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SCEGGS Darlinghurst

Preliminary Course
April Examination 2003

Mathematics Extension 1

General Instructions

- Time allowed – 1 hour
- This paper has **four** questions of equal value
- Attempt **all** questions
- Answer all questions on the pad paper provided
- Begin each question **on a new page**
- Write your name at the start of each page
- Marks will be deducted for careless or badly arranged work
- Approved scientific calculators and mathematical templates may be used

Question	Reasoning	Communication	Marks
1	1 /1	2 /2	10/10
2	2 /2	1 /1	10/10
3		3 /3	10 /10
4	6 /10		6 /10
TOTAL			36 /40

90%

- Answer the questions on the pad paper provided
- Clearly label each part
- Write your name on the top of each page
- START EACH QUESTION ON A NEW PAGE

Question 1 (10 Marks)

Marks

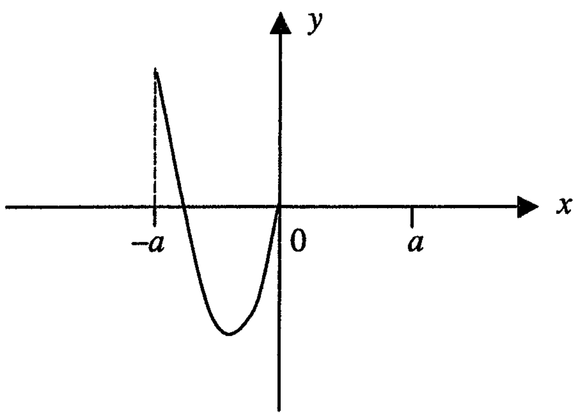
(a) Find, correct to one decimal place, the value of: **1**

$$\frac{2^h - 1}{h} \quad \text{where} \quad h = 0 \cdot 03$$

(b) Factorise, if possible, and simplify: **3**

$$\frac{1}{x-3} - \frac{8x}{x^3 - 27} - \frac{x-3}{x^2 + 3x + 9}$$

(c) The diagram shows the graph of a function $f(x)$ for $-a \le x \le 0$. **1**
It is known that $f(x)$ is an odd function.
Copy the diagram and continue the graph of $f(x)$ for $0 < x \le a$.



(d) (i) Solve simultaneously: **3**

$$xy = 12$$
$$x^2 + y^2 = 25$$

(ii) Draw a neat sketch showing each curve and the point(s) of intersection. **2**

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Question 2 (10 Marks)

Marks

(a) Solve: **3**

$$\frac{2x+5}{x+3} \leq 1$$

(b) (i) Sketch, on the same number plane, the functions $y = |x+1|$ and $y = \frac{1}{2}x - 1$. **2**

(ii) Hence, explain why all real numbers are the solutions of the inequation **1**

$$|x + 1| > \frac{1}{2}x - 1$$

(c) (i) Write expressions for $\sin 2\theta$ and $\cos 2\theta$. **2**

(ii) Hence prove that: **2**

$$\sin 3\theta = 2\sin\theta - 4\sin^3\theta$$

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Question 3 (10 Marks)

Marks

(a) If $t = \tan\frac{x}{2}$, express as simply as possible in term of t, **3**

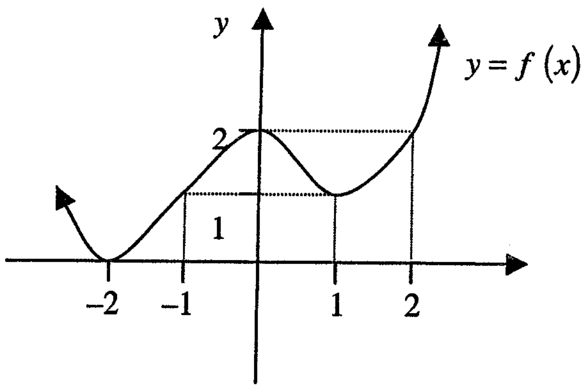
$$\frac{1 - \sin x}{1 + \cos x}$$

(b) (i) Express $\sin x - \sqrt{3}\cos x$ in the form $A\sin(x - \alpha)$ where $A > 0$ and $0° < \alpha < 90°$. **2**

(ii) Hence solve for $0° \leq x \leq 360°$ **2**

$$\sin x - \sqrt{3}\cos x = \frac{2}{\sqrt{2}}$$

(c) A function $y = f(x)$ has been drawn below.



