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SCEGGS Darlinghurst

Preliminary Year, 2003
Semester 2 Examination

Mathematics (Extension)

General Instructions

- Reading time – 5 minutes
- Working time – 1½ hours
- This paper has **5 questions**
- Attempt **all** questions
- Answer all questions on the pad paper provided
- Write your name on every page
- Total marks for all parts (62)
- Approved calculators may be used

Questions 1-5

Total marks (62)

- Attempt all parts of Questions 1-5

	Communication	Calculus	Reasoning	Total
Question 1	3 /3		2 /2	12 /12
Question 2		1 /1	5 /5	12 /12 :)
Question 3	1 /3	2 /2		11 /13
Question 4			9 /9	9 /12
Question 5	3 /3		10 /10	13 /13
TOTAL	7 /9	3 /3	26 /26	57 /62

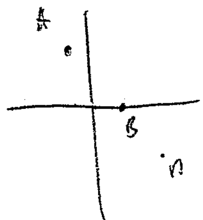
- Answer the questions on the pad paper provided
- Clearly label each part
- Write your name on the top of each page
- START EACH QUESTION ON A NEW PAGE

Question 1 (12 Marks)

Marks

(a) Solve for x: **3**

$$\frac{3x+1}{2-x} \geq 1$$



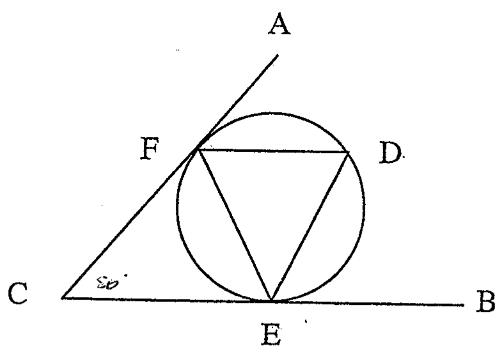
(b) A and B are the points $(-1, 3)$ and $(2, 0)$ respectively:

(i) Find the co-ordinates of the point P which divides AB externally in the ratio $5:2$. **2**

(ii) If the line AB cuts the y axis at Q, find the ratio in which Q divides AB internally without finding the equations of any lines. **2**

(c) In the diagram, AC and BC are tangents to the circle, touching the circle at F and E respectively. $\angle ACB$ equals $50°$. Copy the diagram onto your paper. **3**

Show that $\angle CEF$ is $65°$ and hence find $\angle EDF$.



(d) Prove the identity $\dfrac{1-\tan^2 A}{1+\tan^2 A} = \cos 2A$. **2**

$\cos^2 A - \sin^2 A$

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Question 2 (12 Marks)

Marks

(a) The polynomial $P(x) = x^3 + x^2 + x - 2$ has roots, α, β, γ.

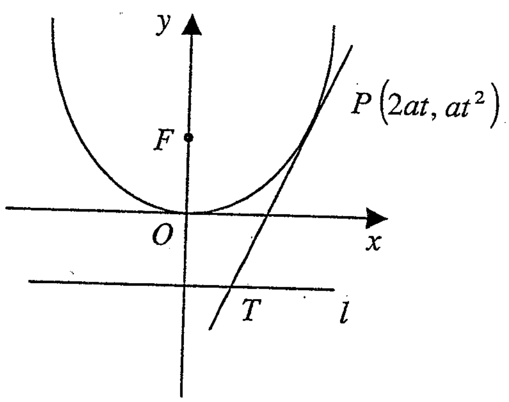
(i) Find the value of $\alpha\beta\gamma$. $\frac{-b}{a}$ $\frac{c}{a}$ **1**

(ii) If $\alpha = 1$, find the value of $\frac{1}{\beta} + \frac{1}{\gamma}$. $= \frac{\beta+\gamma}{\beta\gamma}$ **2**

(b) (i) Find the gradient of the tangent to the curve $y = x^2 + 3$ at the point $(1, 4)$. **1**

(ii) Find the acute angle between the line $y = 3x + 1$ and the curve $y = x^2 + 3$ at the point of intersection $(1, 4)$. Give your answer to the nearest minute. **2**

(c) The tangent $tx - y - at^2 = 0$ at the point $P(2at, at^2)$ on the parabola $x^2 = 4ay$ cuts the directrix l at T. F is the focus of the parabola.



