \frac{dx}{\sqrt{16x^2-1}}$ **2**

(b) Evaluate $\displaystyle\int_1^e x\ln x\,dx$ **3**

(c) (i) Find real numbers a and b such that **2**

$$\frac{5x^2+x+8}{(x+1)(x^2+3)} \equiv \frac{a}{x+1}+\frac{bx-1}{x^2+3}$$

(ii) Hence find $\displaystyle\int \frac{5x^2+x+8}{(x+1)(x^2+3)}\,dx$ **2**

(d) Find $\displaystyle\int \tan^3 x\,dx$ **2**

(e) Using a suitable substitution, or otherwise, evaluate: **4**

$$\int_0^2 \frac{x^2}{\sqrt{4-x^2}}\,dx$$

End of Question 1

Marks

Question 2 (15 marks) Use a SEPARATE writing booklet.

(a) Let $\alpha = 1-\sqrt{3}\,i$.

(i) Find the exact value of $|\alpha|$ and $\arg\alpha$. **2**

(ii) Hence express $\left(1-\sqrt{3}\,i\right)^{10}$ in modulus-argument form. **1**

(b) Express $\sqrt{7-24i}$ in the form $a+ib$, where a and b are real. **3**

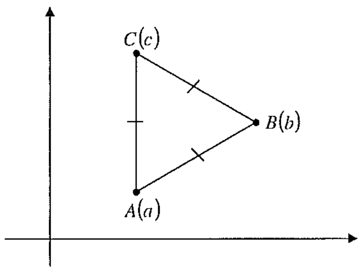
(c) Sketch the region in the complex plane where the two inequalities **3**
$0\le \operatorname{Arg}(z)\le\frac{3\pi}{4}$ and $|z-2i|\ge|z|$ both hold.

(d) Sketch the locus of z satisfying $|z-3|+|z+3| = 10$. **3**
Show any intercepts with the axes.

Question 2 continues on page 4

Question 2 (continued) **Marks**

(e)



The points A, B and C on the Argand diagram represent the complex numbers a, b and c respectively. ΔABC is equilateral.

Let $w = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$.

(i) Show that $\frac{a-b}{c-b} = w$. **1**

(ii) By writing another similar expression for w, prove that **2**

$$a^2 + b^2 + c^2 = ab + bc + ca$$

End of Question 2

Marks

Question 3 (15 marks) Use a SEPARATE writing booklet.

(a) The equation $x^3 + 3x^2 - 5x - 2 = 0$ has roots α, β and γ. **2**

Find a cubic equation with integer coefficients whose roots are $\frac{2}{\alpha}, \frac{2}{\beta}$ and $\frac{2}{\gamma}$.

(b) Consider the curve $x^2 + y^2 + xy = 3$.

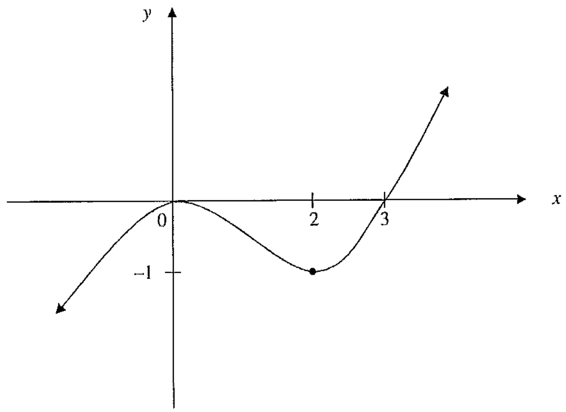
(i) Show that $\frac{dy}{dx} = -\left(\frac{2x+y}{x+2y}\right)$. **1**

(ii) Hence find the coordinates of any stationary points. **2**

Question 3 continues on page 6

Marks

(c) The diagram shows the graph of $y = f(x)$ where $f(x) = \frac{1}{4}x^2(x-3)$.



On the answer page provided, draw separate sketches of the graphs of the following:

(i) $y = \frac{1}{4}x^2 \left| x - 3 \right|$ **1**

(ii) $y = \frac{1}{f(x)}$ **1**

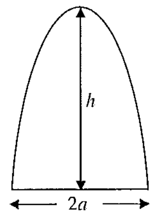
(iii) $y^2 = -f(x)$ **2**

(iv) $y = \tan^{-1}(f(x))$ **2**

Question 3 continues on page 7

Marks

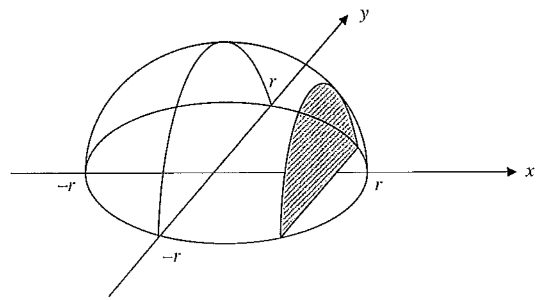
(d) (i) **1**



A parabolic segment has height h and width $2a$.

Use Simpson's Rule with three function values, to show that the exact area of this segment is $\frac{4ah}{3}$.

(ii) **3**



The base of a solid is the region in the xy plane enclosed by the circle $x^2 + y^2 = r^2$.

Each cross-section perpendicular to the x-axis is a parabolic segment with height one half its width.

Show that the volume of the solid is $\frac{16r^3}{9}$ units^3.

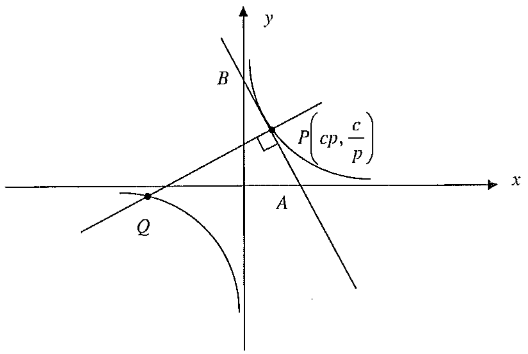
End of Question 3

Marks

Question 4 (15 marks) Use a SEPARATE writing booklet.

(a) The point $P\left(cp, \frac{c}{p}\right)$ is a point on the hyperbola $xy = c^2$.

The tangent to the hyperbola at P intersects the x and y axes at A and B respectively and the normal to the hyperbola at P intersects the second branch at Q.



(i) Show that the equation of the normal at P is $py - c = p^3(x - cp)$. **2**

(ii) Show that the x coordinates of P and Q satisfy the equation **2**

$$x^2 - c\left(p - \frac{1}{p^3}\right)x - \frac{c^2}{p^2} = 0$$

and hence find the coordinates of Q.

(iii) Given the distance $AB = 2c\sqrt{p^2 + \frac{1}{p^2}}$, show that the **2**

area of $\Delta ABQ = c^2\left(p^2 + \frac{1}{p^2}\right)^2$.

(iv) Find the minimum area of ΔABQ. **1**

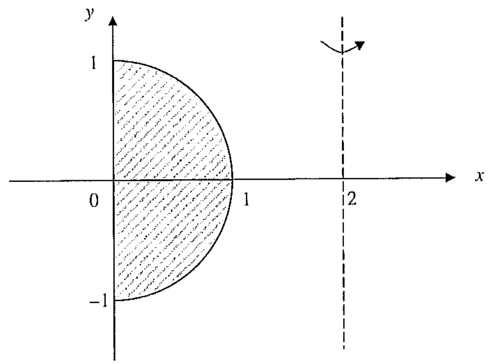
(You may use the inequality $\frac{a}{b} + \frac{b}{a} \geq 2$ for $a, b > 0$.)