Draw each of these functions marking a clear scale on each set of axes. **3**

(i) $y = f(x) + 1$

(ii) $y = f(x - 2)$

(iii) $y = f(-x)$

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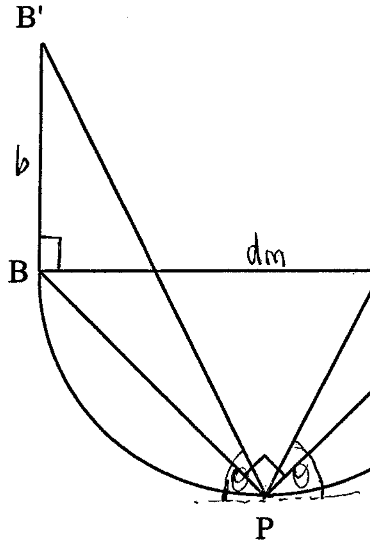
Question 4 (12 Marks)

Marks

(a)



APB is a triangle, right angled at P, in a horizontal semicircle of diameter d metres.

At A and B are vertical posts of height a metres and b metres.

From P, the angle of elevation of the tops of both posts is θ.

(i) Use Pythagoras' rule in ΔAPB to prove that: **3**

$$d^2 = \frac{a^2}{\tan^2\theta} + \frac{b^2}{\tan^2\theta}$$

(ii) From B, the angle of elevation of A' is α and from A, the angle of elevation of B' is β. **4**

Prove that:

$$\tan^2\alpha + \tan^2\beta = \tan^2\theta.$$

Question 4 continues on next page

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Question 4 (continued)

Marks

(b) If $x = \frac{1}{m}$ and $y = \frac{1}{1-x}$ and $z = \frac{y}{y-1}$ **3**

show that $z = m$.

End of paper

①

a) $\frac{2^{0.03}-1}{0.03} = 0.7$ (one d.p.) ✓

b) $\frac{1}{x-3} - \frac{8x}{x^3-27} - \frac{x-3}{x^2+3x+9}$

$= \frac{1}{x-3} - \frac{8x}{(x-3)(x^2+3x+9)} - \frac{x-3}{x^2+3x+9}$ ✓

common denominator ✓

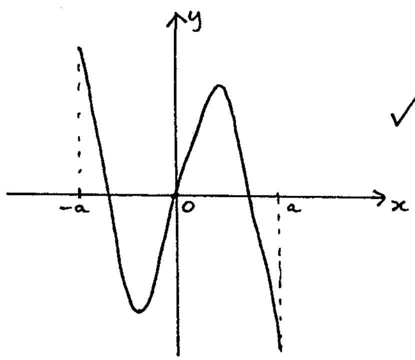
$= \frac{x^2+3x+9 \quad -8x \quad -(x-3)(x-3)}{(x-3)(x^2+3x+9)}$ ← signs!

$= \frac{x^2+3x+9-8x-(x^2-6x+9)}{(x-3)(x^2+3x+9)}$

$= \frac{x^2+3x+9-8x-x^2+6x-9}{(x-3)(x^2+3x+9)}$

$= \frac{x}{(x-3)(x^2+3x+9)}$ ✓

c)



✓ Ⓡ

ODD FUNCTIONS HAVE ROTATIONAL (POINT) SYMMETRY ABOUT THE ORIGIN.