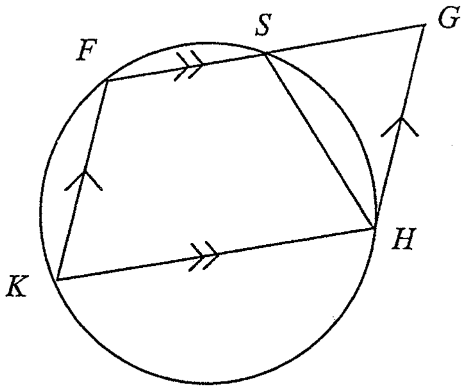
(i) Find the co-ordinates of T. **1**

(ii) Show that TF is perpendicular to PF. **3**

Question 2 continues on the next page

Marks

(d)



2

The figure $FGHK$ is a parallelogram. The point S lies on FG, and F,S,H,K lie on a circle.

Copy or trace the diagram onto your answer paper.

Prove that triangle HGS is isosceles.

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Question 3 (13 Marks)

Marks

(a) Consider the polynomial $P(x) = 6x^3 - 5x^2 - 2x + 1$.

(i) Show that 1 is a zero of $P(x)$. **1**

(ii) Express $P(x)$ as a product of 3 linear factors. **2**

(iii) Solve the inequality $P(x) \leq 0$. **1**

(b) Using the substitution $\dfrac{\sqrt{x+h} - \sqrt{x}}{h} = \dfrac{1}{\sqrt{x+h} + \sqrt{x}}$ find from first principles the derivative of $y = \sqrt{x}$. **2**

$y = x^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

(c) (i) Express $4\sin\theta - 3\cos\theta$ in the form $A\sin(\theta - \alpha)$ where $A > 0$ and $0° < \alpha < 90°$. Give α to the nearest degree. **2**

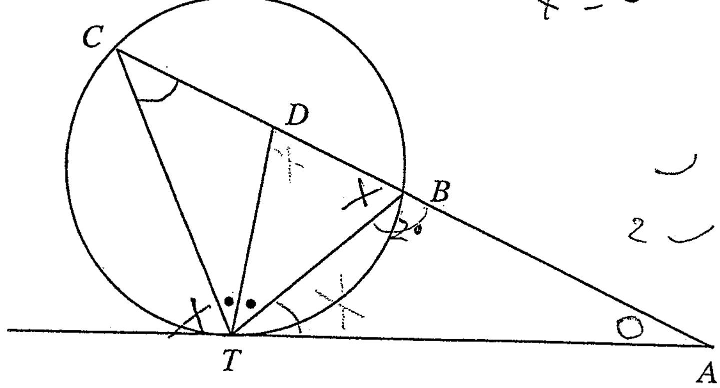
(ii) Find all solutions of $4\sin\theta - 3\cos\theta = 1$ for $0° \leq \theta \leq 360°$. Give θ to the nearest degree. **2**

$= x + 2•$

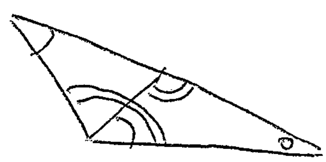
205.46

48.40

(d)



$+ x + 2• = 180°$



TA is a tangent to a circle. Line *ABDC* intersects the circle at *B* and *C*. Line *TD* bisects angle *BTC*. **3**

Prove $AT = AD$.