Question 4 continues on page 9

Marks

Question 4 (continued)

(b)



The shaded semicircle in the diagram above is rotated about the line $x = 2$.

(i) Using the method of cylindrical shells, show that the volume V of the resulting solid is given by **1**

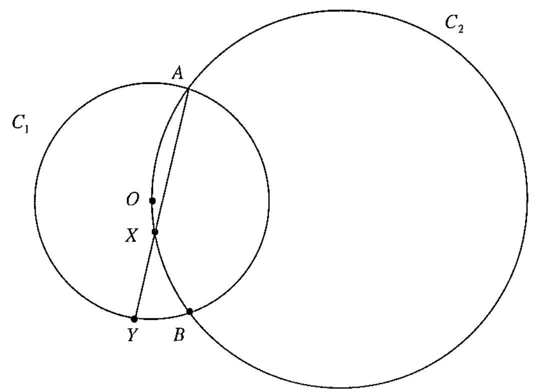
$$V = \int_0^1 4\pi(2 - x)\sqrt{1 - x^2}\,dx$$

(ii) Hence find the volume of the solid. **3**

Question 4 continues on page 10

Question 4 (continued) **Marks**

(c)



Two circles C_1 and C_2 intersect at A and B. C_2 passes through O, the centre of C_1. X lies on the arc AOB and AX intersects C_1 again at Y.

(i) State why $\angle AOB = 2 \times \angle AYB$. **1**

(ii) Prove that $XY = XB$. **3**

End of Question 4

Marks

Question 5 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Show that if α is a double root of $f(x)=0$ then $f(\alpha)=f'(\alpha)=0$. **2**

(ii) Find all roots of the equation $2x^3-5x^2-4x+12=0$ given that two of the roots are equal. **3**

(b) (i) By drawing a diagram, or otherwise, find the solutions of $z^5=1$. **2**

(ii) Show that $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$. **2**

(iii) Hence find the exact value of $\cos\frac{2\pi}{5}$. **2**

(c) 11 persons gather to play basketball by forming 2 teams of 5 to play each other. The remaining person acts as a referee.

(i) In how many ways can the teams be formed? **2**

(ii) If two particular persons are not to be in the same team, how many ways are there then to choose the teams? **2**

End of Question 5

Marks

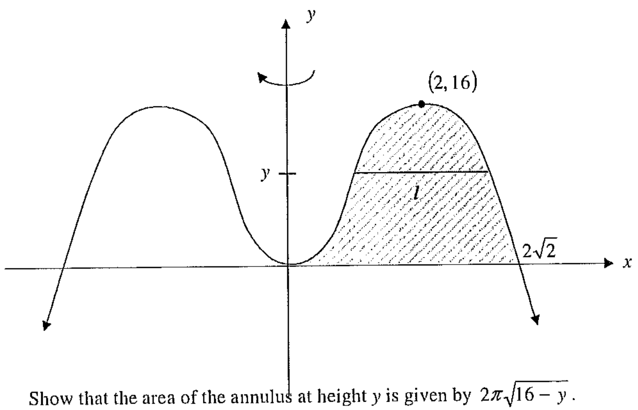
Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) The sequence $\{a_n\}$ is given by: **3**

$$a_1 = 2,\ a_2 = \frac{3}{2} \text{ and } (n+1)\,a_{n+1} = a_{n-1} - (n-2)a_n \text{ for } n > 1.$$

Prove by induction that for $n \geq 1$, $a_n = \dfrac{n+1}{n!}$

(b) The region bound by the curve $y = 8x^2 - x^4$ and the x axis in the first quadrant is rotated about the y axis to form a solid. When the region is rotated, the horizontal line segment l at height y sweeps out an annulus.



(i) Show that the area of the annulus at height y is given by $2\pi\sqrt{16-y}$. **3**

(ii) Find the volume of the solid. **2**

Question 6 continues on page 13

Marks

Question 6 (continued)

(c) (i) Differentiate $x\cos^{n-1}x$. **1**

(ii) Let $I_n = \displaystyle\int_0^{\frac{\pi}{2}} x\cos^n x\, dx$ for $n = 0, 1, 2, \cdots$ **4**

Show that for $n \geq 2$

$$I_n = -\frac{1}{n^2} + \frac{n-1}{n} I_{n-2}$$

(iii) Hence evaluate $\displaystyle\int_0^{\frac{\pi}{2}} x\cos^4 x\, dx$. **2**

End of Question 6

Marks

Question 7 (15 marks) Use a SEPARATE writing booklet.

(a) (i) If $z = \cos\theta + i\sin\theta$, show that $z + \dfrac{1}{z} = 2\cos\theta$ and **2**

$z^n + \dfrac{1}{z^n} = 2\cos n\theta$.

(ii) Hence show that $\cos^5\theta = \dfrac{1}{16}\cos 5\theta + \dfrac{5}{16}\cos 3\theta + \dfrac{5}{8}\cos\theta$. **2**

(iii) Hence find the general solution to the equation **3**

$$16\cos^5\theta = 15\cos 3\theta + \cos 5\theta.$$

Question 7 continues on page 15

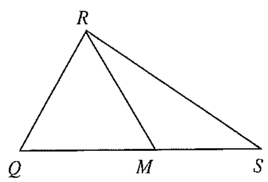
Marks

Question 7 (continued)

For parts (b) and (c) you may use the following identity:

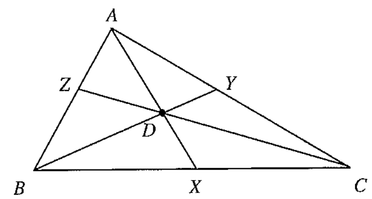
$$\text{If } \frac{P}{Q} = \frac{R}{S}, \text{ then } \frac{P}{Q} = \frac{R}{S} = \frac{P \pm R}{Q \pm S}$$

(b) (i) **1**



Show that $\dfrac{QM}{MS} = \dfrac{\text{Area } \Delta RQM}{\text{Area } \Delta RMS}$.

(ii)



In the diagram, Z, X and Y lie on the sides of ΔABC AB, BC and CA respectively such that AX, BY and CZ are concurrent.
D is the point of concurrency.

(α) Show that $\dfrac{BX}{XC} = \dfrac{\text{Area } \Delta ABD}{\text{Area } \Delta ACD}$. **2**

(β) Hence prove $\dfrac{BX}{XC} \cdot \dfrac{CY}{YA} \cdot \dfrac{AZ}{ZB} = 1$. **2**

Question 7 continues on page 16

Marks

Question 7 (continued)

(c) a, x, y, z are real numbers such that

$$\frac{\cos x + \cos y + \cos z}{\cos(x+y+z)} = \frac{\sin x + \sin y + \sin z}{\sin(x+y+z)} = a$$

(i) Use the identity given earlier to show that **1**

$$a = \frac{cisx + cisy + cisz}{cis(x+y+z)}$$

(ii) Hence show that **2**

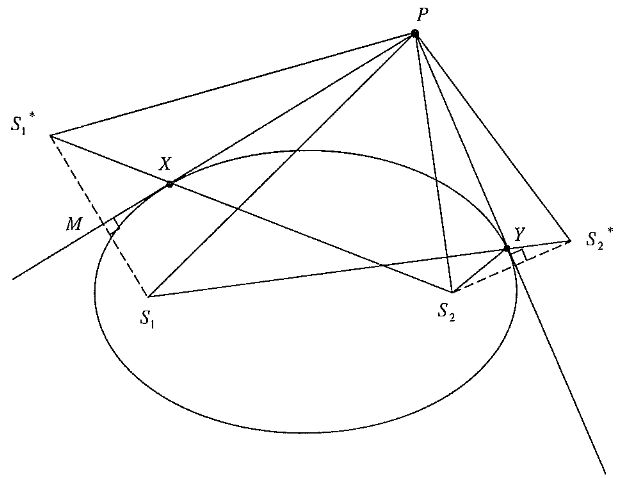
$$a = \cos(y+z) + \cos(x+z) + \cos(x+y)$$

End of Question 7

Marks

Question 8 (15 marks) Use a SEPARATE writing booklet.

