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Centre Number

Student Number

SCEGGS Darlinghurst

2009
HIGHER SCHOOL CERTIFICATE
Assessment Task
Monday, 23rd March

Extension 1 Mathematics

Task Weighting: 30%

General Instructions

- Time allowed - 65 minutes
- Answer on the pad paper provided
- Write your student number at the top of each page
- Start each question on a new page
- Attempt **all** questions
- Marks may be deducted for careless or badly arranged work
- Mathematical templates and geometrical equipment may be used
- Approved scientific calculators should be used
- A table of standard integrals is provided

	Com	Calc	Reas	Marks
Question 1	/3	/2	/9	/14
Question 2	/3	/6	/4	/13
Question 3	/5		/8	/13
TOTAL	/11	/8	/21	/40

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

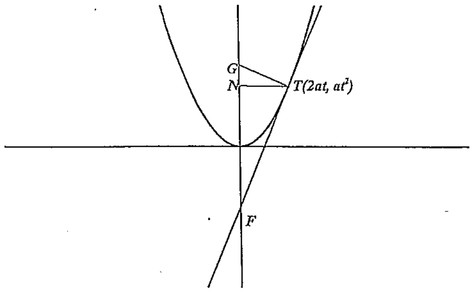
$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1 (14 Marks) **Marks**

a)



In the diagram a tangent is drawn at the point $T(2at, at^2)$ on the parabola $x^2 = 4ay$ to cut the y-axis at F, and a normal is drawn at T to cut the y-axis at G.

i) Find the equation of the tangent at T and hence find the coordinates of F. **3**

ii) If the equation of the normal at T is given by $x + ty = at^3 + 2at$, find the coordinates of G. **1**

iii) Show the focus of the parabola bisects the interval FG. **1**

iv) A line TN is drawn perpendicular to the y-axis and meeting it at N. Show that the interval NG has constant length for all positions of T. **1**

b) Put the following lines in the correct order so the solution reads correctly. **2**

Find $\int x\sqrt{1+x^2}\,dx$ let $u = 1 + x^2$ $du = 2x\,dx$

$$\frac{1}{2}\int \sqrt{u}du$$

$$\frac{1}{2}\int \sqrt{1+x^2}\;2x\,dx$$

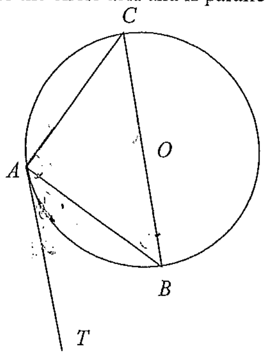
$$\frac{1}{2}u^{3/2} \times \frac{2}{3} + C$$

$$\frac{1}{3}\sqrt{(1+x^2)^3} + C$$

$$\frac{1}{2}\int u^{1/2}\,du$$

Question 1 (continued) **Marks**

c) In the diagram A, B and C are points on the circle with centre O. The line AT is a tangent to the circle at A and is parallel to the diameter CB. $\angle TAB = \alpha°$. **3**



Find the value of $\alpha°$ giving reasons.

d) Use mathematical induction to prove that for all integers $n \geq 0$: **3**

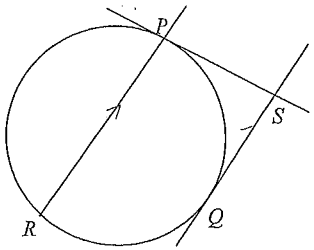
$$13 \times 6^n + 2 \text{ is divisible by } 5.$$

START A NEW PAGE

Question 2 (13 Marks) **Marks**

a) Find $\int \frac{3-x}{(3+x)^3}\,dx$ using the substitution $x = u - 3$. 3

b) P and Q are points on a circle and the tangents to the circle at P and Q meet at S. R is a point on the circle so that the chord PR is parallel to QS.



i) Copy the diagram on to your answer sheet.

ii) Prove $QP = QR$. 3

c) $P(2t, t^2)$ where $t > 0$ is a point on the parabola $x^2 = 4y$. The tangent to the parabola at P cuts the x-axis at T. $\angle OPT = \theta$ where O is the origin.

i) Find the gradient of OP and TP. 2

ii) Show that $\tan\theta = \frac{t}{t^2+2}$. 2

d) Use the substitution $u = 1 - x$ show that for $n > 0$: 3

$$\int_0^1 x(1-x)^n\,dx = \frac{1}{(n+1)(n+2)}$$

START A NEW PAGE

Question 3 (13 Marks) **Marks**

a) Prove by mathematical induction that: 3

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1} \text{ for all integers } n \geq 1$$