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Question 4 (12 Marks)

Marks

(a) (i) Using the $\sin(A-B)$ formula, with $A=60°$ and $B=45°$ (or otherwise), find the value of $\sin 15°$ in surd form. **2**

(ii) Prove the identity: **2**

$$\tan\alpha = \operatorname{cosec} 2\alpha - \cot 2\alpha$$

(iii) Using the results in part (ii), find a value (in surd form) for $\tan 15°$. **2**

(iv) Using the results in parts (i), (ii) and (iii), (or otherwise) show that:. **3**

$$\tan 7\frac{1}{2}^{\circ} = \sqrt{6} + \sqrt{2} - \sqrt{3} - 2$$

9.26

(b) The remainder when $x^3 + ax + b$ is divided by $(x-2)(x+3)$ is $2x+1$. Find the values of a and b. **3**

$\sqrt{18} = 3\sqrt{2}$

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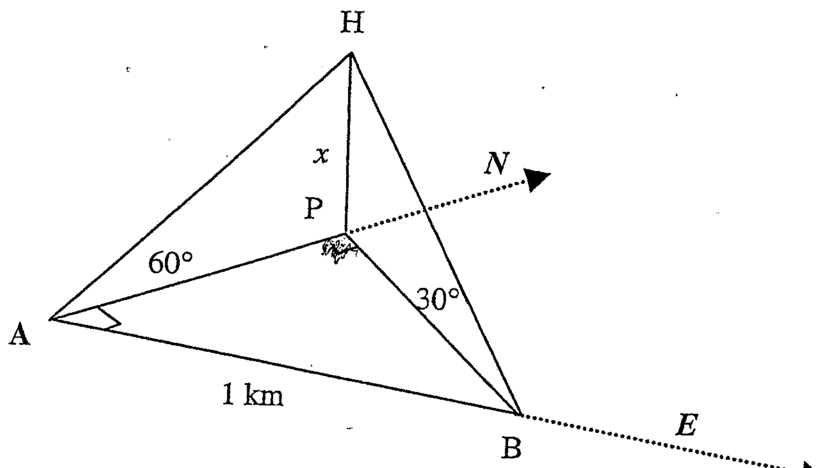
Question 5 (13 Marks)

Marks

(a) On the same set of axes, draw the graphs of $y = |x-2|$ and $y = x+1$. For what values of x is $|x-2| < x+1$? **3**

(b) Anna (A) is standing due south of Phillip (P) who is assisting an injured bushwalker. A rescue helicopter (H) is hovering directly over P and lowering a stretcher. Anna measures the angle of elevation of the helicopter to be $60°$ from her position. Belinda (B) is 1 kilometre due east of A and measures the angle of elevation of the helicopter to be $30°$. The height of the helicopter above P is x metres.



(i) Write expressions for both AP and BP in terms of x. **1**

(ii) Hence or otherwise, find the height of the helicopter (x), correct to the nearest m. **3**

(c) $P(2ap, ap^2)$ is a variable point on a parabola $x^2 = 4ay$. The line PR is perpendicular to the directrix of the parabola with R on the directrix. The tangent to the parabola at P meets the y-axis at T. If M is the midpoint of the interval RT:

(i) find the locus of M. **4**

(ii) show that the directrix of the locus of M is the x-axis. **2**

End of Paper

Extension 1 Mathematics Prelim. Year 2003.