(a)



In the diagram, X and Y are arbitrary points on the ellipse and tangents to the ellipse at X and Y meet at the point P. The points S_1 and S_2 are the foci of the ellipse, and S_1^* and S_2^* are the reflections of S_1 and S_2 across the tangents, as shown. $S_1 S_1^*$ and the tangent at X intersect at the point M.

You may assume, without proof, the following two properties of an ellipse:

1. The sum of the focal lengths from any point on an ellipse is constant.
2. The reflection property:
   Tangents to an ellipse are equally inclined to the focal chords drawn through the point of contact.

(i) Prove $\Delta MXS_1 \equiv \Delta MXS_1^*$ and hence show that $S_1^* XS_2$ is a straight line. [Note that similarly, $S_1 YS_2^*$ is a straight line.] **3**

(ii) Prove that $S_1^* S_2 = S_1 S_2^*$ **2**

(iii) Hence state why $\Delta S_1^* PS_2 \equiv \Delta S_1 PS_2^*$. **1**

(iv) Deduce that $\angle S_1 PX = \angle S_2 PY$. **2**

Question 8 continues on page 18

Question 8 (continued) **Marks**

(b) (i) What value of x maximizes the expression $\log_e x - x + 1$? **1**

(ii) Deduce that $\log_e x \le x - 1$ for $x > 0$. **1**

(iii) Consider the set of n positive numbers **2**

$$p_1, p_2, \ldots, p_n \text{ such that } p_1 + p_2 + \ldots + p_n = 1.$$

Use the result in part (ii) to show that

$$\log_e(np_1) + \log_e(np_2) + \ldots + \log_e(np_n) \le 0$$

(iv) Deduce that $n^n p_1 p_2 \ldots p_n \le 1$. **1**

(v) Let $A = x_1 + x_2 + \ldots + x_n \quad (x_1, x_2, \ldots, x_n \ge 0)$ and set **1**

$p_1 = \frac{x_1}{A},\ p_2 = \frac{x_2}{A}, \ldots, p_n = \frac{x_n}{A}$.

Prove that $\frac{x_1 + x_2 + \ldots + x_n}{n} \ge \sqrt[n]{x_1\, x_2 \ldots x_n}$.

(vi) Show that for $a, b, c, d > 0$, with $abcd = 1$ **1**

$a^2 + b^2 + c^2 + d^2 + ab + ac + ad + bc + bd + cd \ge 10$.

End of Paper

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2008 SCEGGS TRIAL HSC

EXTENSION II

Question 1 (15 marks)

Calc /15

(a) $\int \frac{dx}{\sqrt{16x^2-1}}$

Reverse chain rule !!!

$= \frac{1}{4} \ln\left(4x + \sqrt{16x^2-1}\right) + C$ ✓ ✓

(b) $u = \ln x \qquad dv = x$

$du = \frac{1}{x} \qquad v = \frac{x^2}{2}$

$\int_1^e x \ln x \, dx$

$= \left[\frac{x^2 \ln x}{2}\right]_1^e - \int_1^e \frac{x}{2}\, dx$ ✓

$= \frac{e^2}{2} - \left[\frac{x^2}{4}\right]_1^e$ ✓

$= \frac{e^2}{2} - \left(\frac{e^2}{4} - \frac{1}{4}\right)$

$= \frac{e^2}{4} + \frac{1}{4}$ ✓

Q1 cont.

(c) (i) $a = 3$ ✓

$b = 2$ ✓

(ii) $\int \frac{5x^2 + x + 8}{(x+1)(x^2+3)}\, dx$

$= \int \frac{3}{x+1} + \frac{2x-1}{x^2+3}\, dx$

$= 3\ln(x+1) + \ln(x^2+3) - \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right)$ ✓ ✓

(d) $\int \tan^3 x \, dx$

Unfortunately some silly mistakes here

$= \int \tan x (\sec^2 x - 1)\, dx$

eg. $\tan^2 x \neq \sec^2 x + 1$

$= \int \sec^2 x \tan x - \frac{\sin x}{\cos x}\, dx$

$= \frac{\tan^2 x}{2} + \ln(\cos x) + C$ ✓ ✓

Q1 cont.

(e) $\displaystyle\int_0^2 \frac{x^2}{\sqrt{4-x^2}}\,dx$

$\displaystyle = \int_0^2 \frac{-(4-x^2)}{\sqrt{4-x^2}} + \frac{4}{\sqrt{4-x^2}}\,dx$ ✓

$\displaystyle = \int_0^2 -\sqrt{4-x^2} + \left[4.\ \sin^{-1}\left(\frac{x}{2}\right)\right]_0^2$ ✓

$\displaystyle = -\frac{\pi . 2^2}{4} + \left[4.\frac{\pi}{2} - 0\right]$ ✓ ✓

$= \pi$

Whilst this is the easier method, no one saw to do it this way.

OR let $x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$

$x = 0 \rightarrow \theta = 0$

$x = 2 \rightarrow \theta = \pi/2$ ✓

This should have been an easy substitution to spot

$\displaystyle\int_0^2 \frac{x^2}{\sqrt{4-x^2}}\,dx$

$\displaystyle = \int_0^{\pi/2} \frac{4\sin^2\theta}{2.\cos\theta} . 2\cos\theta\, d\theta$ ✓

$\displaystyle = \int_0^{\pi/2} 4\sin^2\theta\, d\theta$

$\displaystyle = \int_0^{\pi/2} 2(1-\cos 2\theta)\, d\theta$ ✓

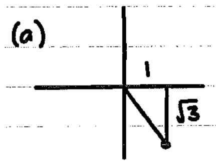
$= \left[2\theta - \sin 2\theta\right]_0^{\pi/2}$

$= \pi$ ✓

Question 2 (15 marks)

Comm /6, Reas /3

(a)



(i) $|\alpha| = 2$ ✓

$\arg\alpha = -\pi/3$ ✓

(ii) $(1-\sqrt{3}i)^{10} = (2\,\text{cis}\,{-\pi/3})^{10}$

$= 2^{10}\,\text{cis}\,(-10\pi/3)$

$= 2^{10}\,\text{cis}\,(2\pi/3)$ ✓

Always simplify the argument to the Principle Argument!