a) $xy = 12$ ①

$x^2+y^2 = 25$ ②

rearrange ①

$y = \frac{12}{x}$ ③

substitute ③ into ②

$x^2 + \left(\frac{12}{x}\right)^2 = 25$

$x^2 + \frac{144}{x^2} = 25$

$x^4 + 144 = 25x^2$

$x^4 - 25x^2 + 144 = 0$ ✓

$(x^2-16)(x^2-9) = 0$

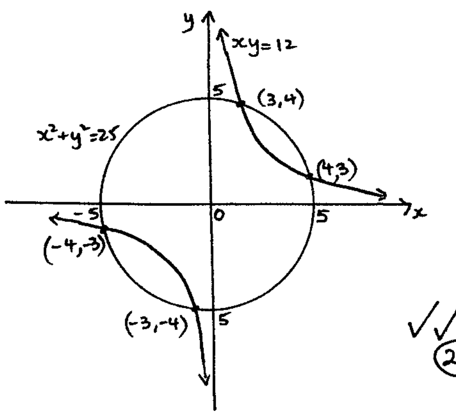
$x^2 = 16$, $x = \pm 4$ $x^2 = 9$, $x = \pm 3$ ✓

when $x = 4$, $y = \frac{12}{4} = 3$ when $x = 3$, $y = \frac{12}{3} = 4$

when $x = -4$, $y = \frac{12}{-4} = -3$ when $x = -3$, $y = \frac{12}{-3} = -4$ ✓

Points of intersection

(4,3) (-4,-3) (3,4) (-3,-4)



✓✓ (2C)

$\frac{2x+5}{x+3} \le 1$

undefined when $x = -3$ ✓

multiply both sides by $(x+3)^2$

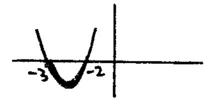
$\frac{2x+5}{x+3} \times (x+3)^2 \le (x+3)^2$

$(2x+5)(x+3) \le (x+3)^2$

$2x^2+6x+5x+15 \le x^2+6x+9$

$x^2+5x+6 \le 0$

$(x+3)(x+2) \le 0$ ✓



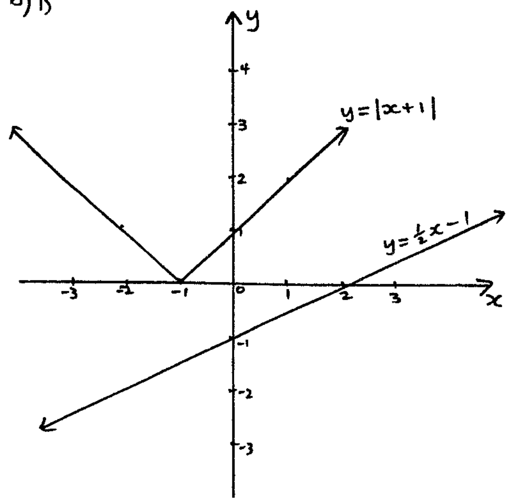
$-3 \le x \le -2$

but undefined when $x = -3$

∴ Solution

$-3 < x \le -2$ ✓

b) i)



ii) using the graph it can be seen that the function $y=|x+1|$ is above the line $y=\frac{1}{2}x-1$ for all real numbers.

∴ $|x+1| > \frac{1}{2}x-1$ for all x. ✓

[NB It is not enough to restate the question "$|x+1|$ is bigger than $\frac{1}{2}x-1$".] (1C)

c) i) $\sin 2\theta = 2\sin\theta\cos\theta$ ✓

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

OR

$= 2\cos^2\theta - 1$ ✓

OR

$= 1 - 2\sin^2\theta$

ii) Prove

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\sin 3\theta = \sin(2\theta+\theta)$

$= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta$

$= 2\sin\theta\cos\theta.\cos\theta + (\cos^2\theta - \sin^2\theta)\sin\theta$ ✓

$= 2\sin\theta\cos^2\theta + \sin\theta\cos^2\theta - \sin^3\theta$

$= 3\sin\theta\cos^2\theta - \sin^3\theta$

$= 3\sin\theta(1-\sin^2\theta) - \sin^3\theta$ ✓

$= 3\sin\theta - 3\sin^3\theta - \sin^3\theta$

$= 3\sin\theta - 4\sin^3\theta$

(2R)