1) a) $\frac{3x+1}{2-x} \geq 1$

undefined when $x = 2$

multiply both sides by $(2-x)^2$

$\frac{3x+1}{\cancel{2-x}} \times (2-x)^{\cancel{2}} \geq 1 \times (2-x)^2$

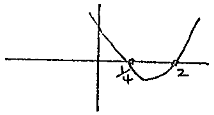
$(3x+1)(2-x) \geq (2-x)^2$

$6x - 3x^2 + 2 - x \geq 4 - 4x + x^2$

$0 \geq 4x^2 - 9x + 2$

$4x^2 - 9x + 2 \leq 0$ ✓

$(4x-1)(x-2) \leq 0$



$\frac{1}{4} \leq x \leq 2$ ✓✓

b) External division $5 : -2$

A $(-1,3)$ B $(2,0)$

5 : -2

$x = \frac{-2\times -1 + 5\times 2}{5-2}$

$= \frac{2+10}{3}$

$= \frac{12}{3}$

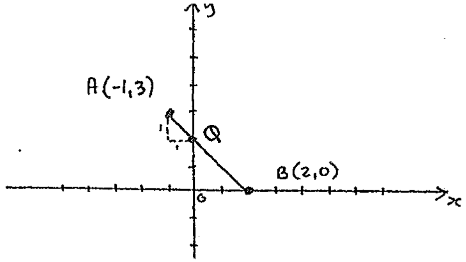
$= 4$

$y = \frac{-2\times 3 + 5\times 0}{5-2}$

$= \frac{-6+0}{3}$

$= -2$

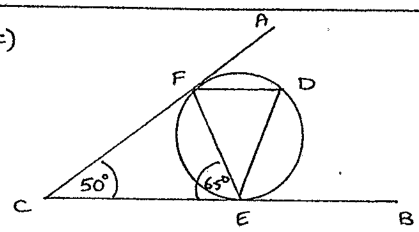
$\therefore P(4,-2)$ ✓✓



By observation

Q divides AB internally in the ratio 1:2 ✓✓

c)



CF = CE

since tangents from an exterior point are equal.

$\therefore \triangle CFE$ is isosceles. ✓

$\therefore \angle CEF = \frac{1}{2}(180° - 50°)$ ($\angle$ sum $\triangle$ = 180°)

$= \frac{1}{2} \times 130°$

$= 65°$ ✓

$\angle EDF = \angle CEF$

$= 65°$ ✓

(Angles in the alternate segments are equal.)

Com 3

d). LHS $= \frac{1 - \tan^2 A}{1 + \tan^2 A}$

$= \frac{1 - \frac{\sin^2 A}{\cos^2 A}}{1 + \frac{\sin^2 A}{\cos^2 A}}$

$= \frac{\frac{\cos^2 A - \sin^2 A}{\cos^2 A}}{\frac{\cos^2 A + \sin^2 A}{\cos^2 A}}$ ✓

$= \frac{\frac{\cos^2 A - \sin^2 A}{\cos^2 A}}{\frac{1}{\cos^2 A}}$

$= \frac{\cos^2 A - \sin^2 A}{\cos^2 A} \times \frac{\cos^2 A}{1}$

$= \cos^2 A - \sin^2 A$ ✓

$= \cos 2A$

Reas 2

2) $P(x) = x^3 + x^2 + x - 2$ (*)

should have been a 3. (otherwise $\alpha \neq 1$!!)

Roots α, β, γ

i) Product of roots

$\alpha\beta\gamma = -d/a$

$= --2/1$

$= 2$ ✓

ii) Sum of Roots

$\alpha + \beta + \gamma = -b/a$

$= -1$

Sum two at a time

$\alpha\beta + \alpha\gamma + \beta\gamma = c/a$

$= 1$

$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$ ✓

put $\alpha = 1$

$1 + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{1}{2}$

$\frac{1}{\beta} + \frac{1}{\gamma} = -\frac{1}{2}$ ✓

b) (i) $y = x^2 + 3$

$y' = 2x$.

At $(1,4)$ gradient tangent ✓

$m_1 = 2$

Calculus 1

ii) $y = 3x + 1$

$m_2 = 3$

Acute angle

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

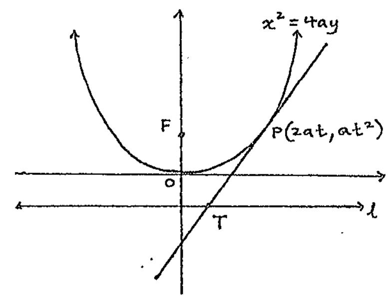
$= \left|\frac{2-3}{1+2\times 3}\right|$

$= \left|\frac{-1}{7}\right|$

$= \frac{1}{7}$ ✓

$\theta \doteqdot 8°8'$ (to nearest minute) ✓

c)