(b) $a+ib = \sqrt{7-24i}$

$(a+ib)^2 = 7-24i$

$(a^2-b^2) + 2ab\,i = 7-24i$ ✓

$a^2-b^2 = 7$ ①

$2ab = -24$ ②

② $\rightarrow$ $b = \dfrac{-12}{a}$

sub in ① $\rightarrow$ $a^2 - \dfrac{144}{a^2} = 7$

$a^4 - 7a^2 - 144 = 0$

$(a^2-16)(a^2+9) = 0$

$a = +4, -4$ (since $a \in \mathbb{R}$) ✓

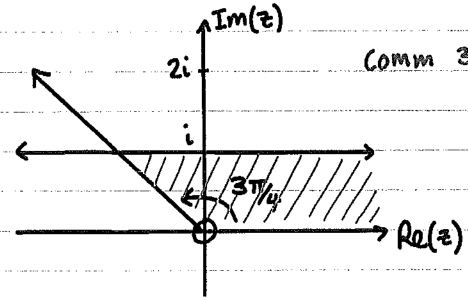
$b = -3, +3$

$\therefore \sqrt{7-24i} = 4-3i, -4+3i$ ✓

It would have been terribly mean of me to have given a quadratic with irrational roots! If your answer doesn't seem correct perhaps you should check over your working.

Q2 cont.

(c)



Comm 3

✓ $0 \le \operatorname{Arg}(z) \le 3\pi/4$
✓ $|z-2i| = |z|$
✓ Correct region

You should be able to sketch $|z-2i| \geq |z|$ by just thinking about. You shouldn't have to resort to algebra

(d) $|z-3| + |z+3| = 10.$

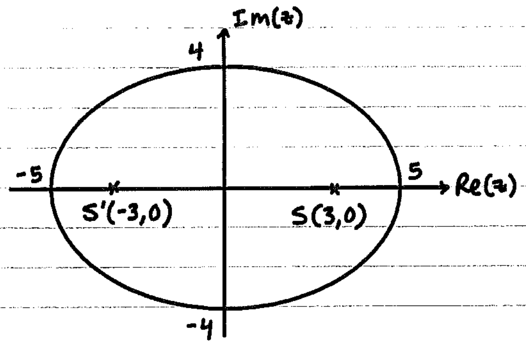
Ellipse → Foci $(\pm 3, 0)$ ie. $ae = 3$
→ $2a = 10$

$\therefore a = 5,\ e = 3/5$

$$b^2 = a^2(1-e^2)$$
$$= 25(1 - 9/25)$$
$$= 16$$
$$b = 4$$

✓ ellipse, foci $(\pm 3, 0)$
✓ x intercepts $= \pm 5$
✓ y intercepts $= \pm 4$

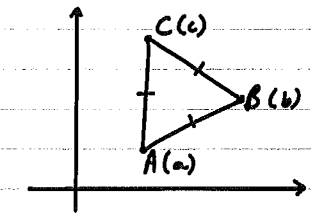
This is a standard question you should know the answer to straight away. You should not have to dive into an algebraic mess.



Comm 3

Q2 cont.

(e)



Reas 3

Never skip a question without even trying, even if it's the style of question you dread most. It might turn out to be quite easy.

(i) $\vec{BA} = \vec{BC} \times \operatorname{cis} \pi/3$
(anticlockwise rotation by $\pi/3$)

$$a - b = (c - b)\,\omega$$
$$\frac{a-b}{c-b} = \omega \quad ✓$$

(ii) Similarly $\dfrac{c-a}{b-a} = \omega$ ✓

$\dfrac{b-c}{a-c} = \omega$ also works

$$\Rightarrow \frac{a-b}{c-b} = \frac{c-a}{b-a}$$
$$(a-b)(b-a) = (c-a)(c-b)$$
$$ab - a^2 - b^2 + ab = c^2 - bc - ac + ab$$
$$ab + bc + ca = a^2 + b^2 + c^2 \quad ✓$$

Question 3 (15 marks)

Comm /6

(a) $x^3+3x^2-5x-2=0$ has roots α,β,γ

let $y=\frac{2}{x}$ (ie $x=\frac{2}{y}$)

for equation with roots $\frac{2}{\alpha},\frac{2}{\beta},\frac{2}{\gamma}$

$$\left(\frac{2}{y}\right)^3+3\left(\frac{2}{y}\right)^2-5\left(\frac{2}{y}\right)-2=0 \checkmark$$

$$8+12y-10y^2-2y^3=0$$

$$y^3+5y^2-6y+4=0 \checkmark$$

(b) $x^2+y^2+xy=3$

(i) $2x+2y\cdot\frac{dy}{dx}+\left(x\cdot\frac{dy}{dx}+y\right)=0$

$$\frac{dy}{dx}(x+2y)=-2x-y$$

$$\frac{dy}{dx}=\frac{-(2x+y)}{x+2y} \checkmark$$

(ii) For S.P. $\frac{dy}{dx}=0$

$$y=-2x$$

sub. into eqn :

$$x^2+(-2x)^2+x(-2x)=3$$

$$3x^2=3$$

$$x=+1,\ -1 \checkmark$$

$$y=-2,\ +2$$

$\therefore$ SP : $(1,-2)$, $(-1,+2)$ ✓

Substitute back into $y=-2x$ to obtain y coord. The original equation gives 4 points — not all of which are stationary.

Comm 6

2008 HIGHER SCHOOL CERTIFICATE TRIAL EXAMINATION

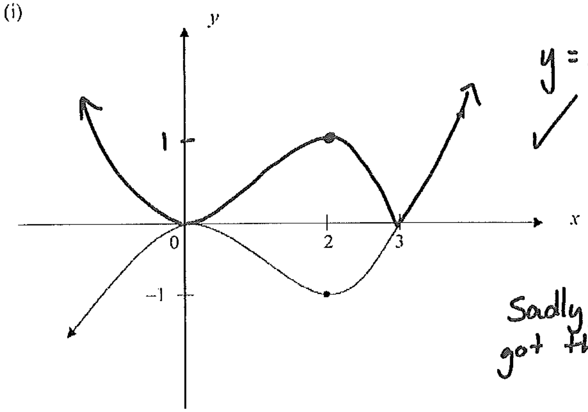
Mathematics Extension 2

Centre Number

Questions 3 (c)

Student Number

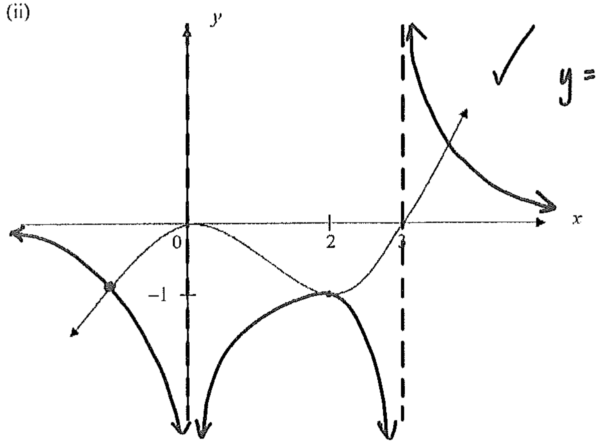
(i)



$y=\frac{1}{4}x^2|x-3|$ ✓

Sadly, not many got this mark.