③ a) $t = \tan\frac{x}{2}$

$\sin x = \frac{2t}{1+t^2}$

$\cos x = \frac{1-t^2}{1+t^2}$

$\frac{1 - \sin x}{1 + \cos x}$

$= \frac{1 - \frac{2t}{1+t^2}}{1 + \frac{1-t^2}{1+t^2}}$ ✓

$= \frac{\frac{1+t^2-2t}{1+t^2}}{\frac{1+t^2+1-t^2}{1+t^2}}$

$= \frac{\frac{t^2-2t+1}{1+t^2}}{\frac{2}{1+t^2}}$ ✓

$= \frac{(t-1)^2}{1+t^2} \times \frac{1+t^2}{2}$

$= \frac{(t-1)^2}{2}$ ✓

OR $\frac{(1-t)^2}{2}$

b) i) $A\sin(x-\alpha)$

$= A\sin x\cos\alpha - A\cos x\sin\alpha$

$= \sin x - \sqrt{3}\cos x$

Equate coefficients

$A\cos\alpha = 1$ ①

$A\sin\alpha = \sqrt{3}$ ②

Both positive ∴ α acute $0° < \alpha < 90°$

Find α : Divide ② ÷ ①

$\frac{A\sin\alpha}{A\cos\alpha} = \frac{\sqrt{3}}{1}$

$\tan\alpha = \sqrt{3}$

$\therefore \alpha = 60°$ ✓

Find A

$A^2 = 1^2 + \sqrt{3}^2$

$A = \sqrt{1+3}$

$= \sqrt{4}$

$= 2$ ✓

$\therefore \sin x - \sqrt{3}\cos x = 2\sin(x - 60°)$

ii)

$\sin x - \sqrt{3}\cos x = \frac{2}{\sqrt{2}}$

$2\sin(x-60°) = \frac{2}{\sqrt{2}}$

$\sin(x-60°) = \frac{1}{\sqrt{2}}$

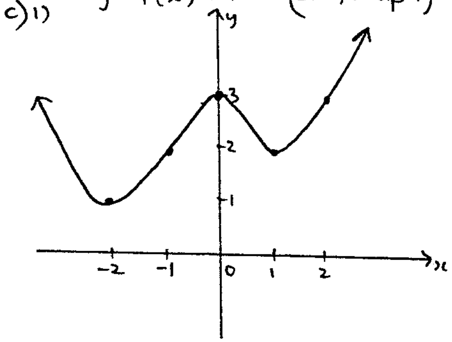
Quad 1 & 2

$x - 60° = 45°, (180-45)°$

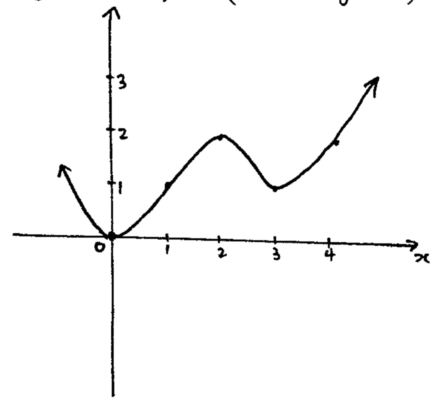
$x - 60° = 45°, 135°$

$\therefore x = 105°, 195°$ ✓✓

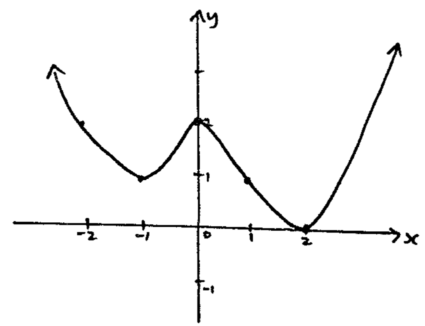
③ c) i) $y = f(x) + 1$ (Shift up 1)



ii) $y = f(x-2)$ (Shift right 2)



iii) $y = f(-x)$ (Reflect in y axis)