Tangent at $P(2at, at^2)$

$tx - y - at^2 = 0$

cuts the directrix l at T

directrix is $y = -a$

Solve simultaneously

$tx - y - at^2 = 0$ ①

$y = -a$ ②

substitute ② into ①

$tx + a - at^2 = 0$

$tx = at^2 - a$

$x = \dfrac{a(t^2-1)}{t}$ ✓

T has coordinates

$\left(\dfrac{a(t^2-1)}{t}, -a\right)$

ii) Focus $F(0, a)$

$P(2at, at^2)$

Gradient TF

$m_1 = \dfrac{y_2 - y_1}{x_2 - x_1}$

$= \dfrac{-a - a}{\dfrac{a(t^2-1)}{t} - 0}$

$= \dfrac{-2\cancel{a}}{\dfrac{\cancel{a}(t^2-1)}{t}}$

$= \dfrac{-2t}{t^2-1}$ ✓

Gradient PF

$m_2 = \dfrac{at^2 - a}{2at - 0}$

$= \dfrac{\cancel{a}(t^2-1)}{2\cancel{a}t}$

$= \dfrac{t^2-1}{2t}$ ✓

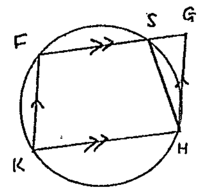
Since $m_1 \times m_2 = \dfrac{-2t}{t^2-1} \times \dfrac{t^2-1}{2t}$

$= -1$ ✓

$\therefore$ TF $\perp$ PF

Reas 3

d)



FGHK is a parallelogram

$\therefore \angle FKH = \angle FGH$ (opposite $\angle$s = in a parm.)

$= x$ ✓

F, S, H, K lie on a circle

$\therefore$ FSHK is a cyclic quadrilateral

$\therefore \angle FKH + \angle FSH = 180°$

(opp. $\angle$s in a cyclic quad. are supplementary)

$\angle FSH = 180° - \angle FKH$

$= 180° - x$

$\angle GSH = 180° - \angle FSH$ ($\angle$ sum straight $\angle = 180°$)

$= 180° - (180° - x)$

$= x$ ✓

Reas 2

$\therefore \angle SGH = \angle GSH = x$

$\therefore \triangle HGS$ is isosceles since it has 2 equal angles.

Q3

a) $P(x) = 6x^3 - 5x^2 - 2x + 1$

i) $P(1) = 6 - 5 - 2 + 1$

$= 0$

$\therefore x = 1$ is a zero of $P(x)$

$\therefore (x-1)$ is a factor of $P(x)$ ✓

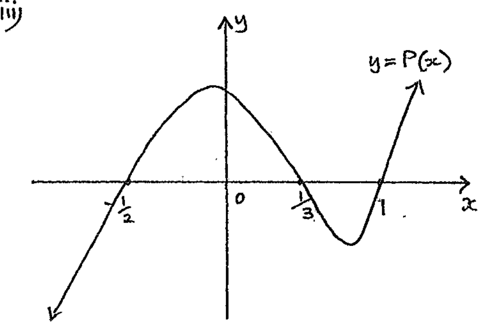
ii)

$$\begin{array}{r l}
 & 6x^2 + x - 1 \\
x-1 \,) & 6x^3 - 5x^2 - 2x + 1 \\
 & 6x^3 - 6x^2 \\
\hline
 & x^2 \\
 & x^2 - x \\
\hline
 & -x \\
 & -x + 1 \\
\hline
 & 0
\end{array}$$

$\therefore P(x) = (x-1)(6x^2 + x - 1)$ ✓

$= (x-1)(3x-1)(2x+1)$ ✓

iii)