(ii)



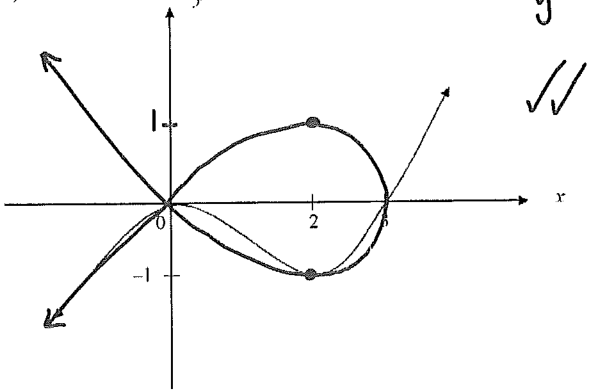
✓ $y=\frac{1}{f(x)}$

Question 3(c) continues on next page

Question 3 (continued)

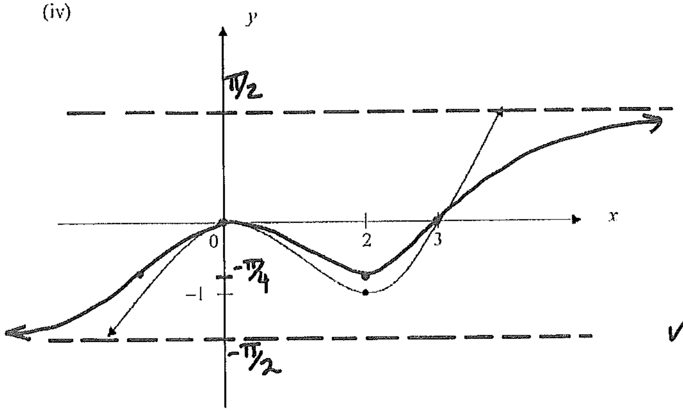
(c) (iii)

$y^2 = -f(x)$



✓✓

(iv)



✓✓

Not many graphs had asymptotes!

Q3 cont.

(d) (i) Area of Parabolic Segment

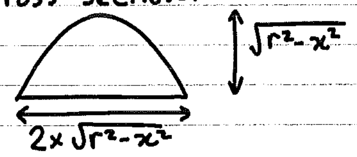
$$= \frac{h}{3}\left(y_1 + 4y_2 + y_3\right)$$

$$= \frac{a}{3}\left(0 + 4 \times h + 0\right) \quad ✓$$

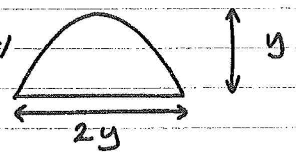
$$= \frac{4ah}{3}$$

Simpson's rule approximates areas by finding the area bound by the parabola through 3 given points. Thus, when used to find the area bound by a parabola it is actually EXACT.

(ii) Cross-section:



OR,



$$\text{Area} = \frac{4ah}{3} = \frac{4(r^2-x^2)}{3}$$

$$\text{Area} = \frac{4ah}{3} = \frac{4y^2}{3} = \frac{4(r^2-x^2)}{3}$$

$$\text{Volume} = 2 \times \int_0^r \frac{4(r^2-x^2)}{3}\,dx \quad ✓$$

$$= \frac{8}{3}\left[r^2x - \frac{x^3}{3}\right]_0^r \quad ✓$$

$$= \frac{8}{3}\left[\frac{2r^3}{3}\right]$$

$$= \frac{16r^3}{9} \text{ units}^3 \quad ✓$$

Question 4 (15 marks)

Calc /4 Reas /5

(a) (i) $x = cp \qquad y = c/p$

$$\underbrace{\frac{dx}{dp} = c \qquad \frac{dy}{dp} = \frac{-c}{p^2}}$$

$$\frac{dy}{dx} = \frac{-c/p^2}{c} = \frac{-1}{p^2} \quad ✓$$

$$m_N = p^2$$

Calc 1

EQN NORMAL:

$$y - \frac{c}{p} = p^2(x - cp)$$

$$py - c = p^3(x - cp) \quad ✓$$

(ii) For points of int. P & Q solve simult.

$$y = \frac{c^2}{x} \quad \& \quad py - c = p^3(x - cp)$$

$$\Rightarrow p\left(\frac{c^2}{x}\right) - c = p^3x - cp^4$$

$$\Rightarrow p^3x^2 - cp^4x + cx - pc^2 = 0$$

$$\Rightarrow x^2 - c\left(p - \frac{1}{p^3}\right)x - \frac{c^2}{p^2} = 0 \quad ✓$$

You can't show the x coord of P & Q satisfy the equation by substituting because you don't even know the coord of Q yet. So you shouldn't even bother starting.

Since x coord of $P(cp)$ & $Q(?)$ are the roots of this equation & Product of roots $= -c^2/p^2$

$$\Rightarrow x \text{ coord } Q = -c/p^3$$

$$\therefore y \text{ coord } Q = \frac{c^2}{x} = -cp^3$$

$$Q\left(-c/p^3, -cp^3\right) \quad ✓$$

Q4 cont.

(iii) $AB = 2c\sqrt{p^2 + \frac{1}{p^2}}$

There was some poor fudging going on here.

$$PQ = \sqrt{\left(cp + \frac{c}{p^3}\right)^2 + \left(\frac{c}{p} + cp^3\right)} \quad ✓$$

$$= c\sqrt{p^2 + \frac{2}{p^2} + \frac{1}{p^6} + \frac{1}{p^2} + 2p^2 + p^6}$$

$$= c\sqrt{p^6 + 3p^2 + \frac{3}{p^2} + \frac{1}{p^6}}$$

$$= c\sqrt{\left(p^2 + \frac{1}{p^2}\right)^3}$$

$$\text{Area } \triangle ABQ = \frac{1}{2} \times AB \times PQ$$

$$= \frac{1}{2} \times 2c\sqrt{p^2 + \frac{1}{p^2}} \times c\sqrt{\left(p^2 + \frac{1}{p^2}\right)^3}$$

$$= c^2\left(p^2 + \frac{1}{p^2}\right)^2 \quad ✓$$

(iv) $\frac{a}{b} + \frac{b}{a} \geq 2$ for $a, b > 0$

Reas 1

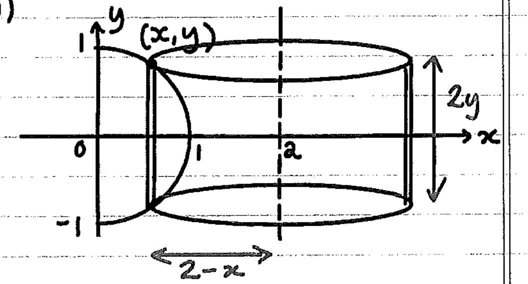
$$\Rightarrow p^2 + \frac{1}{p^2} \geq 2$$

$$\therefore \text{Minimum area} = 4c^2 \quad ✓$$

An easy mark if you understand inequalities & used the hint.

Q4 cont.

(b) (i)



Radius = $2-x$ Height = $2y$

$\delta V = 2\pi r h$

$= 2\pi(2-x)2y$

$= 4\pi(2-x)\sqrt{1-x^2}$

$\therefore V = \int_0^1 4\pi(2-x)\sqrt{1-x^2}$ ✓

(ii) $V = 8\pi \underbrace{\int_0^1 \sqrt{1-x^2}\,dx}_{\text{Area of } 1/4 \text{ circle}} + 2\pi\int_0^1 -2x\sqrt{1-x^2}\,dx$ ✓ ✓

$\int_0^1 \sqrt{1-x^2}$ is $\frac{1}{4}$ of circle (not $\frac{1}{2}$)

$= 8\pi \times \frac{\pi 1^2}{4} + 2\pi\left[\frac{2}{3}(1-x^2)^{3/2}\right]_0^1$

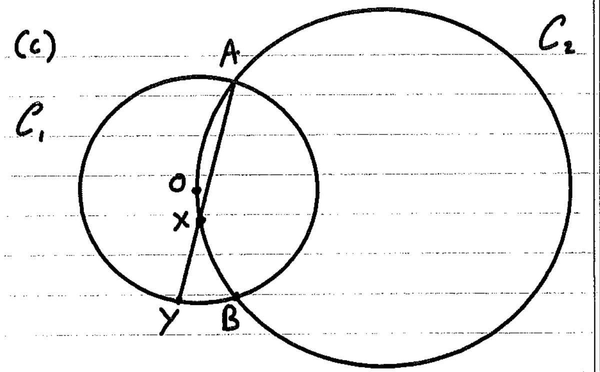
$= 2\pi^2 + 2\pi\left[0 - \frac{2}{3}\right]$

$= 2\pi^2 - \frac{4\pi}{3}$ units^3 ✓

Calc 3

Q4 cont.