3C

④ b) $x = \frac{1}{m}$ ①

$y = \frac{1}{1-x}$ ②

$z = \frac{y}{y-1}$ ③

Show that $z = m$

Substitute ① into ②

$y = \frac{1}{1-x}$

$= \frac{1}{1-\frac{1}{m}}$

$= \frac{1}{\frac{m-1}{m}}$

$= \frac{m}{m-1}$ ✓

Substitute into ③

$z = \frac{y}{y-1}$

$= \frac{\frac{m}{m-1}}{\frac{m}{m-1} - 1}$ ✓

$= \frac{\frac{m}{m-1}}{\frac{m-(m-1)}{m-1}}$

$= \frac{\frac{m}{m-1}}{\frac{m-m+1}{m-1}}$

$= \frac{\frac{m}{m-1}}{\frac{1}{m-1}}$

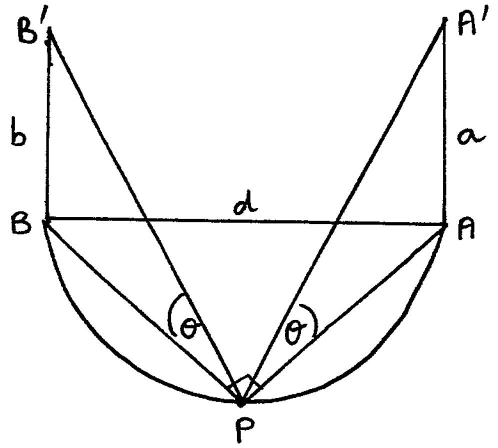
$= \frac{m}{m-1} \times \frac{m-1}{1}$ ✓

$= m$

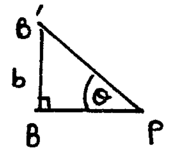
$\therefore z = m$

3R

1a)



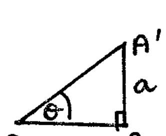
i) <u>In $\Delta PBB'$</u>



$$\tan\theta = \frac{b}{BP}$$

$$BP = \frac{b}{\tan\theta}$$ ✓

<u>In $\Delta PAA'$</u>



$$\tan\theta = \frac{a}{AP}$$

$$AP = \frac{a}{\tan\theta}$$ ✓

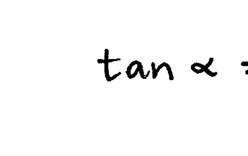
<u>In ΔAPB</u>, using Pythagoras

$$AB^2 = AP^2 + BP^2$$

$$d^2 = \frac{a^2}{\tan^2\theta} + \frac{b^2}{\tan^2\theta}$$ ✓

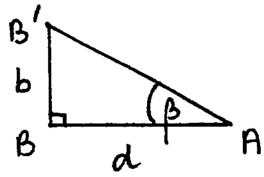
(3R)

ii)



$$\tan\alpha = \frac{a}{d}$$ ✓

$$\therefore \quad a = d\tan\alpha$$



$$\tan\beta = \frac{b}{d}$$

$$\therefore \quad b = d\tan\beta$$ ✓

Substitute $a = d\tan\alpha$

$b = d\tan\beta$

into $$d^2 = \frac{a^2}{\tan^2\theta} + \frac{b^2}{\tan^2\theta}$$ ✓

$$d^2 = \frac{d^2\tan^2\alpha}{\tan^2\theta} + \frac{d^2\tan^2\beta}{\tan^2\theta}$$

$$d^2\tan^2\theta = d^2\tan^2\alpha + d^2\tan^2\beta$$

(divide by d^2)

$$\tan^2\theta = \tan^2\alpha + \tan^2\beta.$$ ✓

(4R)