$P(x) \leq 0$

for

$x \leq -\frac{1}{2}$, $\frac{1}{3} \leq x \leq 1$ ✓

b) $y = \sqrt{x}$

$f(x) = \sqrt{x}$

$f(x+h) = \sqrt{x+h}$

Using first principles

$f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim\limits_{h \to 0} \dfrac{\sqrt{x+h} - \sqrt{x}}{h}$ ✓

using the given substitution

$= \lim\limits_{h \to 0} \dfrac{1}{\sqrt{x+h} + \sqrt{x}}$

$= \dfrac{1}{\sqrt{x+0} + \sqrt{x}}$

Calc 2

$= \dfrac{1}{2\sqrt{x}}$ ✓

c) i) $4\sin\theta - 3\cos\theta$

$= A\sin(\theta - \alpha)$

$= A\sin\theta\cos\alpha - A\cos\theta\sin\alpha$

Match parts

$A\cos\alpha = 4$ ①

$A\sin\alpha = 3$ ②

Find A

$A^2 = 4^2 + 3^2$

$A = \sqrt{16 + 9}$

$= \sqrt{25}$

$= 5$ ✓

Find α

divide ②/①

$\tan\alpha = \frac{3}{4}$

$\therefore \alpha \doteq 37°$

$\therefore\ 4\sin\theta - 3\cos\theta$

$= 5\sin(\theta - 37^\circ)$

ii) $4\sin\theta - 3\cos\theta = 1$

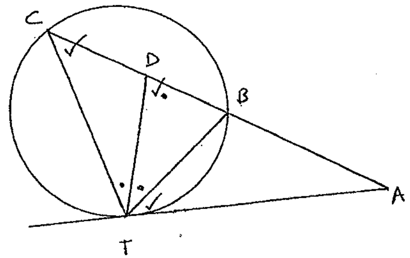
$5\sin(\theta - 37^\circ) = 1$

$\sin(\theta - 37^\circ) = \frac{1}{5}$

$\therefore\ \theta - 37^\circ = 11^\circ 32' \text{ or } 168^\circ 28'$

$\therefore\ \theta \doteqdot 49^\circ \text{ or } 205^\circ$

d)



$\angle CTD = \angle BTD$ (given)

$\angle ATB = \angle TCD$ ($\angle$ in the alt segment = $\angle$ between chord + tagent)

$\angle TDB = \angle CTD + \angle TCD$ (ext. $\angle$ in Δ)

$\therefore\ \angle ATD = \angle ADT$

$\therefore\ AT = AD$ (sides opp = $\angle$ in an isosceles Δ =)

Com 3

(4) a) i) $\sin(A-B)$

$= \sin A\cos B - \cos A\sin B$

$\sin 15^\circ = \sin(45^\circ - 30^\circ)$

$= \sin 45^\circ\cos 30^\circ - \cos 45^\circ \sin 30^\circ$

$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}$

$= \frac{\sqrt{3}-1}{2\sqrt{2}}$

$= \frac{\sqrt{3}-1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{\sqrt{6}-\sqrt{2}}{4}$

ii) Prove $\tan\alpha = \operatorname{cosec} 2\alpha - \cot 2\alpha$

RHS $= \operatorname{cosec} 2\alpha - \cot 2\alpha$

$= \frac{1}{\sin 2\alpha} - \frac{\cos 2\alpha}{\sin 2\alpha}$

$= \frac{1 - \cos 2\alpha}{\sin 2\alpha}$

$= \frac{1 - (\cos^2\alpha - \sin^2\alpha)}{2\sin\alpha\cos\alpha}$

$= \frac{1 - \cos^2\alpha + \sin^2\alpha}{2\sin\alpha\cos\alpha}$

Substituting $1 - \cos^2\alpha = \sin^2\alpha$

$= \frac{\sin^2\alpha + \sin^2\alpha}{2\sin\alpha\cos\alpha}$

$= \frac{\cancel{2}\sin^{\cancel{2}}\alpha}{\cancel{2}\cancel{\sin\alpha}\cos\alpha}$