(c)



Reas 4

This is a really nice, straightforward question that could be written out clearly & efficiently

(i) Angle subtended at the centre is twice the angle subtended at the circumference, by the same arc ✓

(ii) let $\angle XYB = \alpha$

$\angle AOB = 2\alpha$ (from i)

$\angle AXB = \angle AOB = 2\alpha$

(angles in the same segment are equal) ✓

$\angle XBY = \angle AXB - \angle XYB$

$= 2\alpha - \alpha$

$= \alpha$

(exterior angle of Δ equals the sum of the 2 opposite interior angles) ✓

$\therefore XY = XB$

(sides opposite equal angles in a Δ are equal) ✓

Question 5 (15 marks)

Reas /8

(a)(i) Assume α is a double root of $f(x)=0$

then $f(x) = (x-\alpha)^2 Q(x)$ [& $f(\alpha)=0$]

$$f'(x) = (x-\alpha)^2 Q'(x) + Q(x).\,2(x-\alpha)$$
$$= (x-\alpha)\left[(x-\alpha)Q'(x) + 2Q(x)\right]$$
$$\therefore f'(\alpha) = (\alpha-\alpha)\left[(x-\alpha)Q'(x) + 2Q(x)\right]$$
$$= 0$$

$$\therefore f(\alpha) = f'(\alpha) = 0$$

The syllabus certainly states that you need to know this proof.

(ii) $f(x) = 2x^3 - 5x^2 - 4x + 12$

$$f'(x) = 6x^2 - 10x - 4$$
$$= 2(3x^2 - 5x - 2)$$
$$= 2(x-2)(3x+1)$$

Double root is a solution to $f'(x)=0$

$x = 2$ or $x = -1/3$

$f(-1/3) \neq 0 \quad \therefore \quad x = 2$ is double root

Prod. Roots $= -d/a = -6$

$\Rightarrow$ third root $= \dfrac{-6}{2\times 2} = -\dfrac{3}{2}$

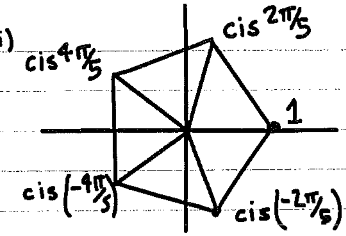
$\therefore$ Roots: $2, 2, -3/2$

What else would you do in a part ii, except use part i ?!?

I was surprised how many started with 'let the roots be α, α, β'

Q5 cont.

(b) (i)



✓ working

One solution of $z^5 = 1$ is $z = 1$ & the other four solutions are evenly spaced around the unit circle

Solutions: $z = 1,\ \text{cis}\frac{2\pi}{5},\ \text{cis}\frac{4\pi}{5},\ \text{cis}\frac{-2\pi}{5},\ \text{cis}\frac{-4\pi}{5}$

(ii) Sum roots $= -b/a$

Reas 4 (ii & iii)

$$1 + \text{cis}\tfrac{2\pi}{5} + \text{cis}\tfrac{-2\pi}{5} + \text{cis}\tfrac{4\pi}{5} + \text{cis}\tfrac{-4\pi}{5} = 0$$

$$1 + \left(\cos\tfrac{2\pi}{5} + i\sin\tfrac{2\pi}{5}\right) + \left(\cos\tfrac{2\pi}{5} - i\sin\tfrac{2\pi}{5}\right) + \left(\cos\tfrac{4\pi}{5} + i\sin\tfrac{4\pi}{5}\right) + \left(\cos\tfrac{4\pi}{5} - i\sin\tfrac{4\pi}{5}\right) = 0$$

$$\Rightarrow 1 + 2\cos\tfrac{2\pi}{5} + 2\cos\tfrac{4\pi}{5} = 0$$
$$\cos\tfrac{2\pi}{5} + \cos\tfrac{4\pi}{5} = -\tfrac{1}{2}$$

Parts ii & iii were poorly done. But really, it should have been quite clear on how you should have proceeded

(iii)
$$\cos\tfrac{2\pi}{5} + \cos\tfrac{4\pi}{5} = -\tfrac{1}{2}$$
$$\cos\tfrac{2\pi}{5} + \left(2\cos^2\tfrac{2\pi}{5} - 1\right) = -\tfrac{1}{2}$$
$$4\cos^2\tfrac{2\pi}{5} + 2\cos\tfrac{2\pi}{5} - 1 = 0$$
$$\cos\tfrac{2\pi}{5} = \frac{-2 \pm \sqrt{20}}{8}$$
$$= \frac{-1 \pm \sqrt{5}}{4}$$

$\cos\frac{2\pi}{5} = \dfrac{-1+\sqrt{5}}{4}$ since $\frac{2\pi}{5}$ is acute & so $\cos\frac{2\pi}{5} > 0$

Q5 cont.

(c) (i)

$$\frac{\overbrace{\binom{11}{5}}^{\text{ways to choose the first team of 5 from 11}} \times \overbrace{\binom{6}{5}}^{\text{ways to choose the 2nd team of 5 from 6 left}} \times \overbrace{\binom{1}{1}}^{\text{1 choice for ref}}}{2} = 1386$$ ✓ ✓

overcounted by a factor of 2! → 2

(ii) Both fussy players play

$\rightarrow \binom{9}{4} \times \binom{5}{4} \times \binom{1}{1} = 630$ ✓

One of fussy players is a ref

$\rightarrow \dfrac{\binom{2}{1} \times \binom{10}{5} \times \binom{5}{5}}{2} = 252$ ✓

Total # ways $= 630 + 252$
$= 882$

Reas 4

Note: we divide by 2 because the two teams of 5 playing eachother are equivalent

Some really good attempts but unfortunately not many marks scored in this Q

Question 6 (15 marks)

Calc /9, Reas /6

(a) Required To Prove: $a_n = \dfrac{n+1}{n!}$

Reas 3

When $n=1$: $\dfrac{1+1}{1!} = 2 = a_1$

When $n=2$: $\dfrac{2+1}{2!} = \dfrac{3}{2} = a_2$ ✓

This is an absolutely straight forward induction. Too many gave up on this question too early!

The statement is true for $n=1$ & $n=2$ & so let $k-1$ & k be integers for which the statement is true

ie. $a_{k-1} = \dfrac{k}{(k-1)!}$ & $a_k = \dfrac{k+1}{k!}$

Then

$$(k+1)a_{k+1} = a_{k-1} - (k-2)a_k$$

$$= \frac{k}{(k-1)!} - \frac{(k-2)(k+1)}{k!}$$ ✓

(USING ASSUMPTION)

$$= \frac{k^2 - (k^2 - k - 2)}{k!}$$

$$= \frac{k+2}{k!}$$

$$a_{k+1} = \frac{k+2}{(k+1)!}$$ ✓

thus the statement is true for the next integer, $k+1$.