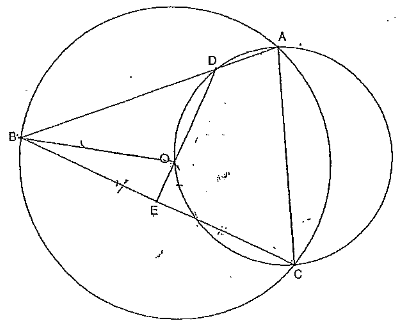
b) $P(2at, at^2)$ is a variable point on the parabola $x^2 = 4ay$, where the focus is S. $Q(x, y)$ divides the interval from P to S in the ratio $t^2 : 1$.

i) Find x and y in terms of a and t. 2

ii) Verify that $\frac{y}{x} = t$. 1

iii) Prove that, as P moves on the parabola, the locus of Q is a circle and state its centre and radius. 2

c) ABC is a triangle inscribed in a circle with centre O. A second circle through the points A, C, O cuts AB at D. DO is produced meets BC at E.



i) Copy the diagram.

ii) Prove that $\angle BOE = \angle BAC$. 2

iii) Show that $BE = CE$. 3

End of Paper

SCEGGS Darlinghurst Extension 1 Mathematics Ass. Tsk 1 2009 – Solutions

Q1 a) i) $x^2 = 4ay$

$y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{x}{2a}$

$\therefore$ when $x = 2at$

$\frac{dy}{dx} = \frac{2at}{2a}$

$= t$

$\therefore m_{tang} = t$ ✓

using $y - y_1 = m(x - x_1)$

$y - at^2 = t(x - 2at)$

$y - at^2 = tx - 2at^2$

$y = tx - at^2 \ldots (1)$ ✓

to find F let $x = 0$ and sub into (1)

$y = t \times 0 - at^2$

$= -at^2$

$F(0, -at^2)$ ✓

This was done well by most candidates. The presentation of solutions were clear and well explained.

ii) to find G sub $x = 0$ into $x + ty = at^3 + 2at$

$\therefore 0 + ty = at^3 + 2at$

$y = at^2 + 2a$

$\therefore G(0, at^2 + 2a)$ ✓

Well done

iii) midpoint of FG $\left(\frac{0+0}{2}, \frac{-at^2 + at^2 + 2a}{2}\right)$

$= (0, a)$ ✓ which is the focus of $x^2 = 4ay$

$\therefore$ focus bisects FG.

Most students recognised they needed to use the midpoint but it is important to explain that $(0,a)$ is the focus.

iv) $N(0, at^2)$

$NG = \sqrt{(0-0)^2 + (at^2 + 2a - at^2)^2}$

$= \sqrt{0^2 + (2a)^2}$

$= 2a$ $\therefore$ the length of NG is independent of t and hence constant for all ✓

(Reas – 6)

Many students didn't recognise how to do this question.

b) $\int x\sqrt{1+x^2}\,dx$ let $u = 1 + x^2$ $du = 2x\,dx$

$= \frac{1}{2}\int \sqrt{1+x^2}\; 2x\,dx$

$= \frac{1}{2}\int \sqrt{u}\,du$

$= \frac{1}{2}\int u^{1/2}\,du$

$= \frac{1}{2}u^{3/2} \times \frac{2}{3} + C$

$= \frac{1}{3}\sqrt{(1+x^2)^3} + C$ ✓✓ Calc – 2

Done very well by all students

c) $\angle TAB = \angle ACB = \alpha$ (angle between tangent and chord equals the angle in the alt. segment) ✓

$\angle TAB = \angle CBA = \alpha$ (alt. angles are eq. in parallel lines)

$\angle CAB = 90°$ (angle in a semi-circle) ✓

$\therefore$ In ΔABC

$\alpha + \alpha + 90 = 180$ (angle sum of a triangle)

$2\alpha + 90 = 180$

$2\alpha = 90$

$\alpha = 45$ ✓

Comm – 3

This was done well by most students. It was pleasing to see the clear reasons given.

d) 1. Show true for $n = 0$

$13 \times 6^0 + 2 = 15$

$= 5 \times 3$ ✓

$\therefore$ divisible by 5.

$\therefore$ result is true for $n = 0$

A few students didn't read the question and started with $n = 1$

2. Assume the result is true for $n=k$

i.e. $13\times 6^{k}+2 = 5M$ where M is an integer.