$= \frac{\sin\alpha}{\cos\alpha}$

$= \tan\alpha$

$=$ LHS

(OR by t results)

iii) $\tan 15^\circ = \operatorname{cosec} 30^\circ - \cot 30^\circ$

$= \frac{1}{\sin 30^\circ} - \frac{1}{\tan 30^\circ}$

$= \frac{1}{\frac{1}{2}} - \frac{1}{\frac{1}{\sqrt{3}}}$

$= 2 - \sqrt{3}$

(OR $\frac{\sqrt{3}-1}{\sqrt{3}+1}$)

iv) $\tan 7\frac{1}{2}^\circ = \operatorname{cosec} 15^\circ - \cot 15^\circ$

$= \frac{1}{\sin 15^\circ} - \frac{1}{\tan 15^\circ}$

$= \frac{1}{\frac{\sqrt{6}-\sqrt{2}}{4}} - \frac{1}{2-\sqrt{3}}$

$= \frac{4}{\sqrt{6}-\sqrt{2}} - \frac{1}{2-\sqrt{3}}$

Rationalise each denominator

$\frac{4}{\sqrt{6}-\sqrt{2}} \times \frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}+\sqrt{2}} = \frac{4(\sqrt{6}+\sqrt{2})}{6-2}$

$= \sqrt{6}+\sqrt{2}$

$\frac{1}{2-\sqrt{3}} \times \frac{2+\sqrt{3}}{2+\sqrt{3}} = \frac{2+\sqrt{3}}{4-3}$

$= 2+\sqrt{3}$

$\therefore\ = \sqrt{6}+\sqrt{2} - (2+\sqrt{3})$

$= \sqrt{6}+\sqrt{2} - \sqrt{3} - 2$

Reas /9

b) $P(x) = A(x).Q(x) + R(x)$

$x^3 + ax + b = (x-2)(x+3).Q(x) + 2x+1.$

Substitute $x = 2$ $\quad P(2) = 5$

$8 + a + b = 0 + 4 + 1$

$a + b = -3$ ①

substitute $x = -3$ $\quad P(-3) = -5$

$-27 - 3a + b = 0 - 6 + 1$

$-3a + b = 22$ ②

Solve simultaneously

$2a + b = -3$ ①

$-3a + b = 22$ ②

Subtract

$5a = -25$

$a = -25/5$

$a = -5$

subst. into ①

$-10 + b = -3$

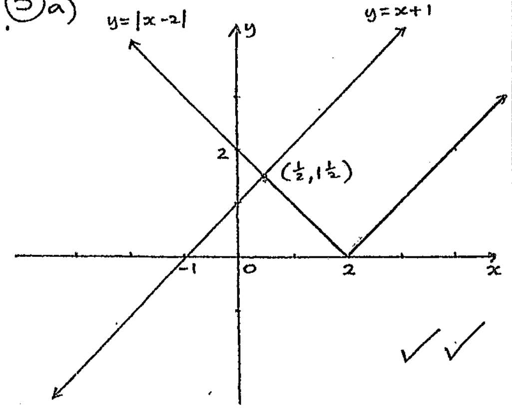
$b = 7$

Solution

$\therefore\ a = -5$

$b = 7$

⑤ a)



One point of intersection only when

$$x+1 = -(x-2)$$
$$x+1 = -x+2$$
$$2x = 1$$
$$x = 1/2$$

$(\frac{1}{2}, 1\frac{1}{2})$

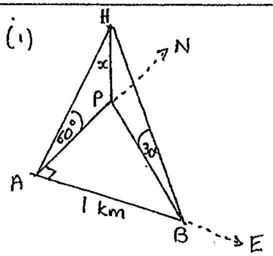
from the graph

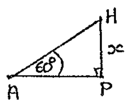
$|x-2| < x+1$

for $x > \frac{1}{2}$

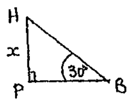
$\frac{6}{3}$

b) (i)