Hence, by strong induction the statement is true for integers $n \geq 1$

Q6 cont.

(b)(i) $y = 8x^2 - x^4$

$x^4 - 8x^2 = -y$

$(x^2-4)^2 = 16-y$

$x^2 = \pm\sqrt{16-y} + 4$

$x = \pm\sqrt{\pm\sqrt{16-y}+4}$ ✓

let x_1 & x_2 be the end points of l with $0 \leq x_1 \leq x_2$, then

$x_2 = +\sqrt{+\sqrt{16-y}+4}$

$x_1 = +\sqrt{-\sqrt{16-y}+4}$ ✓

Area $= \pi(x_2^2 - x_1^2)$

$= \pi\left(\left(\sqrt{16-y}+4\right) - \left(-\sqrt{16-y}+4\right)\right)$

$= 2\pi\sqrt{16-y}$ ✓

Reas 3

Poorly done! So many didn't even know where to start.

(ii) Volume $= \int_0^{16} 2\pi\sqrt{16-y}\, dy$

$= \left[\dfrac{-4\pi(16-y)^{3/2}}{3}\right]_0^{16}$ ✓

$= 0 - \dfrac{-4\pi \times 4^3}{3}$

$= \dfrac{4^4\pi}{3}$ units^3 ✓

Calc 2

An easy integral that could be done without (i). Unfortunately some silly mistakes & some fudging going on

Q6 cont.

(c)(i) $u = x\cos^{n-1}x$

$du = x(n-1)\cos^{n-2}x \cdot -\sin x + \cos^{n-1}x$

$= -(n-1)x\sin x\cos^{n-2}x + \cos^{n-1}x$ ✓

(ii) $I_n = \int_0^{\pi/2} \underbrace{x\cos^{n-1}x}_{u} \cdot \underbrace{\cos x}_{dv}\, dx$

$= \left[x\cos^{n-1}x \cdot \sin x\right]_0^{\pi/2}$

$-\int_0^{\pi/2} \sin x\left(-(n-1)x\sin x\cos^{n-2}x + \cos^{n-1}x\right)dx$

$= [0] + (n-1)\int_0^{\pi/2} x\sin^2 x\cos^{n-2}x\, dx$ ✓

$- \int_0^{\pi/2} \sin x\cos^{n-1}x\, dx$ ✓

$= (n-1)\int_0^{\pi/2} x\cos^{n-2}x - x\cos^n x\, dx$

$+ \left[\frac{1}{n}\cos^n x\right]_0^{\pi/2}$ ✓

$I_n = (n-1)\, I_{n-2} - (n-1)\, I_n - \frac{1}{n}$

$nI_n = (n-1)\, I_{n-2} - \frac{1}{n}$

$I_n = \frac{-1}{n^2} + \frac{n-1}{n} I_{n-2}$ ✓

(iii) $I_0 = \int_0^{\pi/2} x\, dx = \left[\frac{x^2}{2}\right]_0^{\pi/2} = \frac{\pi^2}{8}$ ✓

$I_4 = \frac{-1}{16} + \frac{3}{4} I_2$

$= \frac{-1}{16} + \frac{3}{4}\left(\frac{-1}{4} + \frac{1}{2} I_0\right)$

$= \frac{-1}{4} + \frac{3\pi^2}{64}$ ✓

Calc 7

I actually just can't believe how many people couldn't take a hint in (i)!

Also, particularly in recurrence questions like this, you need to concentrate really hard and be SO careful not to make algebraic errors.

Question 7 (15 marks)

Reas /15

(a) (i) $z = \cos\theta + i\sin\theta$ ①

$\frac{1}{z} = z^{-1} = \cos(-\theta) + i\sin(-\theta)$ (by de Moivre)

$= \cos\theta - i\sin\theta$ ②

①+② $\Rightarrow z + \frac{1}{z} = 2\cos\theta$ ✓

$z = \cos\theta + i\sin\theta$

$z^n = \cos n\theta + i\sin n\theta$ ③ (by de Moivre)

$\frac{1}{z^n} = z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$ (by de Moivre)

$= \cos n\theta - i\sin n\theta$ ④

③+④ $\Rightarrow z^n + \frac{1}{z^n} = 2\cos n\theta$ ✓

(ii) $\left(z+\frac{1}{z}\right)^5 = z^5 + 5z^3 + 10z + 10.\frac{1}{z} + 5.\frac{1}{z^3} + \frac{1}{z^5}$ ✓

$(2\cos\theta)^5 = \left(z^5 + \frac{1}{z^5}\right) + 5\left(z^3 + \frac{1}{z^3}\right) + 10\left(z + \frac{1}{z}\right)$

$32\cos^5\theta = 2\cos 5\theta + 5.2\cos 3\theta + 10.2\cos\theta$

$\cos^5\theta = \frac{1}{16}\cos 5\theta + \frac{5}{16}\cos 3\theta + \frac{5}{8}\cos\theta$ ✓

Follow the lead in part (i). You don't want to start with the expansion of $(c+is)^5$ — this will give a nice expression for $\cos 5\theta$, not $\cos^5\theta$

(iii) $16\cos^5\theta = 15\cos 3\theta + \cos 5\theta$

$\cancel{\cos 5\theta} + 5\cos 3\theta + 10\cos\theta = 15\cos 3\theta + \cancel{\cos 5\theta}$

$10\cos\theta = 10\cos 3\theta$

$\cos\theta = \cos 3\theta$ ✓

$\theta = 3\theta + 2\pi n, \quad \theta = -3\theta + 2\pi n$ ✓

$-2\theta = 2\pi n, \quad 4\theta = 2\pi n$

$\theta = -\pi n, \quad \theta = \frac{\pi n}{2}$ for $n \in \mathbb{Z}$ ✓

This should have been doable without having done i & ii !

[Note: together this is simply $\theta = \frac{\pi n}{2}, n \in \mathbb{Z}$]

Q7 cont.

(b) (i) $\frac{\text{Area } \Delta RQM}{\text{Area } \Delta RMS} = \frac{\frac{1}{2} \times QM \times h}{\frac{1}{2} \times MS \times h}$

$= \frac{QM}{MS}$ ✓

where h is $\perp$ distance from R to line QMS

(ii)(α) $\frac{BX}{XC} = \frac{\text{Area } \Delta ABX}{\text{Area } \Delta ACX}$

$= \frac{\text{Area } \Delta DBX}{\text{Area } \Delta DCX}$ ✓

$= \frac{\text{Area } \Delta ABX - \text{Area } \Delta DBX}{\text{Area } \Delta ACX - \text{Area } \Delta DCX}$

$= \frac{\text{Area } \Delta ABD}{\text{Area } \Delta ACD}$ ✓

A pity not many attempted this question because it's so nice.

This question had nothing to do with similar Δs

(β) Similarly $\frac{CY}{YA} = \frac{\text{Area } \Delta BCD}{\text{Area } \Delta BAD}$

$\frac{AZ}{ZB} = \frac{\text{Area } \Delta CAD}{\text{Area } \Delta CBD}$ ✓

$\therefore \frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB}$

$= \frac{\text{Area } \Delta ABD}{\text{Area } \Delta ACD} \times \frac{\text{Area } \Delta BCD}{\text{Area } \Delta BAD} \times \frac{\text{Area } \Delta CAD}{\text{Area } \Delta CBD}$

$= 1$ ✓

Q7 cont.