3. Show the result is true for $n=k+1$

i.e. $13\times 6^{k+1}+2 = 5Q$ where Q is an integer.

$$\begin{aligned}
\text{LHS} &= 13\times 6^{k+1}+2\\
&= 13\times 6\times 6^{k}+2\\
&= 6(5M-2)+2 \quad \text{from step 2} \ \checkmark\\
&= 30M-12+2\\
&= 30M-10\\
&= 5(6M-2) \quad \checkmark\\
&= 5Q \quad \text{as } M \text{ is an integer.}
\end{aligned}$$

$\therefore$ the result is true for $n=k+1$

Done fairly well except some students had trouble manipulating $13\times6\times6^k+2$ eg a few of you thought that the $+2$ was also multiplied by the 6.

4. Assuming the result is true for $n=k$ we have proven it true for $n=k+1$. Since the result is true for $n=0$ it is true for $n=1$ and so on for all integers $n\geq 0$

Reas - 3

Q2a)

$$\int \frac{3-x}{(3+x)^3}\,dx \qquad x=u-3,\ dx=du$$

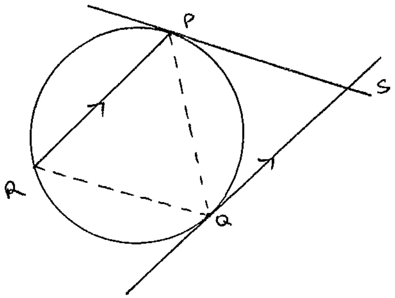
$$\begin{aligned}
&= \int \frac{3-(u-3)}{u^3}\,du \quad \checkmark\\
&= \int \frac{6-u}{u^3}\,du\\
&= \int 6u^{-3}-u^{-2}\,du \quad \checkmark\\
&= -3u^{-2}+u^{-1}+c\\
&= \frac{-3}{(x+3)^2}+\frac{1}{x+3}+c \quad \checkmark
\end{aligned}$$

Calc - 3

Mostly very well done - even the converting back to x in the end.

The biggest issue in this question was BRACKETS! If you subtract a bunch of terms you need to remember the brackets & expand that minus in.

b)



Proof: $PS=QS$ (tangents meeting at an external point are equal in length) ✓

$\angle SPQ = \angle SQP$ (angles opp equal sides are equal)

$\angle SPQ = \angle PRQ$ (angle between chord and tangent equals angle in alt. segment) ✓

$\angle SQP = \angle RPQ$ (alt. angles in parallel lines are equal)

$\therefore \angle RPQ = \angle PRQ$ ✓

$\therefore QR = QP$ (sides opp. equal angles are equal)

Comm - 3

The most efficient proof doesn't even use the fact that PS is a tangent; the property $QR=QP$ is only dependent on the fact RP is parallel to the tangent at Q

There are no congruent triangles in this diagram! There are similar Δ's... but it's not an efficient method of proof.

← This was not always written correctly and lost people marks.

c) i) $M_{OP} = \dfrac{t^2-0}{2t-0}$ ✓ $\qquad M_{TP} = t$ ✓ (gradient of the tangent)

$= \dfrac{t}{2}$

ii) $\tan\theta = \left|\dfrac{M_1-M_2}{1+M_1M_2}\right|$ (note: as $t>0$, $\angle OPT=\theta$ is acute)

$$\begin{aligned}
&= \left|\frac{t-t/2}{1+t\times t/2}\right| \quad \checkmark\\
&= \left|\frac{\frac{t}{2}}{1+t^2/2}\right|\\
&= \left|\frac{t/2}{\frac{2+t^2}{2}}\right| \quad \checkmark\\
&= \left|\frac{t}{\cancel{2}}\times\frac{\cancel{2}}{2+t^2}\right|\\
&= \frac{t}{t^2+2} \quad \text{as } t>0
\end{aligned}$$

Reas - 4

You use the formula with absolute value signs when θ is acute. You really need to justify that θ is acute and justify any disappearance of any absolute value signs.