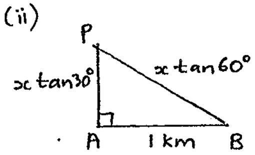


$$\tan 60° = \frac{x}{AP}$$
$$AP = \frac{x}{\tan 60°}$$
or $AP = x \tan 30°$



$$\tan 30° = \frac{x}{BP}$$
$$BP = \frac{x}{\tan 30°}$$
or $BP = x\tan 60°$

(ii)

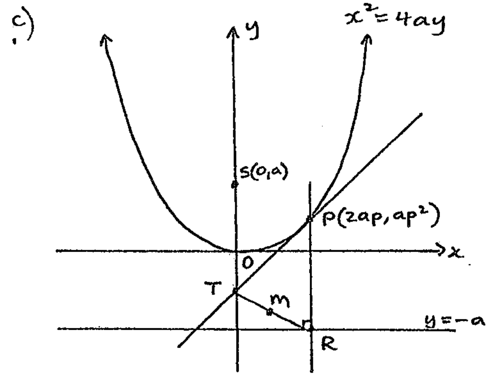


Using Pythagoras rule

$$BP^2 = AP^2 + AB^2$$
$$x^2\tan^2 60° = x^2 \tan^2 30° + 1^2$$
$$x^2\tan^2 60° - x^2\tan^2 30° = 1$$
$$x^2(\tan^2 60° - \tan^2 30°) = 1$$
$$x^2 = \frac{1}{\tan^2 60° - \tan^2 30°}$$
$$x = \sqrt{\frac{1}{\tan^2 60° - \tan^2 30°}}$$
$$\doteq \sqrt{\frac{1}{3 - 1/3}}$$
$$= \sqrt{\tfrac{3}{8}}$$
$$\doteq 0.612372\ldots \text{ km}$$
$$\doteq 612 \text{ m}$$

(to nearest metre)

c)



$x^2 = 4ay$

directrix $y = -a$

Coordinates of R

$(2ap, -a)$

Equation of tangent at P

$$y = \frac{x^2}{4a}$$
$$y' = \frac{2x}{4a}$$
$$= \frac{x}{2a}$$

At $P(2ap, ap^2)$

tangent $m_1 = \frac{2ap}{2a}$

$$= p$$
$$y - y_1 = m(x - x_1)$$
$$y - ap^2 = p(x - 2ap)$$
$$y - ap^2 = px - 2ap^2$$
$$y = px - ap^2$$

$\frac{10}{10}$

Coordinates of T

Tangent cuts y axis when $x=0$

$$y = px - ap^2$$
$$y = 0 - ap^2$$
$$y = -ap^2$$

$\therefore T(0, -ap^2)$

midpoint of TR

$T(0, -ap^2)$ $R(2ap, -a)$

$$m = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$
$$= \left(\frac{0+2ap}{2}, \frac{-ap^2-a}{2}\right)$$
$$= \left(ap, \frac{-a(p^2+1)}{2}\right)$$

Locus of m.

$$\begin{cases} x = ap \\ y = -\frac{a}{2}(p^2+1) \end{cases}$$

substitute $p = x/a$ into y

$$y = -\frac{a}{2}\left(\left(\frac{x}{a}\right)^2+1\right)$$
$$y = -\frac{a}{2}\left(\frac{x^2}{a^2}+1\right)$$
$$y = -\frac{x^2}{2a} - \frac{a}{2}$$

The locus is a parabola.

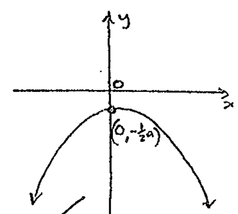
(x2a) $2ay = -x^2 - a^2$

$$x^2 = -2ay - a^2$$
$$x^2 = -2a\left(y + \tfrac{1}{2}a\right)$$
$$x^2 = -4 \cdot \tfrac{1}{2}a\left(y + \tfrac{1}{2}a\right)$$

The locus is a parabola.

Vertex $(0, -\frac{1}{2}a)$

Focal length $= \frac{1}{2}a$



Directrix $y = 0$

$\therefore$ The directrix is the x-axis.