(c) $a = \dfrac{\cos x + \cos y + \cos z}{\cos(x+y+z)}$

$= \dfrac{\sin x + \sin y + \sin z}{\sin(x+y+z)}$

$= \dfrac{i \times (\sin x + \sin y + \sin z)}{i \times (\sin(x+y+z))}$

$= \dfrac{\cos x + \cos y + \cos z + i(\sin x + \sin y + \sin z)}{\cos(x+y+z) + i(\sin(x+y+z))}$

$= \dfrac{\text{cis}\, x + \text{cis}\, y + \text{cis}\, z}{\text{cis}(x+y+z)}$ ✓

Also a really nice question lots of people didn't attempt

USING THE IDENTITY

(ii) $a = \dfrac{\text{cis}\, x + \text{cis}\, y + \text{cis}\, z}{\text{cis}(x+y+z)}$

$= \text{cis}(x-(x+y+z)) + \text{cis}(y-(x+y+z)) + \text{cis}(z-(x+y+z))$

$= \text{cis}(-y-z) + \text{cis}(-x-z) + \text{cis}(-x-y)$ ✓

Taking the real part of both sides (& given $a \in \mathbb{R}$)

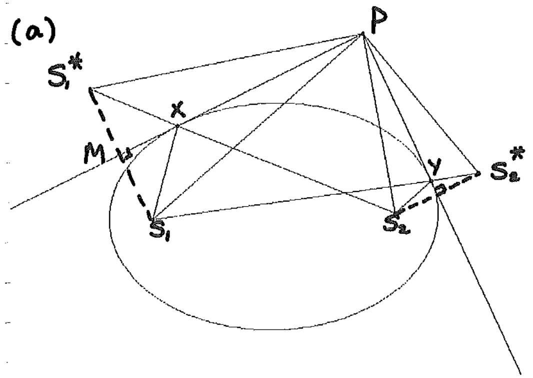
$a = \cos(-y-z) + \cos(-x-z) + \cos(-x-y)$

$a = \cos(y+z) + \cos(x+z) + \cos(x+y)$ ✓

Question 8 (15 marks)

Reas /15

(a)



There's lots of words here you should have read really carefully in reading time. It would have been impossible to do this question in a rush at the 2½ hour mark if you weren't familiar with the diagram already.

(i) In ΔMXS_1 & ΔMXS_1^*

MX (common)

$\angle S_1MX = \angle S_1^*MX = 90°$ & $MS_1 = MS_1^*$

(given S_1^* is a reflection of S_1)

$\therefore \Delta MXS_1 \equiv \Delta MXS_1^*$ (SAS) ✓

$\therefore \angle S_1XM = \angle S_1^*XM$

(matching angles in congruent Δs =) ✓

$\angle S_1XM = \angle S_2XP$

(using property ②) ✓

$\Rightarrow \angle S_1^*XM = \angle S_2XP$

$\therefore$ Since MXP is a straight line & opposite angles are equal, $S_1^*XS_2$ is a straight line.

Note: similarly $S_1YS_2^*$ is a straight line

Q8 cont.

(ii) $S_1^*S_2 = S_1^*X + XS_2$ (since $S_1^*XS_2$ is a straight line)

$= S_1X + XS_2$ (matching sides $S_1X = S_1^*X$ in congruent Δs ΔS_1MX & ΔS_1^*MX)

$= S_1Y + YS_2$ (using property ①)

$= S_1Y + YS_2^*$ (matching sides $S_2Y = S_2^*Y$ in congruent Δs ΔS_2NY & ΔS_2^*NY)

$= S_1S_2^*$ ✓ (since $S_1YS_2^*$ is a straight line) ✓

(iii) SSS ✓

[Proof: In $\Delta S_1^*PS_2$ & $\Delta S_1PS_2^*$

$S_1^*S_2 = S_1S_2^*$ (from part (i))

$S_1^*P = S_1P$ (matching sides = in congruent Δs ΔS_1^*MP & ΔS_1MP (SAS))

$S_2P = S_2^*P$ (matching sides = in congruent Δs ΔS_2^*NP & ΔS_2NP (SAS))

$\therefore \Delta S_1^*PS_2 \equiv \Delta S_1PS_2^*$ (SSS)

(iv) $\angle S_1^*PS_2 = \angle S_1PS_2^*$ ✓ (matching angles in congruent Δs =)

$\angle S_1^*PS_2 - \angle S_1PS_2 = \angle S_1PS_2^* - \angle S_1PS_2$

$\angle S_1^*PS_1 = \angle S_2PS_2^*$

$\Rightarrow \angle S_1PX = \angle S_2PY$ (since matching angles $\angle S_1PX$ & $\angle S_1^*PX$ and $\angle S_2PY$ & $\angle S_2^*PY$ are equal in congruent triangles) ✓

Q8 cont.

(b) (i) $f(x) = \log_e x - x + 1$

$f'(x) = \dfrac{1}{x} - 1$

$f''(x) = \dfrac{-1}{x^2}$

For max/min $f'(x) = 0$

$\Rightarrow x = 1$

$f''(1) = -1 < 0$

$\therefore$ maximum occurs at $x = 1$ ✓

(ii) Maximum value $= f(1)$

$= \log_e 1 - 1 + 1$

$= 0$

$\therefore f(x) \leq 0$ for x in the domain

$\Rightarrow \log_e x - x + 1 \leq 0$ for $x > 0$

$\Rightarrow \log_e x \leq x - 1$ for $x > 0$ ✓

(iii) $\log_e(np_1) + \log_e(np_2) + \cdots + \log_e(np_n)$

$\leq (np_1 - 1) + (np_2 - 1) + \cdots + (np_n - 1)$ ✓

$= n(p_1 + p_2 + \cdots + p_n) - n$

$= n \times 1 - n$

$= 0$

$\therefore \log_e(np_1) + \log_e(np_2) + \cdots + \log_e(np_n) \leq 0$ ✓

(iv) $\log_e(np_1) + \log_e(np_2) + \cdots + \log_e(np_n) \leq 0$

$\log_e(n^n p_1 p_2 \cdots p_n) \leq 0$

$n^n p_1 p_2 \cdots p_n \leq 1$ ✓

This question was much easier to do in a rush than part (a)

(v) $n^n p_1 p_2 \cdots p_n \leq 1$

$$n^n \frac{x_1}{A} \cdot \frac{x_2}{A} \cdots \frac{x_n}{A} \leq 1$$

$$\frac{n^n}{A^n} x_1 x_2 \cdots x_n \leq 1$$

$$x_1 x_2 \cdots x_n \leq \frac{A^n}{n^n}$$

$$\sqrt[n]{x_1 x_2 \cdots x_n} \leq \frac{A}{n}$$

$$\sqrt[n]{x_1 x_2 \cdots x_n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n} \checkmark$$

(vi) $$\frac{a^2+b^2+c^2+d^2+ab+ac+ad+bc+bd+cd}{10} \geq \sqrt[10]{a^5 b^5 c^5 d^5}$$

$$a^2+b^2+c^2+d^2+ab+ac+ad+bc+bd+cd \geq 10 \quad \text{since } abcd = 1 \checkmark$$

Some tried to do (vi) just using pure inequalities without even thinking to use (v). It's a good idea to just try & get the last mark – but do it sensibly!