(No marks were deducted for not doing this - but you should have done it!).

$$\int_0^1 x(1-x)^n\,dx \qquad u = 1-x \quad x=0\ u=1$$
$$du = -dx \quad x=1\ u=0$$

$$= -\int_0^1 x(1-x)^n \cdot -dx$$

$$= -\int_1^0 (1-u)u^n\,du \quad ✓$$

$$= \int_0^1 u^n - u^{n+1}\,du$$

$$= \left[\frac{u^{n+1}}{n+1} - \frac{u^{n+2}}{n+2}\right]_0^1 \quad ✓$$

$$= \left(\frac{1^{n+1}}{n+1} - \frac{1^{n+2}}{n+2}\right) - 0$$

$$= \frac{1}{n+1} - \frac{1}{n+2}$$

$$= \frac{n+2-(n+1)}{(n+1)(n+2)} \quad ✓$$

$$= \frac{1}{(n+1)(n+2)}$$

Calc - 3

This idea is worth remembering. That is,

$$-\int_a^b f(x)\,dx = \int_b^a f(x)\,dx$$

Note:
This is ~~not~~ an induction question!

Q3a) 1. Show true for $n=1$

$LHS = \frac{d}{dx}(x) = 1 \qquad RHS = 1 \times x^{1-1} = 1$

$\therefore$ result is true for $n=1$ ✓

2. Assume true for $n=k$

i.e. $\frac{d}{dx}x^k = kx^{k-1}$

3. Show result is true for $n=k+1$

i.e. $\frac{d}{dx}x^{k+1} = (k+1)x^k$

$LHS = \frac{d}{dx}x^{k+1}$

$= \frac{d}{dx}(x \times x^k)$

$= \frac{d}{dx}(x) \times x^k + x \times \frac{d}{dx}(x^k)$ ✓

$= 1 \times x^k + x \times kx^{k-1}$ from step 2

$= x^k + kx^k$ ✓

Most students were unaware of the requirements of this questions (and therefore fudged the solution!

Whenever you need to differentiate functions that are multiplied together think product rule!

$= x^k(k+1)$

$\therefore$ result is true for $n=k+1$

Assuming the result is true for $n=k$ we have proven it true for $n=k+1$. Since the result is true for $n=1$ it is true for $n=2$ and so on for all integers $n \geq 1$

Reas - 3

b) i) $P(2at, at^2) \quad S(0,a)$

$t^2 : 1$

$x = \frac{2at + t^2 \times 0}{t^2+1} \qquad y = \frac{at^2 + a \times t^2}{t^2+1}$

$Q\left(\frac{2at}{t^2+1}, \frac{2at^2}{t^2+1}\right)$ ✓✓

This question challenged students to remember work from the preliminary course. Those who used the ~~x~~ method were more successful than those who used the formula i.e. $x = \frac{mx_2 + nx_1}{m+n}$

ii) $\frac{y}{x} = \frac{2a\cancel{t^2}}{\cancel{t^2+1}} \times \frac{\cancel{t^2+1}}{\cancel{2at}}$ ✓

$= t$

iii) $x = \frac{2at}{t^2+1}$

$x = \frac{2a\left(\frac{y}{x}\right)}{\left(\frac{y}{x}\right)^2+1}$ ✓

$x = \frac{\frac{2ay}{x}}{\frac{y^2}{x^2}+1}$

$x = \frac{\frac{2ay}{x}}{\frac{y^2+x^2}{x^2}}$

$x = \frac{2ay}{\cancel{x}} \times \frac{x^{\cancel{2}}}{y^2+x^2}$

$x(x^2+y^2) = 2ayx$

$x^2+y^2 = 2ay$

$x^2+y^2-2ay = 0$

$x^2+y^2-2ay+a^2 = a^2$

Most students didn't attempt this question which is fair enough as it was tough!

$$x^2 + (y-a)^2 = a^2$$

$\therefore$ the locus of Q is a circle centre $(0,a)$ radius a ✓ Reas - 5

c) ii) $\angle COE = \angle BAC$ (exterior angle of a cyclic quad ADOC is equal to opp. interior angle) ✓

Some students did not recognise that it was ODAC that was cyclic

$\angle BOC = 2\angle BAC$ (angle subtended by arc BC at centre is twice that at circumf. of circle ABC) ✓

$\therefore \angle BOE = \angle BAC$ ($\angle BOC$ is sum of adjacent angles $\angle BOE, \angle COE$)

iii) In $\triangle BOE$, $\triangle COE$

$\angle BOE = \angle COE$ (each proven equal to $\angle BAC$ above) ✓

$BO = CO$ (radii of circle ABC are equal) ✓

OE is common

$\therefore \triangle BOE \equiv \triangle COE$ (SAS) ✓

$\therefore BE = CE$ (corresponding sides of congruent Δ's are equal).

Comm - 5

This question was easy to do even if you could not do ii). It is important to remember with the later questions to try and get marks anyway you can. Assuming something you haven't shown is OK to prove another result.